Power Systems Analysis

Chapter 1 Basic concepts

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Outline

1. Phasor representation
2. KCL, KVL, Ohm's law, Tellegen's theorem
3. Balanced three-phase systems
4. Complex power

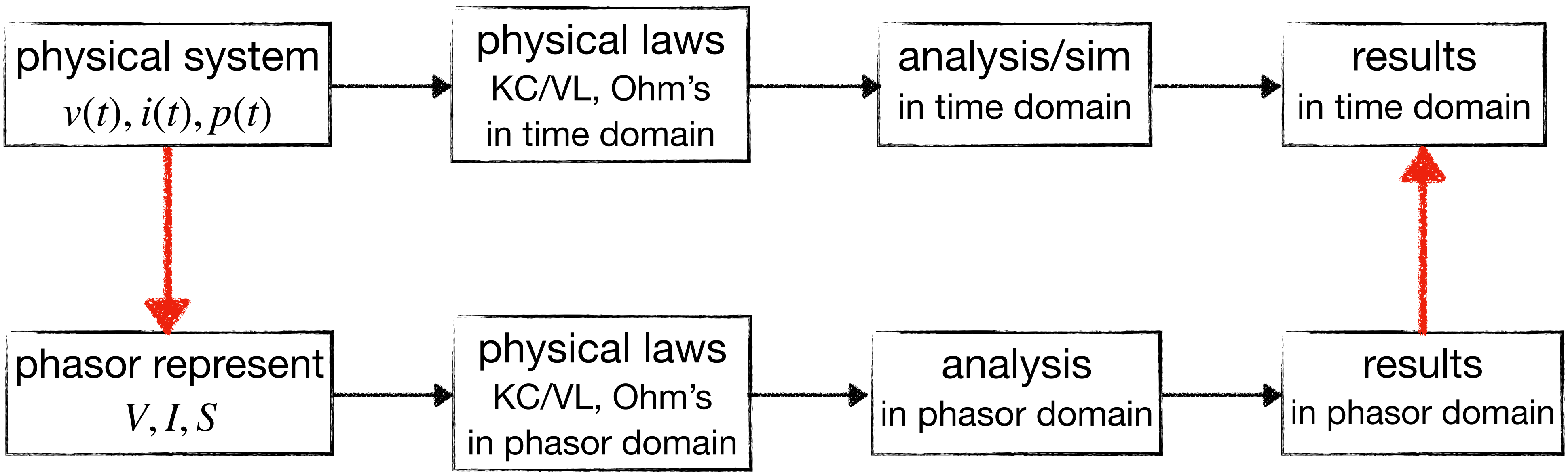
Outline

1. Phasor representation
   - Voltage & current phasors
   - Single-phase devices
2. KCL, KVL, Ohm's law, Tellegen's theorem
3. Balanced three-phase systems
4. Complex power

Physical quantities

1. Voltage, current, power, energy
2. All are sinusoidal functions of time
3. Steady state:
    - Frequencies at all points are nominal: $\omega = 60$ Hz in US, 50 Hz in China, Europe
    - Reasonable model at timescales of minutes and up

Power system analysis



Voltage phasor

1. Voltage: $v(t) = V_{\text{max}} \cos(\omega t + \theta_V) = \text{Re}\left\{V_{\text{max}} e^{i\theta_V} \cdot e^{i\omega t}\right\}$
   - ω : nominal system frequency
   - V_{max} : amplitude
   - θ_V : phase (angle)
2. Phasor: $V := \dfrac{V_{\text{max}}}{\sqrt{2}} e^{i\theta_V}$ volt (V)
3. Relationship: $v(t) = \text{Re}\{\sqrt{2} V \cdot e^{i\omega t}\}$

Current phasor

1. Current: $i(t) = I_{\text{max}} \cos(\omega t + \theta_I) = \text{Re}\left\{ I_{\text{max}} e^{i\theta_I} \cdot e^{i\omega t} \right\}$

2. Phasor: $I := \frac{I_{\text{max}}}{\sqrt{2}} e^{i\theta_I}$ ampere (A)

3. Relationship: $i(t) = \text{Re}\{\sqrt{2} I \cdot e^{i\omega t}\}$

Single-phase devices

1. Impedance Z
2. Voltage source (E, Z)
3. Current source (J, Y)
4. Transmission/distribution line (Chapter 2)
5. Transformer (Chapter 3)

Impedance

$$v(t) = R \cdot i(t)$$

$$v(t) = L \cdot \frac{di}{dt}(t)$$

$$i(t) = C \cdot \frac{dv}{dt}(t)$$

these are main circuit elements to model the grid

Impedance

$$v(t) = R \cdot i(t)$$

$$V = R \cdot I$$

$$v(t) = L \cdot \frac{di}{dt}(t)$$

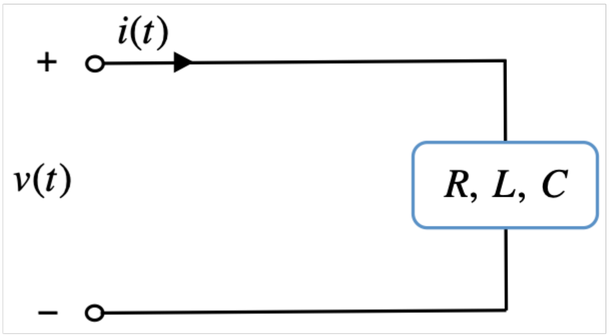
$$V = j\omega L \cdot I$$

$$i(t) = C \cdot \frac{dv}{dt}(t)$$

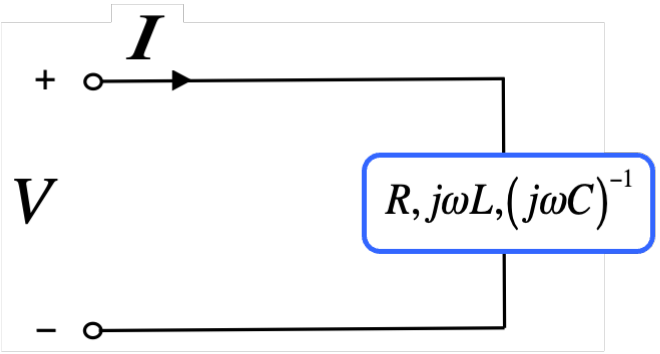
$$V = (j\omega C)^{-1} \cdot I$$

Impedance

time domain



phasor domain



Impedance

In general, impedance $Z = R + iX$

- R : resistance Ω
- X : reactance Ω

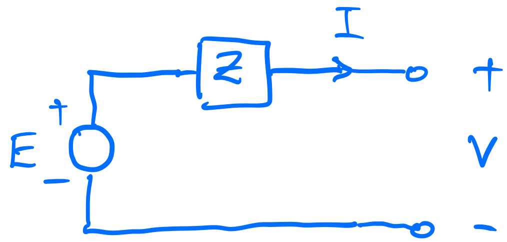
Admittance $Y := Z^{-1} =: G + iB$

- G : conductance Ω^{-1}
- B : susceptance Ω^{-1}

Voltage source

Voltage source (E, Z)

- E : internal voltage
- Z : internal impedance
- Internal model



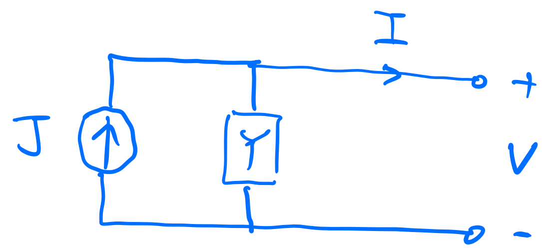
External model

- V : terminal voltage
- I : terminal current
- Relation between (V, I) : $V = E - ZI$

Current source

Current source (J, Y)

- J : internal current
- Y : internal admittance
- Internal model



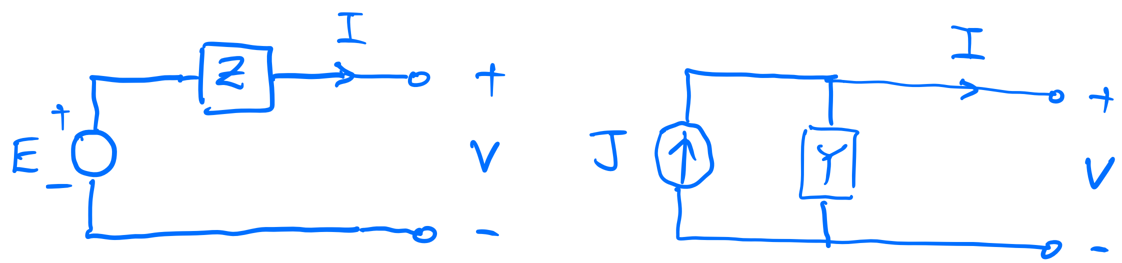
External model

- V : terminal voltage
- I : terminal current
- Relation between (V, I) : $I = J - YV$

Equivalent source

A nonideal voltage source (E, Z) and current source (J, Y) are equivalent if

- $J = \dfrac{E}{Z}, \qquad Y = Z^{-1}$
- They have the same external model



Circuit models

These are circuit models of physical devices

Device	Circuit model
Generator	Voltage source, current source
Load	Impedance, voltage source, current course
Line	Impedance (Chapter 2)
Transformer	Impedance, voltage/current gain (Chapter 3)

Outline

1. Phasor representation
2. Linear circuit analysis
    - KCL, KVL, Ohm's law, Tellegen's theorem
3. Balanced three-phase systems
4. Complex power

Circuit analysis: review

A brief review of circuit analysis for EE students

Mathematical background required

- Basic algebraic graph theory (Chapter 26.2 of Draft Notes)

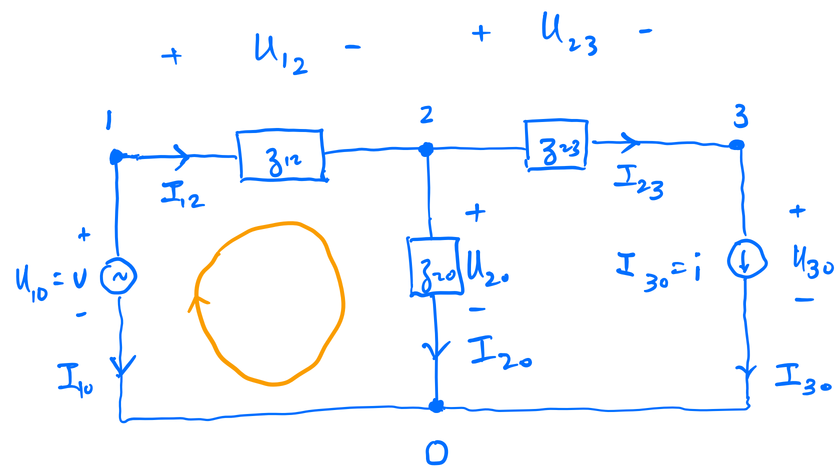
Notation

Directed graph $G := (N, E)$

- Arbitrary orientation
- Link (j, k) or $j \rightarrow k$ in E
- Reference node 0 with $V_0 := 0$ by definition

Variables

- Nodal voltage V_j at node j wrt reference node 0
- Branch voltage $U_{jk} := V_j - V_k$ across link (j, k)
- Branch current I_{jk} across link $j \rightarrow k$



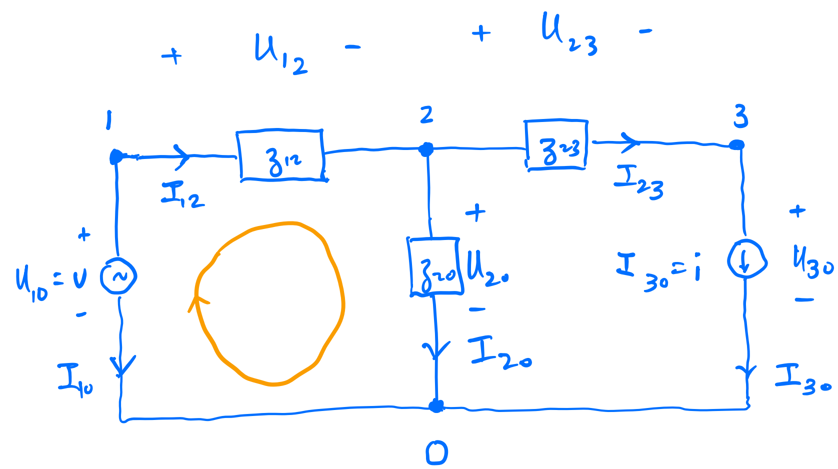
KCL, KVL

KCL: incident currents at any node j sum to zero

$$-\sum_{i:i\to j\in E} I_{ij} + \sum_{k:j\to k\in E} I_{jk} = 0$$

KVL: voltage drops around any cycle c sum to zero

$$\sum_{l\in c} U_l - \sum_{-l\in c} U_l = 0$$

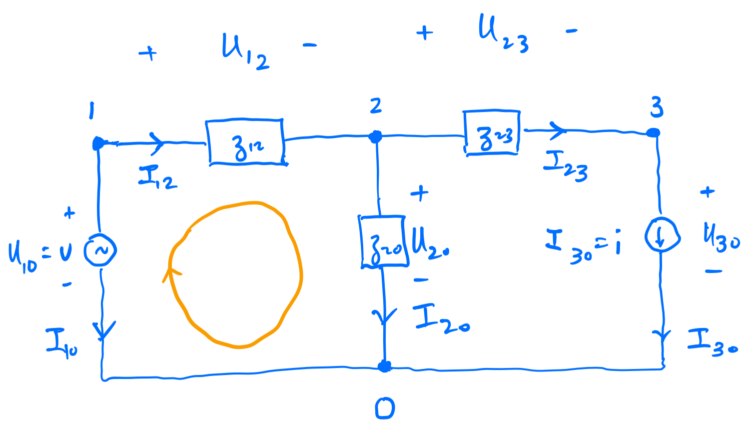


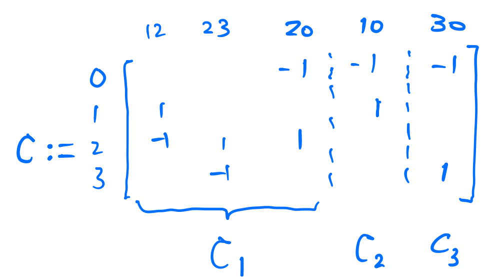
KCL, KVL

Incident matrix C

$|N| \times |E|$ incident matrix

$$C_{jl} \;:=\; \begin{cases} 1 & \text{if } l = j \rightarrow k \text{ for some bus } k \\ -1 & \text{if } l = i \rightarrow j \text{ for some bus } i \\ 0 & \text{otherwise} \end{cases}, \qquad j \in N,\; l \in E$$





KCL, KVL

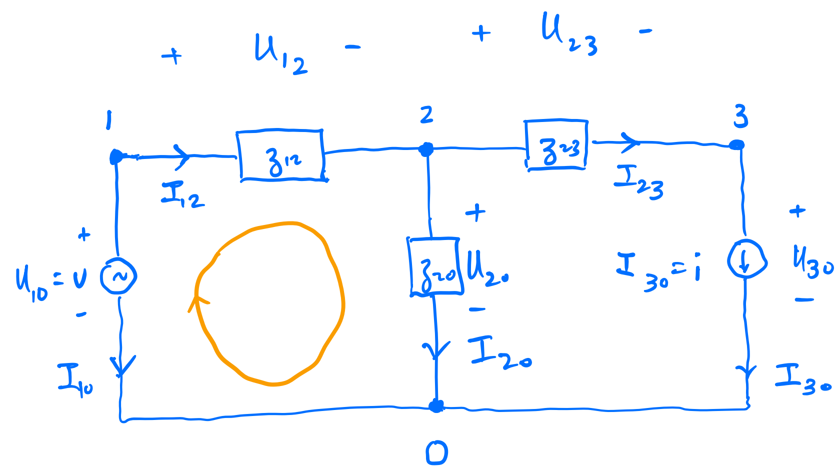
Vector form

KCL: incident currents at any node j sum to zero

$$C I \; = \; 0$$

KVL: there exists nodal voltages $V \in \mathbb{C}^{|N|}$ s.t.

$$U \; = \; C^{\mathsf{T}} V$$



$|N| + |E|$ equations in $|N| + 2|E|$ variables (V, U, I)

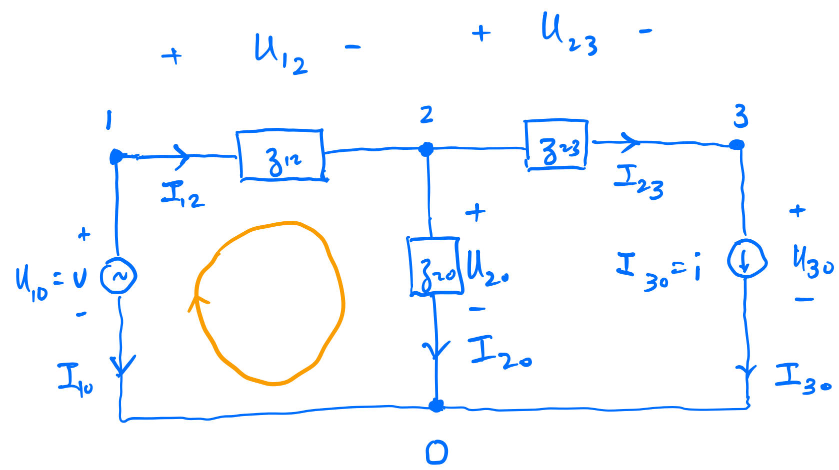
- C has rank $N - 1$, $V_0 := 0$
- $|N| + |E| - 1$ linearly independent equations in $|N| + 2|E| - 1$ variables (V_{-0}, U, I)

Need another $|E|$ equations

Device specification

Across each link (j, k) is exactly one device

1. Impedance with given z_{jk} : $U_{jk} \;=\; z_{jk} I_{jk}$
2. Voltage source with given v_{jk} : $U_{jk} \;=\; v_{jk}$
3. Current source with given i_{jk} : $I_{jk} \;=\; i_{jk}$

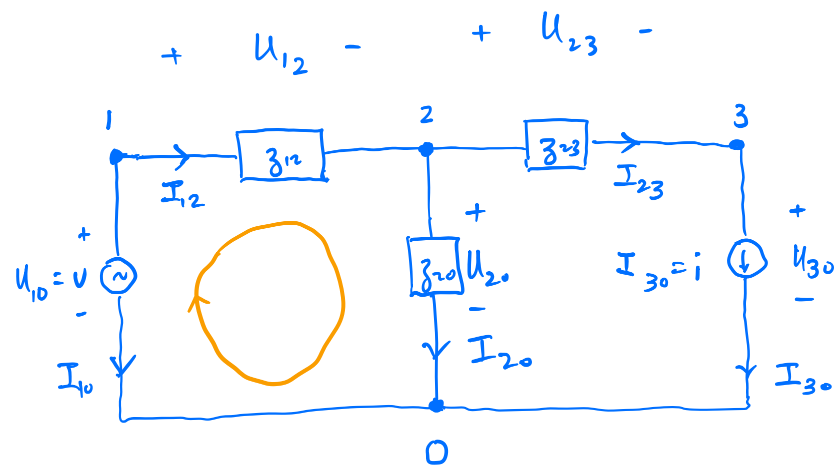


These device specifications provide additional $|E|$ equations

Circuit analysis

Solve for (V, U, I)

- Impedance: $U_{jk} \;=\; z_{jk} I_{jk}$
- Voltage source: $U_{jk} \;=\; v_{jk}$
- Current source: $I_{jk} \;=\; i_{jk}$
- KCL: $CI \;=\; 0$
- KVL: $U \;=\; C^{\mathsf{T}} V$
- Reference voltage: $V_0 := 0$



Tellegen's theorem

Tellegen's theorem is consequence of 3 facts

- $C^{|E|} \;=\;$ null$(C) \oplus$ range $\left(C^{\mathsf{T}}\right)$ is direct sum
- KCL: $CI = 0$, i.e., $I \in$ null(C)
- KVL: $U = C^{\mathsf{T}} V$, i.e., $U \in$ range $\left(C^{\mathsf{T}}\right)$

Therefore branch currents I and branch voltages U are orthogonal:

- $I^{\mathsf{H}} U \;=\; 0$ (Tellegen's theorem)

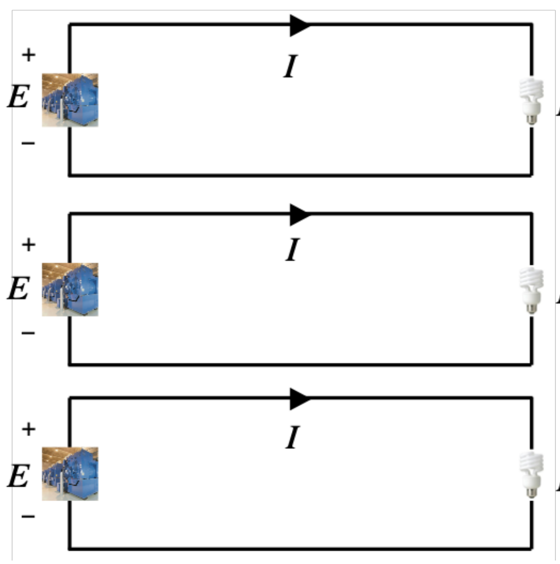
I and U can be from different networks as long as they have the same incidence matrix C !

Outline

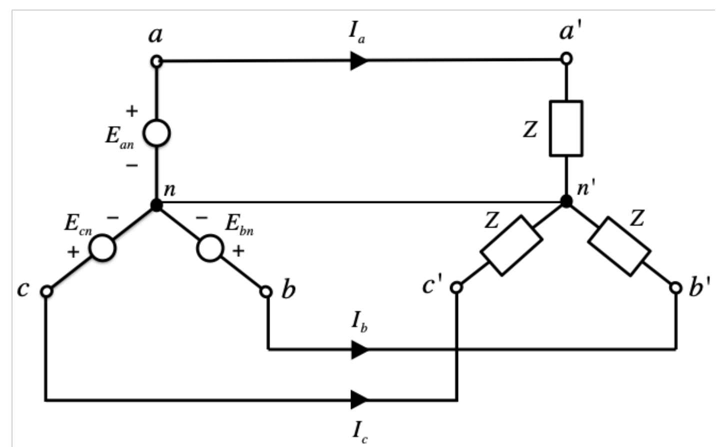
1. Phasor representation
2. Linear circuit analysis
3. Balanced three-phase systems
   - Y and Δ configuration
   - Balanced vectors and conversion matrices
   - Balanced systems in Y and Δ configurations
   - Δ-Y transformation
   - Per-phase analysis
4. Complex power

Balanced 3-phase system

3 single-phase system:



single 3-phase system:



Y configuration
Internal variables

Each single-phase device can be arbitrary

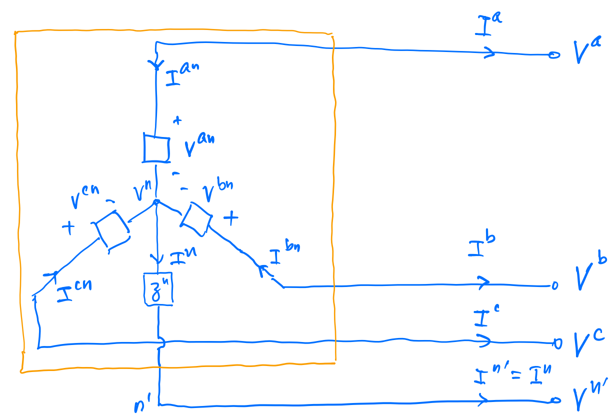
- Voltage source, current source, impedance, ideal or not

Internal voltages and currents across single-phase devices:

$$V^Y := \begin{bmatrix} V^{an} \\ V^{bn} \\ V^{cn} \end{bmatrix}, \quad I^Y := \begin{bmatrix} I^{an} \\ I^{bn} \\ I^{cn} \end{bmatrix}$$

neutral voltage (wrt common reference pt) $V^n \in \mathbb{C}$

neutral current (away from neutral) $I^n \in \mathbb{C}$



- Neutral line may or may not be present
- Device may or may not be grounded
- Neutral impedance z^n may or may not be zero

For single-phase models, we sometimes assume $V^n = 0$

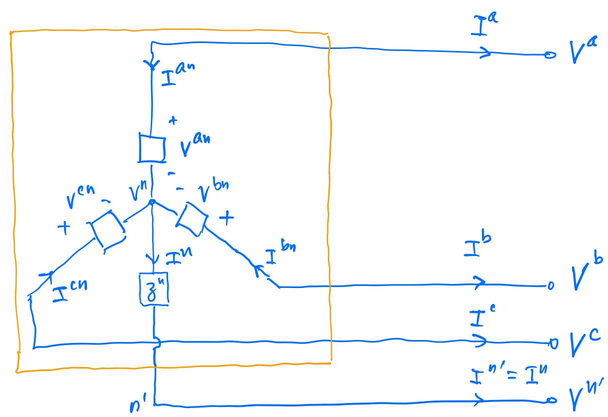
Y configuration

Terminal variables

Terminal voltages and currents:

$$V := \begin{bmatrix} V^a \\ V^b \\ V^c \end{bmatrix}, \quad I := \begin{bmatrix} I^a \\ I^b \\ I^c \end{bmatrix}$$

- V is with respect to common reference, e.g. ground
- I is in direction out of device



Conversion from internal to terminal variables

$$V = V^Y + v^n 1, \quad I = -I^Y$$

- $V = V^Y$ if $V^n = 0$, i.e., if neutral is directly grounded and ground is the reference

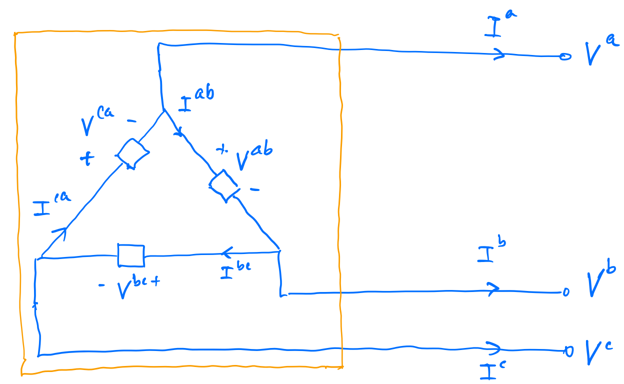
Δ configuration
Internal variables

Each single-phase device can be arbitrary

- Voltage source, current source, impedance, ideal or not

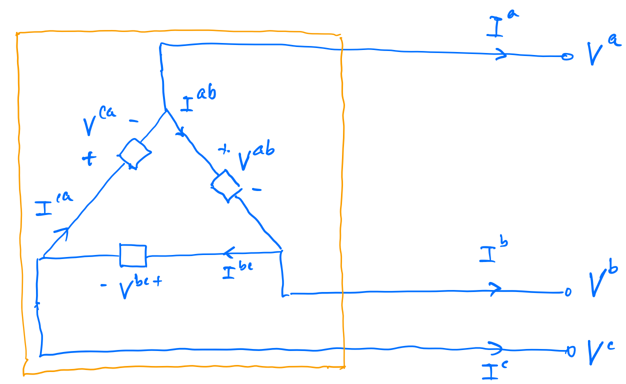
Internal voltages and currents across single-phase devices:

$$V^{\Delta} := \begin{bmatrix} V^{ab} \\ V^{bc} \\ V^{ca} \end{bmatrix}, \quad I^{\Delta} := \begin{bmatrix} I^{ab} \\ I^{bc} \\ I^{ca} \end{bmatrix}$$



Δ configuration
Terminal variables

Terminal voltages and currents:



$$V := \begin{bmatrix} V^a \\ V^b \\ V^c \end{bmatrix}, \quad I := \begin{bmatrix} I^a \\ I^b \\ I^c \end{bmatrix}$$

- Same for Y configured devices

Conversion between internal and terminal variables

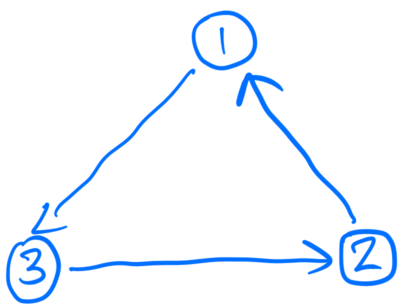
$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}}_{\Gamma} \begin{bmatrix} V^a \\ V^b \\ V^c \end{bmatrix} = \begin{bmatrix} V^{ab} \\ V^{bc} \\ V^{ca} \end{bmatrix}, \qquad \begin{bmatrix} I^a \\ I^b \\ I^c \end{bmatrix} = -\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{\Gamma^{\mathsf{T}}} \begin{bmatrix} I^{ab} \\ I^{bc} \\ I^{ca} \end{bmatrix}$$

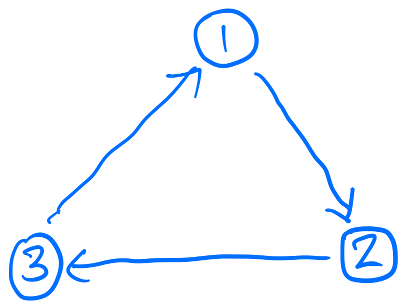
- In vector form: $\Gamma V = V^{\Delta}, \qquad I = -\Gamma^{\mathsf{T}} I^{\Delta}$

Conversion matrices

Conversion matrices

$$\Gamma \; := \; \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \qquad \Gamma^{\mathsf{T}} \; := \; \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$





Spectral properties of $(\Gamma, \Gamma^{\mathsf{T}})$ underlie much of three-phase (**balanced** or unbalanced) systems

Balanced systems

Balanced vector

Definition

A vector $x := \left(x_1, x_2, x_3\right)$ with $x_j = |x_j| e^{i\theta_j} \in \mathbb{C}$ is called balanced if

- $|x_1| \;=\; |x_2| \;=\; |x_3|$
- Either $\theta_2 - \theta_1 = -\frac{2\pi}{3}$ and $\theta_3 - \theta_1 = \frac{2\pi}{3}$ (positive sequence)

  or $\theta_2 - \theta_1 = \frac{2\pi}{3}$ and $\theta_3 - \theta_1 = -\frac{2\pi}{3}$ (negative sequence)

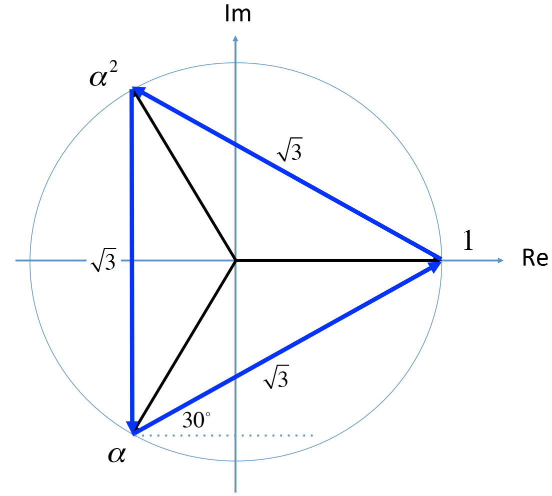
Balanced systems

Standard balanced vectors

Let $\alpha := e^{-i2\pi/3}$

Standard positive-sequence vector $\alpha_+ := \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \end{bmatrix}$

Standard negative-sequence vector $\alpha_- := \begin{bmatrix} 1 \\ \alpha^2 \\ \alpha \end{bmatrix}$



All balanced positive-seq vectors are in span(α_+)

All balanced negative-seq vectors are in span(α_-)

A **balanced system** is one in which all voltages and currents are in span(α_+) (WLOG)

Balanced systems

Transformation by $(\Gamma, \Gamma^{\mathsf{T}})$

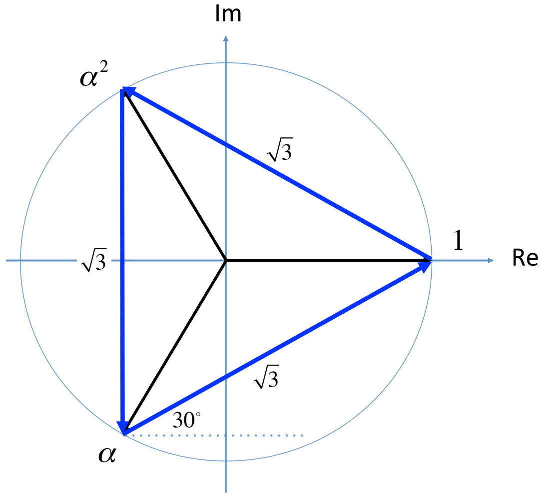
Theorem (Transformation of balanced vectors by $(\Gamma, \Gamma^{\mathsf{T}})$)

1. Eigenvalues and eigenvectors of Γ are

$$\Gamma 1 = 0, \qquad \Gamma \alpha_+ = (1-\alpha)\alpha_+, \qquad \Gamma \alpha_- = (1-\alpha^2)\alpha_-$$

2. Eigenvalues and eigenvectors of Γ^{T} are

$$\Gamma^{\mathsf{T}} 1 = 0, \qquad \Gamma^{\mathsf{T}} \alpha_+ = (1-\alpha^2)\alpha_+, \qquad \Gamma^{\mathsf{T}} \alpha_- = (1-\alpha)\alpha_-$$



These properties will be used repeatedly, for both balanced and unbalanced systems

Balanced systems

Transformation by $(\Gamma, \Gamma^{\mathsf{T}})$

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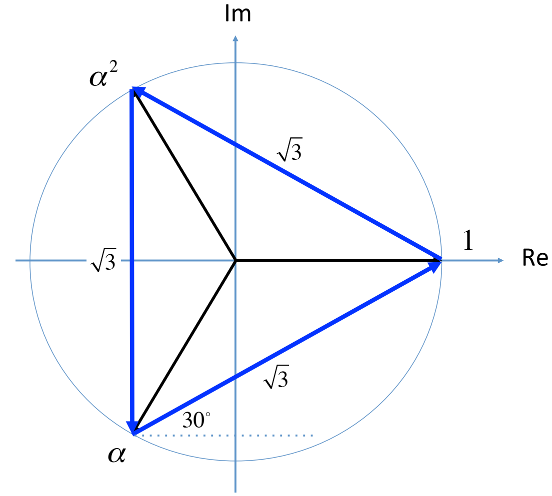


Application to balanced systems have the following implications ….

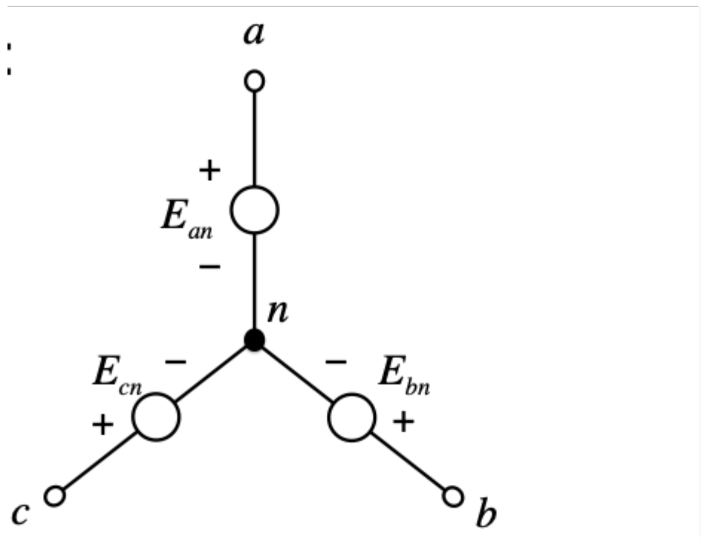
Balanced systems

Implications

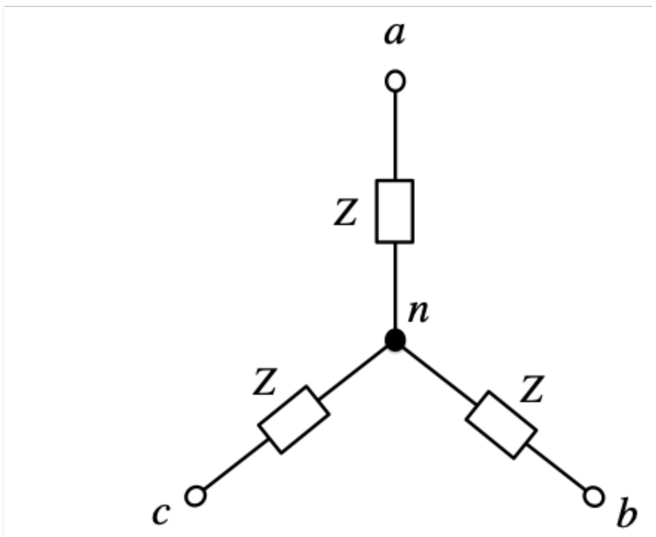
1. Informally, a balanced system is one in which all voltages and currents are in $\text{span}(\alpha_+)$ (WLOG)
2. Balanced voltage and current sources are in $\text{span}(\alpha_+)$
3. Voltages and currents at every point in a network can be written as linear combination of transformed source voltages and source currents, transformed by $(\Gamma, \Gamma^{\mathsf{T}})$
4. But α_+ are eigenvectors of $(\Gamma, \Gamma^{\mathsf{T}}) \implies$ transformation by $(\Gamma, \Gamma^{\mathsf{T}})$ reduces to scaling by $1-\alpha$ and $1-\alpha^2$ respectively (provided impedances & lines are balanced)
5. $\implies$ all voltages and currents remain in $\text{span}(\alpha_+)$

Formal statement and proof need to wait till Part II where we study unbalanced systems

Y configuration
Balanced system

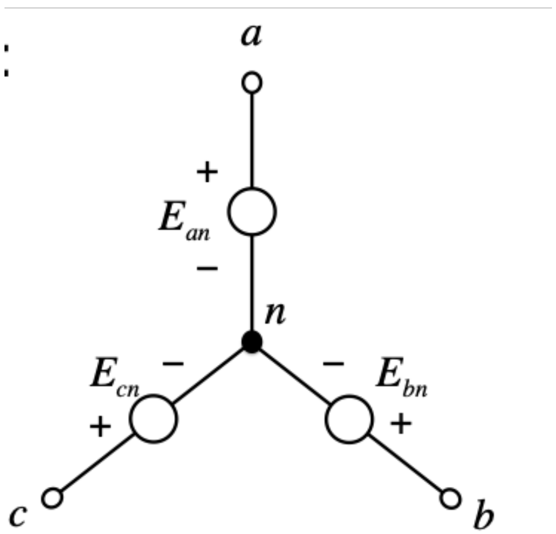


3ϕ voltage source



3ϕ impedance

Y configuration
Balanced system

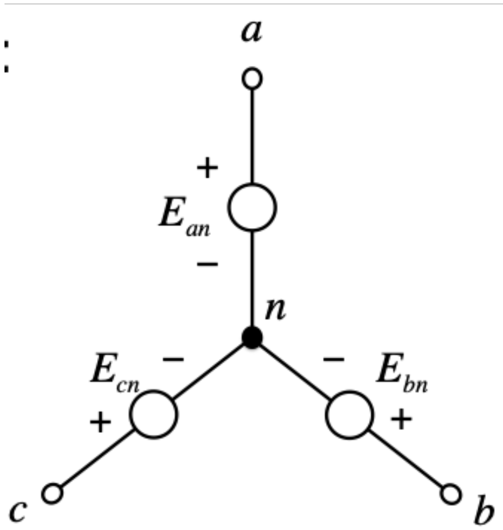


E_{an}, E_{bn}, E_{cn} : line-to-neutral or phase voltages E^Y

E_{ab}, E_{bc}, E_{ca} : line-to-line or line voltages E^{line}

Y configuration

Balanced system

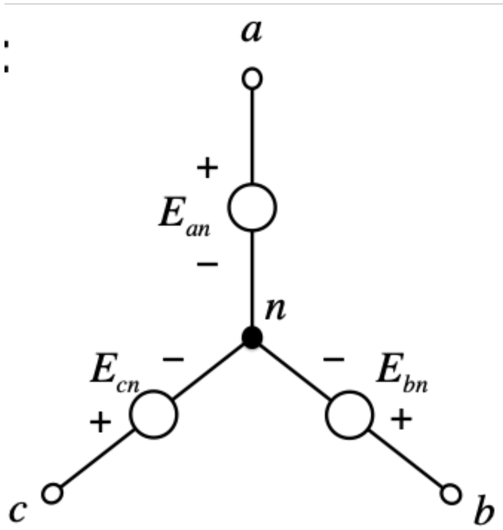


Voltage source is balanced (in positive-seq set) if $E^Y \in \text{span}\left(\alpha_+\right)$:

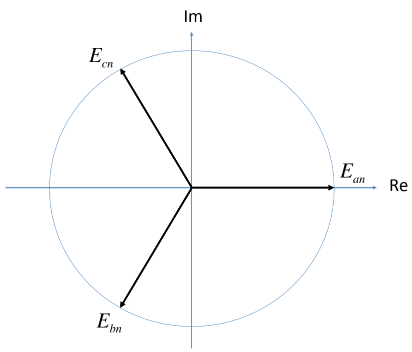
$$E_{an} = 1\angle\theta, \quad E_{bn} = 1\angle\theta - 120^\circ, \quad E_{cn} = 1\angle\theta + 120^\circ$$

Y configuration

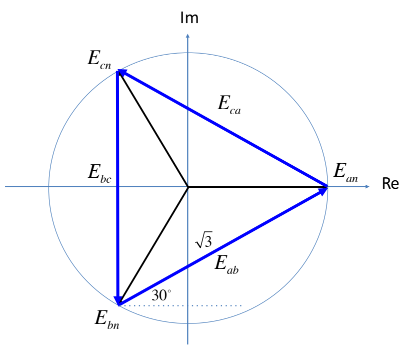
Implications of Theorem



1. Sum to zero: $E_{an} + E_{bn} + E_{cn} = 0$
   - $1^{\mathsf{T}} E^Y = 1^{\mathsf{T}} \alpha_+ = 0$
2. Line voltages are balanced positive sequence
   - $V^{\text{line}} = \Gamma E^Y = (1 - \alpha)\alpha_+$
3. Phases are decoupled



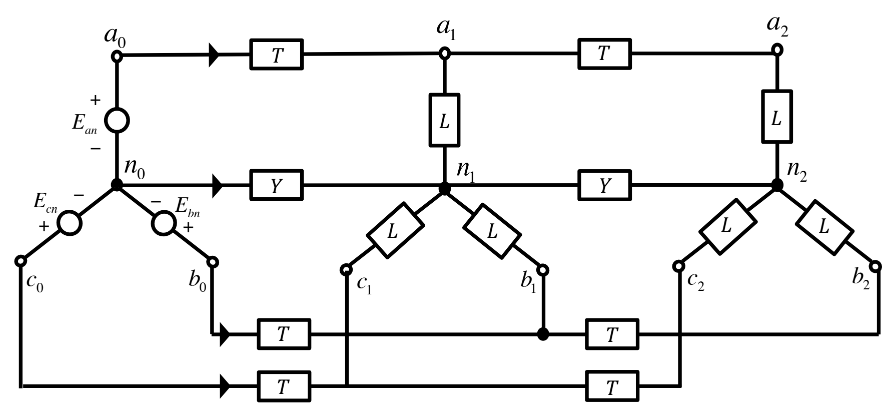
(a) Phase voltages



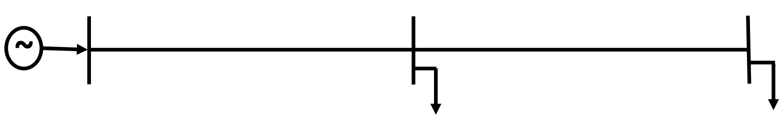
(b) Phase and line voltages

Y configuration
Phase decoupling

Example



One line diagram:



Show:

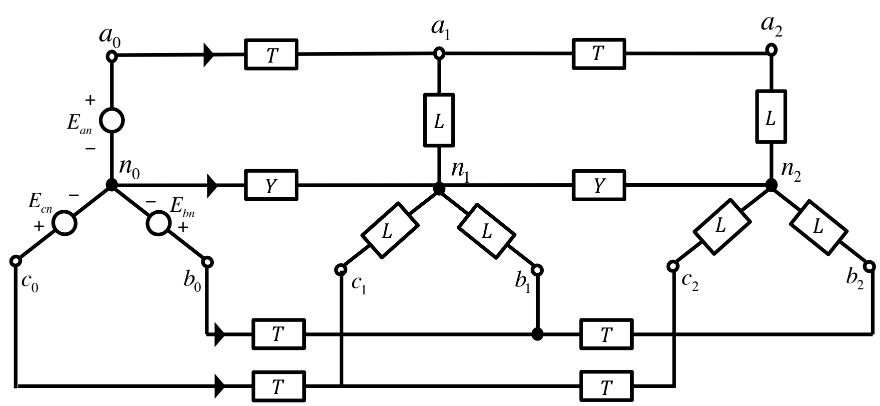
1. $V_{n_0 n_1} = V_{n_1 n_2} = 0$
2. All currents and voltages are balanced positive sequence sets
3. Phases are decoupled, i.e.,

$$E_{a_0 n_0} = V_{a_0 a_1} + V_{a_1 n_1}$$
$$V_{a_1 n_1} = V_{a_1 a_2} + V_{a_2 n_2}$$

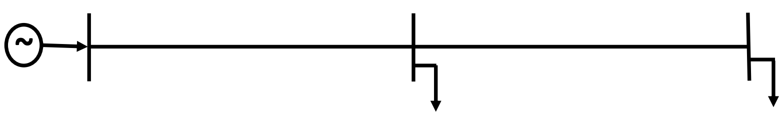
Y configuration

Phase decoupling

Example



One line diagram:



Show:

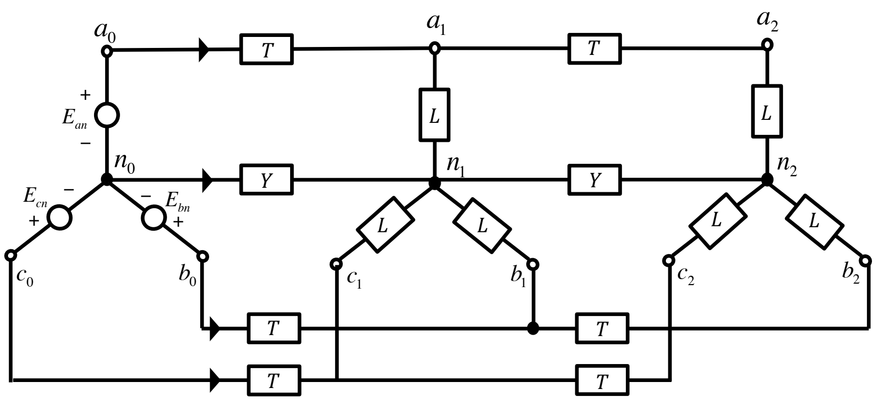
1. $V_{n_0 n_1} = V_{n_1 n_2} = 0$

Implications:

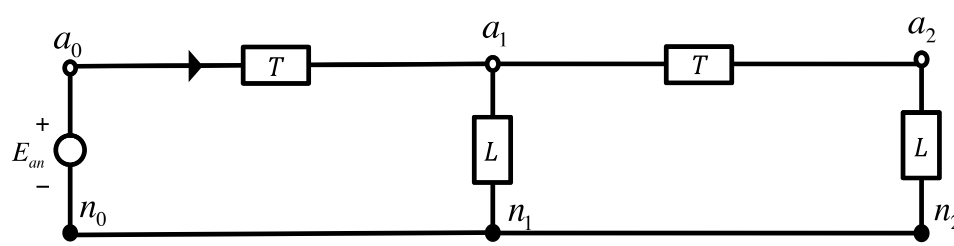
- Zero currents on neutral lines even if present $\Rightarrow$ can assume neutrals are connected or not for analysis
- No physical wires necessary for return currents, saving materials

Y configuration
Per-phase circuit

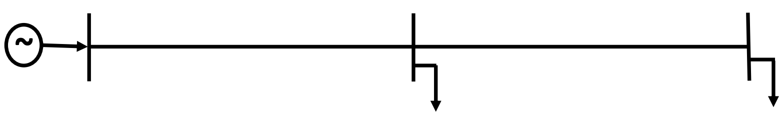
Example



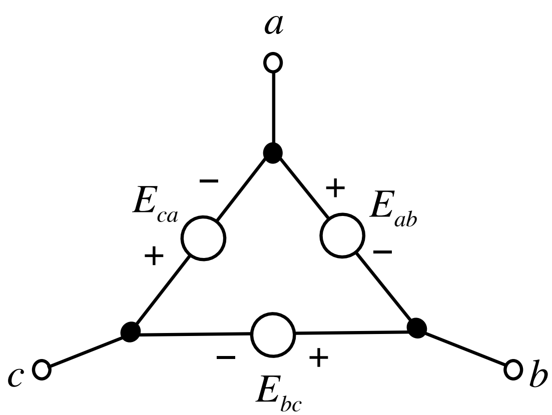
Per-phase equivalent circuit:



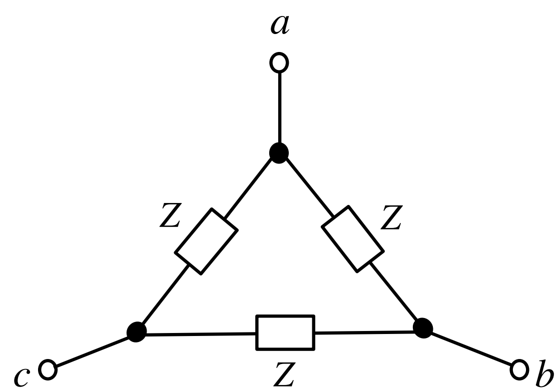
One line diagram:



Δ configuration
Balanced system

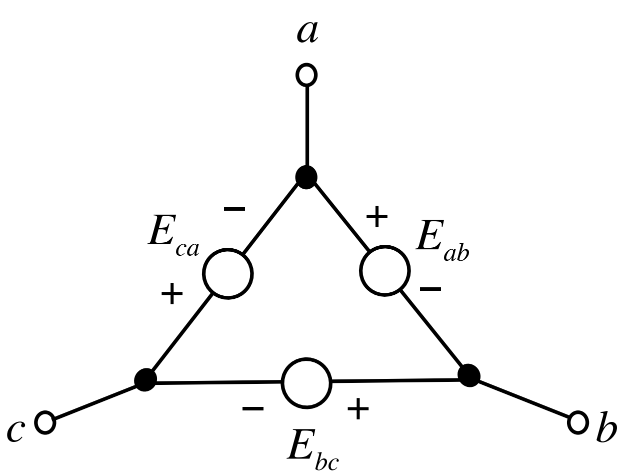


3ϕ voltage source



3ϕ impedance

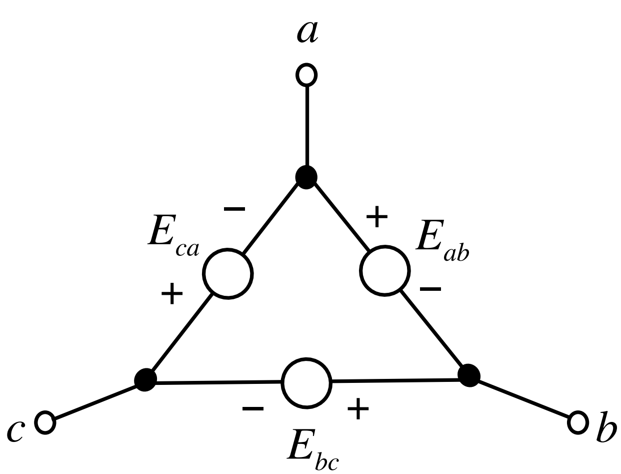
Δ configuration
Balanced system



Voltage source is balanced (in positive-seq set) if $E^{\Delta} \in \text{span}\left(\alpha_+\right)$:

$$E_{bc} = e^{-i2\pi/3} E_{ab}, \qquad E_{ca} = e^{i2\pi/3} E_{ab}$$

Δ configuration
Implications of Theorem



1. Sum to zero: $E_{ab} + E_{bc} + E_{ca} = 0$
   - $1^\mathsf{T} E^\Delta = 1^\mathsf{T} \alpha_+ = 0$
2. Y equivalent voltage source is balanced
3. Phases are decoupled

Δ and Y transformation

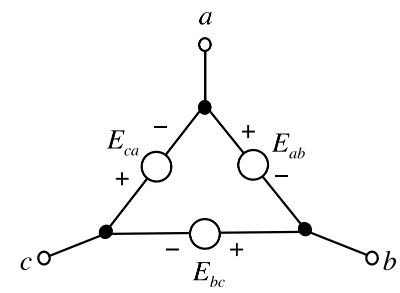
1. They are equivalent if they have the same external behavior
2. Given $\left(V^{\Delta}, I^{\Delta}\right)$, the internal $\left(V^{Y}, I^{Y}\right)$ of the Y equivalent must satisfy

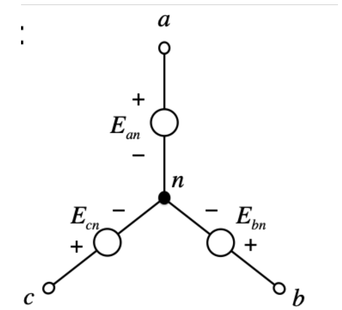
$$\Gamma\, V^Y \;=\; V^{\Delta}, \qquad I^Y \;=\; \Gamma^{\mathsf{T}} I^{\Delta}$$

3. If $\left(V^{\Delta}, I^{\Delta}\right)$ are balanced vectors then

$$V^Y \;=\; \frac{1}{1-\alpha}\, V^{\Delta} \;=\; \frac{1}{\sqrt{3}\, e^{i\pi/6}}\, V^{\Delta}$$

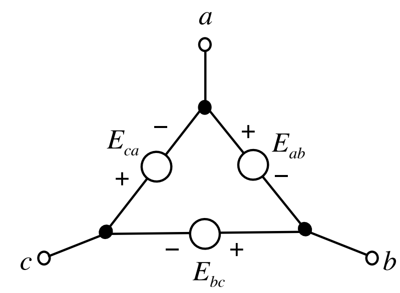
$$I^Y \;=\; \left(1-\alpha^2\right) I^{\Delta} \;=\; \frac{\sqrt{3}}{e^{i\pi/6}}\, I^{\Delta}$$





Δ and Y transformation

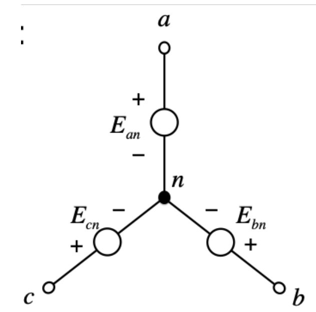
1. They are equivalent if they have the same external behavior



2. Given $\left(V^\Delta, I^\Delta\right)$, the internal $\left(V^Y, I^Y\right)$ of the Y equivalent must satisfy

$$\Gamma V^Y = V^\Delta, \qquad I^Y = \Gamma^\mathsf{T} I^\Delta$$

3. If $\left(V^\Delta, I^\Delta\right)$ are balanced vectors then

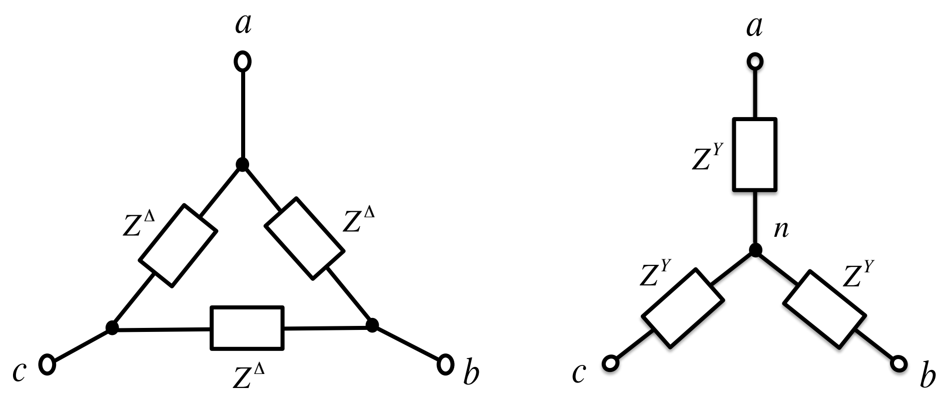


$$V^Y = \frac{1}{1-\alpha} V^\Delta = \frac{1}{\sqrt{3}\, e^{i\pi/6}} V^\Delta$$

$$I^Y = \left(1-\alpha^2\right) I^\Delta = \frac{\sqrt{3}}{e^{i\pi/6}} I^\Delta$$

Assume $V^n = 0$ (neutral is common reference node)

Δ and Y transformation



They are equivalent if they have the same external behavior:
When the same line voltages are applied to both configuration, they have the same line currents

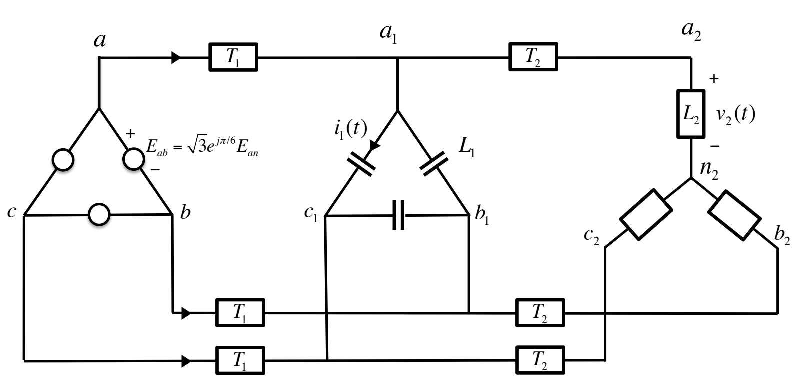
$$I_a^\Delta = \frac{V_{ab} - V_{ca}}{Z^\Delta} \quad = \quad I_a^Y = \frac{V_{an}^Y}{Z^Y} \qquad \Rightarrow \qquad Z^Y = \frac{Z^\Delta}{3}$$

Per-phase analysis

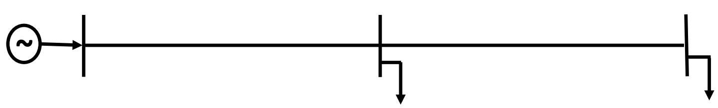
1. Convert all sources and loads in Δ configuration into equivalent Y config.
2. Solve for phase a variables using per-phase circuit
3. For positive sequence, phase b or phase c variables are determined by subtracting 120° and 240° from corresponding phase a variables
4. For variables in the internal of Δ configuration, derive them from original circuit

Per-phase analysis

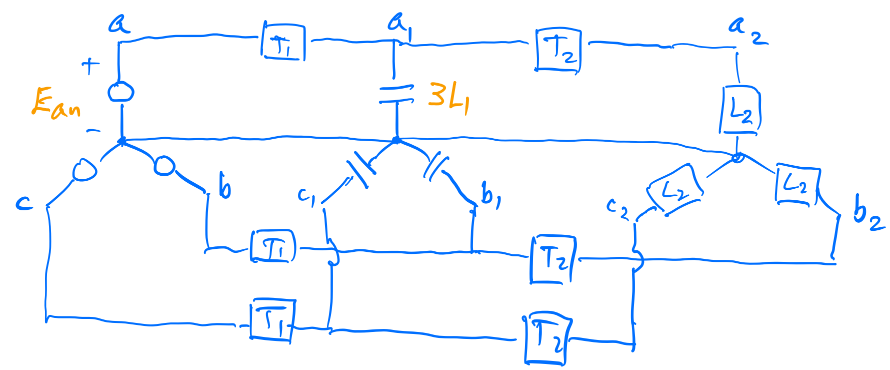
Example 1.3

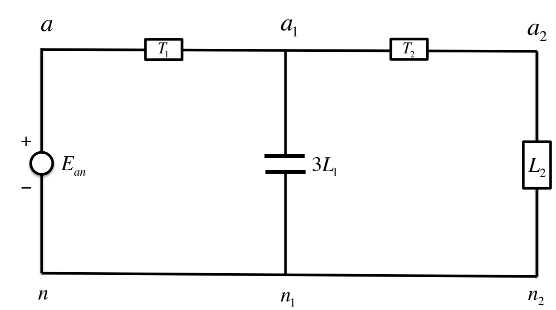


One line diagram:



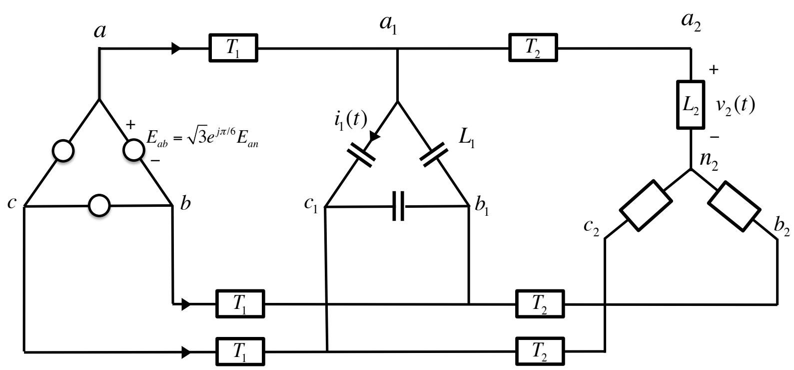
(a) Balanced three-phase system

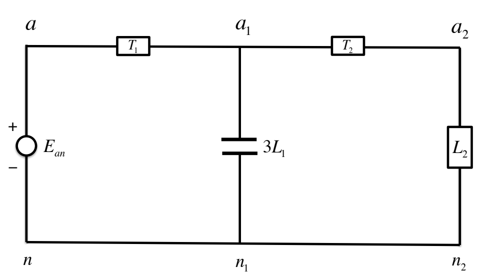




Per-phase analysis

Example 1.3





Solution:

1. Using per-phase circuit, solve for $V_{a_1n_1}$ and $V_{a_2n_2}$
2. $v_2(t) \;=\; \sqrt{2}\,\left|V_2\right|\,\cos\left(\omega t + \angle V_2\right)$
3. $i_1(t) \;=\; \sqrt{2}\,\left|I_{a_1c_1}\right|\,\cos\left(\omega t + \angle I_{a_1c_1}\right)$
4. To calculate $I_{a_1c_1}$, obtain $V_{a_1b_1} \;=\; \sqrt{3}\,e^{i\pi/6}\,V_{a_1n_1}$
5. Obtain $I_{a_1b_1} \;=\; L_1V_{a_1b_1} \;=\; \sqrt{3}\,L_1\,e^{i\pi/6}\,V_{a_1n_1}$
6. Obtain $I_{a_1c_1} \;=\; -\,I_{a_1b_1}\,e^{i2\pi/3} \;=\; 3\sqrt{3}\,e^{-i\pi/6}\,L_1V_{a_1n_1}$

Per-phase analysis

1. Convert all sources and loads in Δ configuration into equivalent Y config.
2. Solve for phase a variables using per-phase circuit
3. For positive sequence, phase b or phase c variables are determined by subtracting 120° and 240° from corresponding phase a variables
4. For variables in the internal of Δ configuration, derive them from original circuit

Can this approach be extended, and justified, for general networks ?

Yes, see Ch 9.3 on Unbalanced Multiphase Networks

Outline

1. Phasor representation
2. Linear circuit analysis
3. Balanced three-phase systems
4. Complex power
   - Single-phase power
   - Three-phase power
   - Advantages of 3ϕ power

Single-phase power

Instantaneous power:

$$p(t) \;:=\; v(t)i(t)$$
$$= \frac{V_{\text{max}} I_{\text{max}}}{2} \left(\cos(\theta_V - \theta_I) + \cos(2\omega t + \theta_V + \theta_I) \right)$$

Average power

$$\frac{1}{T}\int_0^T p(t)dt \;=\; \frac{V_{\text{max}} I_{\text{max}}}{2} \cos(\theta_V - \theta_I)$$

$\phi := \theta_V - \theta_I$: power factor angle

Single-phase power

Complex power:

$$S \ := \ VI^* \ = \ \frac{V_{\max} I_{\max}}{2} \ e^{i(\theta_V - \theta_I)} \ = \ |V| \, |I| \, e^{i\phi}$$

Active and reactive power

$$P \ := \ |V| \, |I| \cos\phi \quad \text{kW} \qquad Q := |V| \, |I| \sin\phi \quad \text{var}$$

Apparent power

$$|S| \ = \ |V| \, |I| \ = \ \sqrt{P^2 + Q^2} \quad \text{VA}$$

Instantaneous and complex power

Relationship:

$$p(t) \;=\; P \;+\; P\cos 2(\omega t + \theta_I) - Q\sin 2(\omega t + \theta_I)$$

Average power

$$P \;=\; \frac{1}{T}\int_0^T p(t)dt$$

Power delivered to impedance

Voltage and current across impedance are related

$$V = ZI$$

Complex power

$$S \;=\; |Z|\,|I|\,e^{i\phi}, \qquad \phi := \angle Z = \theta_V - \theta_I$$

	$\lvert Z\rvert$	$\phi = \angle Z$	P	Q
Resistor $Z = R$	R	0	$R\lvert I\rvert^2$	0
Inductor $Z = \mathbf{i}\omega L$	ωL	$\pi/2$	0	$\omega L\lvert I\rvert^2$
Capacitor $Z = (\mathbf{i}\omega C)^{-1}$	$(\omega C)^{-1}$	$-\pi/2$	0	$-(\omega C)^{-1}\lvert I\rvert^2$

Power delivered to impedance

Instantaneous power delivered to

$$\begin{aligned} \text{resistor } R: \quad & p(t) = P\left(1 + \cos 2\left(\omega t + \theta_I\right)\right) \\ \text{inductor } i\omega L: \quad & p(t) = -Q \sin 2\left(\omega t + \theta_I\right) \\ \text{capacitor } (i\omega C)^{-1}: \quad & p(t) = Q \sin 2\left(\omega t + \theta_V\right) \end{aligned}$$

Three-phase power

Per-phase power: $S \; := \; V_{an} I_{an}^*$

Three-phase power: $S_{3\phi} \; := \; V_{an} I_{an}^* + V_{bn} I_{bn}^* + V_{cn} I_{cn}^* \; = \; 3S$

because $V_{bn} = e^{-2\pi/3} V_{an}, \; I_{bn} = e^{-2\pi/3} I_{an} \quad \Rightarrow \quad V_{bn} I_{bn}^* = S$

Three-phase power

Instantaneous 3ϕ power is constant

$$p_{3\phi}(t) := v_a(t)i_a(t) + v_b(t)i_b(t) + v_c(t)i_c(t) \quad = \quad 3P$$

A 3ϕ motor receives constant torque

Instantaneous 1ϕ power is sinusoidal

$$p(t) \;=\; P \;+\; P\cos 2(\omega t + \theta_I) - Q\sin 2(\omega t + \theta_I)$$

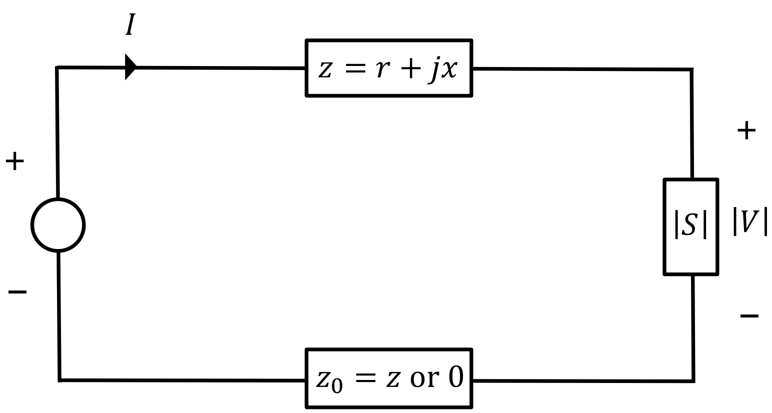
Three-phase power

Instantaneous 3ϕ power is constant

$$
\begin{aligned}
p_{3\phi}(t) &:= v_a(t)i_a(t) + v_b(t)i_b(t) + v_c(t)i_c(t) \\
&= |V_a||I_a| \left(\cos\phi \ + \ \cos(2\omega t + \theta_V + \theta_I)\right) \\
&\quad + \ |V_a||I_a| \left(\cos\phi \ + \ \cos(2\omega t + (\theta_V - 2\pi/3) + (\theta_I - 2\pi/3))\right) \\
&\quad + \ |V_a||I_a| \left(\cos\phi + \cos(2\omega t + (\theta_V + 2\pi/3) + (\theta_I + 2\pi/3))\right) \\
&= 3|V_a||I_a|\cos\phi \ + \ |V_a||I_a| \underbrace{\left(\cos\theta(t) \ + \ \cos(\theta(t) - 4\pi/3) \ + \ \cos(\theta(t) + 4\pi/3)\right)}_{=0} \\
&= 3P
\end{aligned}
$$

Savings from 3ϕ system

Example



Spec:

- Supply load with power $|S|$ at voltage $|V|$
- Distance between generator & load: d
- Line impedance $z = r + ix$ ohm/meter
- Resistance / unit length $r = \frac{\rho}{\text{area}}$
- Line current $\leq \delta\,$area

Savings:

- Material required: $m_{3\phi} = \frac{1}{2} m_{1\phi}$
- Active power loss: $l_{3\phi} = \frac{1}{2} l_{1\phi}$