Power System Analysis

Chapter 13 Semidefinite relaxations: BIM

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Outline

1. Relaxation of QCQP
2. Application to OPF
3. Exactness condition: linear separability
4. Exactness condition: small angle difference
5. Condition for global optimality

Outline

1. Relaxation of QCQP
   - SDP relaxation
   - Partial matrices and completions
   - Feasible sets
   - Relaxations and solution recovery
   - Tightness of relaxations
2. Application to OPF
3. Exactness condition: linear separability
4. Exactness condition: small angle difference
5. Condition for global optimality

QCQP

Quadratically constrained quadratic program:

$$\begin{aligned} \min_{x\in\mathbb{C}^n} \quad & x^{\mathsf{H}} C_0 x \\ \text{s.t.} \quad & x^{\mathsf{H}} C_l x \ \le \ b_l, \qquad l = 1,\ldots,L \end{aligned}$$

- C_l : $n \times n$ Hermitian matrix
- $b_l \in \mathbb{R}$
- Homogeneous QCQP : all monomials are of degree 2
- OPF can be formulated as (nonconvex) QCQP

QCQP

Equivalent problem

Using $x^{\mathsf{H}} C_l x = \text{tr}\left(C_l x x^{\mathsf{H}}\right)$, this is equivalent to:

$$\begin{aligned} \min_{X \in \mathbb{S}^n, x \in \mathbb{C}^n} \quad & \text{tr}\left(C_0 X\right) \\ \text{s.t.} \quad & \text{tr}\left(C_l X\right) \ \le \ b_l, \quad l = 1, \ldots, L \\ & X \ = \ x x^{\mathsf{H}} \end{aligned}$$

- Any psd rank-1 matrix $X \in \mathbb{S}_+^{n \times n}$ has a spectral decomposition $X = x x^{\mathsf{H}}$ for some $x \in \mathbb{C}^n$
- x is unique up to a rotation, i.e., x satisfies $X = x x^{\mathsf{H}}$ $x\, e^{j\theta}$ for any $\theta \in \mathbb{R}$
- Therefore can eliminate x

QCQP

Equivalent problem

Eliminating x —> minimization over psd matrices X:

$$\begin{aligned} \min_{X \in \mathbb{S}^n} \quad & \text{tr}\left(C_0 X\right) \\ \text{s.t.} \quad & \text{tr}\left(C_l X\right) \ \leq \ b_l, \quad l = 1,\ldots,L \\ & X \ \succeq \ 0, \quad \text{rank}(X) \ = \ 1 \end{aligned}$$

- tr $\left(C_l X\right) \leq b_l$ is linear in X
- $X \succeq 0$ is convex in X
- rank$(X) = 1$ is nonconvex in X

Removing rank constraint yields SDP relaxation

SDP relaxation

SDP relaxation of QCQP

$$\begin{aligned} \min_{X \in \mathbb{S}^n} \quad & \text{tr}\left(C_0 X\right) \\ \text{s.t.} \quad & \text{tr}\left(C_l X\right) \ \le \ b_l, \quad l = 1, \ldots, L \\ & X \ \succeq \ 0 \end{aligned}$$

- This is a standard semidefinite program which is a convex problem
- Solution strategy:
  - Solve SDP for an optimal solution X^{opt}
  - If rank$\left(X^{\text{opt}}\right) = 1$, then $x^{\text{opt}} \in \mathbb{C}^n$ from spectral decomposition from $X^{\text{opt}} = x^{\text{opt}} \left(x^{\text{opt}}\right)^{\text{H}}$
- If rank$\left(X^{\text{opt}}\right) > 1$, then, in general, no feasible solution of QCQP can be directly obtained

SDP relaxation

SDP relaxation of QCQP

$$\min_{X \in \mathbb{S}^n} \quad \text{tr}\left(C_0 X\right)$$

$$\text{s.t.} \quad \text{tr}\left(C_l X\right) \ \le \ b_l, \quad l = 1, \ldots, L$$

$$X \ \succeq \ 0$$

- Even though SDP is convex, for large networks, it is still computationally impractical
- How to exploit sparsity of large networks to reduce computational burden?

Ans: partial matrices and completions !

Partial matrices

A QCQP instance specified by $(C_0, C_l, b_l, l = 1, \ldots, L)$ induces graph $F := (N, E)$

- N : n nodes (where $C_l \in \mathbb{C}^{n\times n}$)
- $E \subseteq N \times N$: m links $(j, k) \in E$ iff $\exists l \in \{0,1,\ldots,L\}$ s.t. $[C_l]_{jk} = [C_l]_{kj}^{\mathsf{H}} \neq 0$

A partial matrix X_F is a set of $n + 2m$ complex numbers defined on $F = (N, E)$

$$X_F \; := \; \left\{ \; [X_F]_{jj}, [X_F]_{jk}, [X_F]_{kj} \; : \; j \in N, (j,k) \in E \; \right\}$$

- X_F can be interpreted as matrix with entries partially specified, or a partial matrix
- If F is complete graph, then X_F is full $n \times n$ matrix

A completion X of X_F is a full $n \times n$ matrix that agrees with X_F on graph F

$$[X]_{jj} \; = \; [X_F]_{jj}, \qquad [X]_{jk} \; = \; [X_F]_{jk}, \qquad [X]_{kj} \; = \; [X_F]_{kj}$$

Partial matrices

If q is clique (fully connected subgraph) of F, then $X_F(q)$ is fully specified principal submatrix of X_F on q :

$$[X(q)]_{jj} \;:=\; [X_F]_{jj}, \qquad [X(q)]_{jk} \;:=\; [X_F]_{jk}, \qquad [X(q)]_{kj} \;:=\; [X_F]_{kj},$$

Hermitian, psd, rank-1, trace

Partial matrix

A partial matrix X_F is

- Hermitian ($X_F = X_F^{\mathsf{H}}$) if $[X_F]_{kj} = [X_F]_{jk}^{\mathsf{H}}$
- psd ($X_F \succeq 0$) if X_F is Hermitian and $X_F(q) \succeq 0$ for all cliques q of F
- rank-1 if $\text{rank}\left(X_F(q)\right) = 1$ for all cliques q of F

Hermitian, psd, rank-1, trace

Partial matrix

A partial matrix X_F is

- Hermitian ($X_F = X_F^{\mathsf{H}}$) if $[X_F]_{kj} = [X_F]_{jk}^{\mathsf{H}}$
- psd ($X_F \succeq 0$) if X_F is Hermitian and $X_F(q) \succeq 0$ for all cliques q of F
- rank-1 if rank $\left(X_F(q)\right) = 1$ for all cliques q of F

- 2×2 psd if $X_F(j,k)$ is psd for all $(j,k) \in E$
- 2×2 rank-1 if $X_F(j,k)$ is rank-1 for all $(j,k) \in E$

  where $X_F(j,k) \ := \ \begin{bmatrix} [X_F]_{jj} & [X_F]_{jk} \\ [X_F]_{kj} & [X_F]_{kk} \end{bmatrix}$

Hermitian, psd, rank-1, trace

Partial matrix

For partial matrix X_F

$$\text{tr}\left(C_l X_F\right) := \sum_{j \in N} [C_l]_{jj} [X_F]_{jj} + \sum_{(j,k) \in E} \left([C_l]_{jk} [X_F]_{kj} + [C_l]_{kj} [X_F]_{jk} \right)$$

If both C_l and X_F are Hermitian, then tr $\left(C_l X_F\right)$ is real:

$$\text{tr}\left(C_l X_F\right) = \sum_{j \in N} [C_l]_{jj} [X_F]_{jj} + 2 \sum_{(j,k) \in E} \text{Re}\left([C_l]_{jk} [X_F]_{kj} \right)$$

Chordal graph & extensions

F is a chordal graph if

- Either F has no cycles, or
- All minimal cycles (ones without chords) are of length 3

A chordal extension $c(F)$ of F is a chordal graph that contains F

- $X_{c(F)}$ is a chordal extension of X_F

Every graph has a (generally nonunique) chordal extension

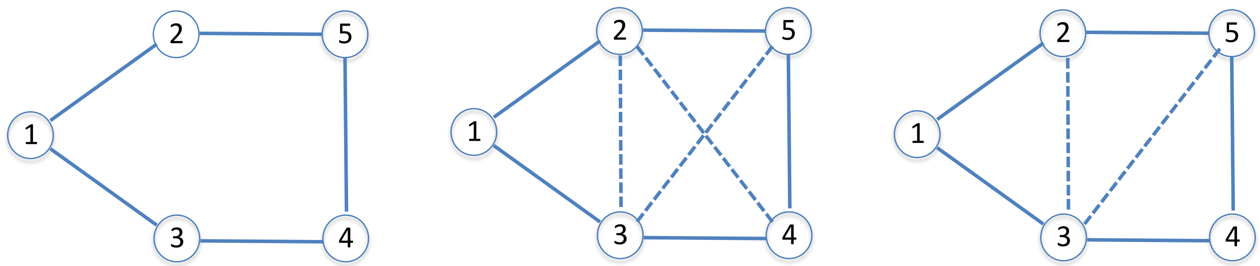
- Complete supergraph of F is a $c(F)$

Theorem [Grone et al 1984]: every psd partial matrix has a psd completion iff underlying graph is chordal

- We will extend this to psd rank-1 submatrices

Partial matrix & chordal extensions

Example



$$W_F = \begin{bmatrix} x_{11} & x_{12} & x_{13} & & \\ x_{21} & x_{22} & & & x_{25} \\ x_{31} & & x_{33} & x_{34} & \\ & & x_{43} & x_{44} & x_{45} \\ & x_{52} & & x_{54} & x_{55} \end{bmatrix} \quad W_{c(F)} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & & \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\ & x_{42} & x_{43} & x_{44} & x_{45} \\ & x_{52} & x_{53} & x_{54} & x_{55} \end{bmatrix} \quad W_{c(F)} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & & \\ x_{21} & x_{22} & x_{23} & & x_{25} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\ & & x_{43} & x_{44} & x_{45} \\ & x_{52} & x_{53} & x_{54} & x_{55} \end{bmatrix}$$

2 cliques $W_{c(F)}(q)$

3 cliques $W_{c(F)}(q)$

Rank-1 characterization

Equivalent conditions

$$\begin{aligned}
&\text{C1:} && X \succeq 0, && \text{rank}(X) = 1 \\
&\text{C2:} && X_{c(F)} \succeq 0, && \text{rank}(X_{c(F)}) = 1 \\
&\text{C3:} && X_F(j,k) \succeq 0, && \text{rank}(X_F(j,k)) = 1, \qquad (j,k) \in E \\
& && \sum_{(j,k)\in c} \angle [X_F]_{jk} = 0 && \text{mod } 2\pi \qquad \textcolor{blue}{\text{cycle condition}}
\end{aligned}$$

Theorem

Suppose $X_{jj} > 0, \ \left[X_{c(F)}\right]_{jj} > 0, \ \left[X_F\right]_{jj} > 0.$ Then C1 $\Longleftrightarrow$ C2 $\Longleftrightarrow$ C3.

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Feasible sets

Feasible set of QCQP

$$\mathbb{V} \;:=\; \{x \in \mathbb{C}^n \,|\, x^{\mathsf{H}} C_l x \leq b_l,\ l = 1,\ldots,L\}$$

psd rank-1 matrices X

$$\mathbb{X} \;:=\; \{\ X \in \mathbb{S}^n \mid X \text{ satisfies } \mathrm{tr}(C_l X) \leq b_l,\ \text{C1}\ \}$$

psd rank-1 chordal extensions $X_{c(F)}$

$$\mathbb{X}_{c(F)} \;:=\; \{\ X_{c(F)} \mid X_{c(F)} \text{ satisfies } \mathrm{tr}\left(C_l X_{c(F)}\right) \leq b_l,\ \text{C2}\ \}$$

psd rank-1 partial matrices X_F

$$\mathbb{X}_F \;:=\; \{\ X_F \mid X_F \text{ satisfies } \mathrm{tr}\left(C_l X_F\right) \leq b_l,\ \text{C3}\ \}$$

Feasible sets

Equivalence

Corollary

Fix any connected F. Any partial matrix $X_{c(F)} \in \mathbb{X}_{c(F)}$ or $X_F \in \mathbb{X}_F$ has a unique psd rank-1 completion $X \in \mathbb{X}$

Definition: Two sets A and B are equivalent ($A \equiv B$) if there is a bijection between them

Theorem

$\mathbb{V} \equiv \mathbb{X} \equiv \mathbb{X}_{c(F)} \equiv \mathbb{X}_F$

Implication: A feasible $x \in \mathbb{V}$ can be recovered from any partial matrix $X_{c(F)} \in \mathbb{X}_{c(F)}$ or $X_F \in \mathbb{X}_F$ through spectral decomposition (but there is a simpler way to compute $x \in \mathbb{V}$ than completion)

Equivalent problems

QCQP

$$\min_{x \in \mathbb{C}^n} \quad x^{\mathsf{H}} C_0 x \qquad \text{subject to} \qquad x \in \mathbb{V}$$

is equivalent to min over matrices and partial matrices:

$$\min_{X} \quad x^{\mathsf{H}} C_0 x \qquad \text{subject to} \qquad X \in \hat{\mathbb{X}}$$

where $\hat{\mathbb{X}} \; := \; \left\{ \mathbb{X}, \mathbb{X}_{c(F)}, \mathbb{X}_F \right\}$

Implications:
Instead of solving for $X \in \mathbb{X}$, solve for $X_{c(F)} \in \mathbb{X}_{c(F)}$ or $X_F \in \mathbb{X}_F$ which are much smaller for large sparse networks

Equivalent problems

QCQP

$$\min_{x \in \mathbb{C}^n} \quad x^{\mathsf{H}} C_0 x \qquad \text{subject to} \qquad x \in \mathbb{V}$$

is equivalent to min over matrices and partial matrices:

$$\min_{X} \quad x^{\mathsf{H}} C_0 x \qquad \text{subject to} \qquad X \in \hat{\mathbb{X}}$$

where $\hat{\mathbb{X}} \ := \ \left\{ \mathbb{X}, \mathbb{X}_{c(F)}, \mathbb{X}_F \right\}$

Computational challenges remain:
$\mathbb{X}$, $\mathbb{X}_{c(F)}$, $\mathbb{X}_F$ are all nonconvex

Semidefinite relaxations

Convex supersets

$$\mathbb{X}^+ := \{X \in \mathbb{S}^n \mid X_F \text{ satisfies } \mathrm{tr}(C_l X) \le b_l,\ X \succeq 0\}$$

$$\mathbb{X}^+_{c(F)} := \{X_{c(F)} \mid X_F \text{ satisfies } \mathrm{tr}\left(C_l X_{c(F)}\right) \le b_l,\ X_{c(F)} \succeq 0\}$$

$$\mathbb{X}^+_F := \{X_F \mid X_F \text{ satisfies } \mathrm{tr}\left(C_l X_F\right) \le b_l,\ X_F(j,k) \succeq 0,\ (j,k) \in E\}$$

Semidefinite relaxations:

QCQP-sdp : $\min_{X} \ C\left(X_F\right)$ s.t. $X \in \mathbb{X}^+$ most complex

QCQP-ch : $\min_{X_{c(F)}} \ C\left(X_F\right)$ s.t. $X_{c(F)} \in \mathbb{X}^+_{c(F)}$

QCQP-socp : $\min_{X_F} \ C\left(X_F\right)$ s.t. $X_F \in \mathbb{X}^+_F$ simplest

Semidefinite relaxations

Solution recovery

If a feasible / optimal solution X of semidefinite relaxation lies in $\mathbb{X}$, $\mathbb{X}_{c(F)}$, or $\mathbb{X}_F$, then can recover feasible / optimal $x \in \mathbb{V}$ of QCQP

Recovery procedure: given $X_F \in \mathbb{X}_F$

1. Set $|x_1| := \sqrt{[X_F]_{11}}$ and $\angle x_1$ to arbitrary value

2. For $j = 1, \ldots, n$,

$$|x_j| \ := \ \sqrt{[X_F]_{jj}}, \qquad \angle x_j \ := \ \angle V_1 \ - \sum_{(i,k)\in\mathbb{P}_j} \angle [X_F]_{ik}$$

where $\mathbb{P}_j$: path from bus 1 to bus j in an arbitrary spanning tree rooted at bus 1

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Semidefinite relaxations

Convex supersets

$$\mathbb{X}^+ := \{X \in \mathbb{S}^n \mid X_F \text{ satisfies } \mathrm{tr}(C_l X) \le b_l,\ X \succeq 0\}$$

$$\mathbb{X}^+_{c(F)} := \{X_{c(F)} \mid X_F \text{ satisfies } \mathrm{tr}\left(C_l X_{c(F)}\right) \le b_l,\ X_{c(F)} \succeq 0\}$$

$$\mathbb{X}^+_F := \{X_F \mid X_F \text{ satisfies } \mathrm{tr}\left(C_l X_F\right) \le b_l,\ X_F(j,k) \succeq 0,\ (j,k) \in E\}$$

Semidefinite relaxations:

$$\text{QCQP-sdp}: \quad \min_{X} \ C\left(X_F\right) \quad \text{s.t.} \quad X \in \mathbb{X}^+$$

most complex

$$\text{QCQP-ch}: \quad \min_{X_{c(F)}} \ C\left(X_F\right) \quad \text{s.t.} \quad X_{c(F)} \in \mathbb{X}^+_{c(F)}$$

$$\text{QCQP-socp}: \quad \min_{X_F} \ C\left(X_F\right) \quad \text{s.t.} \quad X_F \in \mathbb{X}^+_F$$

simplest

Tightness

Definition

1. A is an effective subset of B ($A \sqsubseteq B$) if given any $a \in A$, $\exists\ b \in B$ with same cost $C_A(a) = C_B(b)$
2. A is similar to B ($A \simeq B$) if $A \sqsubseteq B$ and $B \sqsubseteq A$

Theorem [Tightness]

1. $\mathbb{V} \sqsubseteq \mathbb{X}^+ \simeq \mathbb{X}^+_{c(F)} \sqsubseteq \mathbb{X}^+_F$
2. If F is a tree, then $\mathbb{V} \sqsubseteq \mathbb{X}^+ \simeq \mathbb{X}^+_{c(F)} \simeq \mathbb{X}^+_F$

Corollary [Optimal values]

1. $C^{\text{qcqp}} \geq C^{\text{sdp}} = C^{\text{ch}} \geq C^{\text{socp}}$
2. If F is a tree, then $C^{\text{qcqp}} \geq C^{\text{sdp}} = C^{\text{ch}} = C^{\text{socp}}$

Semidefinite relaxations

Implications

1. Radial networks: Solve QCQP-socp
   - Simplest computationally
   - Same tightness as QCQP-ch and QCQP-SDP
2. Meshed networks: Solve QCQP-ch or QCQP-socp
   - QCQP-ch strictly tighter than QCQP-socp, and same tightness as QCQP-sdp
   - QCQP-ch can be orders of magnitude simpler computationally than QCQP-sdp for large sparse networks
   - QCQP-ch is as complex as QCQP-sdp in the worst case

Outline

1. Relaxation of QCQP
2. Application to OPF
    - Single-phase networks
    - Defintion: exact relaxation
3. Exactness condition: linear separability
4. Exactness condition: small angle difference
5. Condition for global optimality

OPF as QCQP

Recall

$$
\begin{aligned}
\min_{V\in\mathbb{C}^{N+1}} \quad & V^{\mathsf{H}} C_0 V \\
\text{s.t.} \quad & p_j^{\min} \ \le\ \mathrm{tr}\left(\Phi_j V V^{\mathsf{H}}\right) \ \le\ p_j^{\max}, && j \in \overline{N} \\
& q_j^{\min} \ \le\ \mathrm{tr}\left(\Psi_j V V^{\mathsf{H}}\right) \ \le\ q_j^{\max}, && j \in \overline{N} \\
& v_j^{\min} \ \le\ \mathrm{tr}\left(J_j V V^{\mathsf{H}}\right) \ \le\ v_j^{\max}, && j \in \overline{N} \\
& \mathrm{tr}\left(\hat{Y}_{jk} V V^{\mathsf{H}}\right) \ \le\ \bar{I}_{jk}^{\max}, && (j,k) \in E \\
& \mathrm{tr}\left(\hat{Y}_{kj} V V^{\mathsf{H}}\right) \ \le\ \bar{I}_{kj}^{\max}, && (j,k) \in E
\end{aligned}
$$

abbreviated as:
$\mathrm{tr}\left(C_l V V^{\mathsf{H}}\right) \le b_l,\ l = 1,\ldots,L$

Constraints

Given $V \in \mathbb{C}^{N+1|}$, define partial matrix W_G by

$$[W_G]_{jj} := |V_j|^2, \qquad j \in \overline{N}$$

$$\left[W_G\right]_{jk} := V_j V_k^{\mathsf{H}} =: [W_G]_{kj}^{\mathsf{H}}, \qquad (j,k) \in E$$

Constraints in terms of W_G

$$p_j^{\min} \le \operatorname{tr}\left(\Phi_j W_G\right) \le p_j^{\max}$$

$$q_j^{\min} \le \operatorname{tr}\left(\Psi_j W_G\right) \le q_j^{\max}$$

$$v_j^{\min} \le \operatorname{tr}\left(J_j W_G\right) \le v_j^{\max}$$

$$\operatorname{tr}\left(\hat{Y}_{jk} W_G\right) \le I_{jk}^{\max}$$

$$\operatorname{tr}\left(\hat{Y}_{kj} W_G\right) \le I_{kj}^{\max}$$

abbreviated as:
$\operatorname{tr}\left(C_l W_G\right) \le b_l,\ l = 1,\ldots,L$

OPF and relaxations

OPF as QCQP

$$\min_{V} \quad C_0(V) \qquad \text{s.t.} \quad \mathrm{tr}\left(C_l V V^{\mathsf{H}}\right) \le b_l, \quad l = 1,\ldots,L$$

Semidefinite relaxations:

$$\text{OPF-sdp}: \quad \min_{W \in \mathbb{S}^{N+1}} \; C_0(W_G) \quad \text{s.t.} \quad \mathrm{tr}\left(C_l W\right) \le b_l, \; l = 1,\ldots,L, \qquad W \succeq 0$$

$$\text{OPF-ch}: \quad \min_{W_{c(G)}} \; C_0(W_G) \quad \text{s.t.} \quad \mathrm{tr}\left(C_l W_{c(G)}\right) \le b_l, \; l = 1,\ldots,L, \qquad W_{c(G)} \succeq 0$$

$$\text{OPF-socp}: \quad \min_{W_G} \; C_0(W_G) \quad \text{s.t.} \quad \mathrm{tr}\left(C_l W_G\right) \le b_l, \; l = 1,\ldots,L, \qquad W_G(j,k) \succeq 0, \; (j,k) \in E$$

Exact relaxation

Definition

1. OPF-sdp is exact if every optimal solution W^{sdp} of OPF-sdp is psd rank-1
2. OPF-ch is exact if every optimal solution $W^{\text{ch}}_{c(G)}$ of OPF-ch is psd rank-1
3. OPF-socp is exact if every optimal solution W^{socp}_G of OPF-docp
   - is 2×2 psd rank-1, i.e., $W^{\text{socp}}_G(j,k)$ are psd rank-1 for all $(j,k) \in E$, and
   - satisfies cycle condition, i.e., $\sum_{(j,k)\in c} \angle [W^{\text{socp}}_G]_{jk} \; = \; 0 \mod 2\pi$

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2. Application to OPF
3. Exactness condition: linear separability
   - Sufficient condition for QCQP
   - Application to OPF
4. Exactness condition: small angle difference
5. Condition for global optimality

QCQP and SOCP relaxation

QCQP:

$$\min_{x \in \mathbb{C}^n} \quad x^{\mathsf{H}} C_0 x$$
$$\text{s.t.} \quad x^{\mathsf{H}} C_l x \ \leq \ b_l, \qquad l = 1, \ldots, L$$

SOCP relaxation:

$$\min_{X_G} \qquad \mathrm{tr}\left(C_0 X_G\right)$$
$$\text{s.t.} \qquad \mathrm{tr}\left(C_l X_G\right) \ \leq \ b_l, \quad l = 1, \ldots, L$$
$$X_G(j,k) \ \succeq \ 0, \quad (j,k) \in E$$

- $C_l : n \times n$ Hermitian matrix, $b_l \in \mathbb{R}$

Sufficient condition

C13.1: C_0 is positive definite

C13.2: for every $(j, k) \in E,\ \exists \alpha_{jk}$ s.t. $\angle \left[C_l \right]_{jk} \in [\alpha_{ij}, \alpha_{ij} + \pi]$ for all $l = 0, \ldots, L$

Theorem

Suppose G is a tree and C13.2 holds. Then

1. $C^{\text{opt}} = C^{\text{socp}}$
2. An optimal solution of QCQP can be recovered from every optimal solution of its SOCP relaxation

An optimal solution of SOCP relaxation may not be 2×2 rank-1
when optimal solutions of SOCP relaxation are nonunique

Sufficient condition

C13.1: C_0 is positive definite

C13.2: for every $(j, k) \in E$, $\exists \alpha_{jk}$ s.t. $\angle \left[C_l\right]_{jk} \in [\alpha_{ij}, \alpha_{ij} + \pi]$ for all $l = 0, \ldots, L$

Corollary

Suppose G is a tree and both C13.1 and C13.2 hold. Then SOCP relaxation is exact, i.e., every optimal solution W_G^{socp} is 2×2 psd rank-1

- Cycle condition is vacuous since G is a tree

Application to OPF

Recall OPF as QCQP

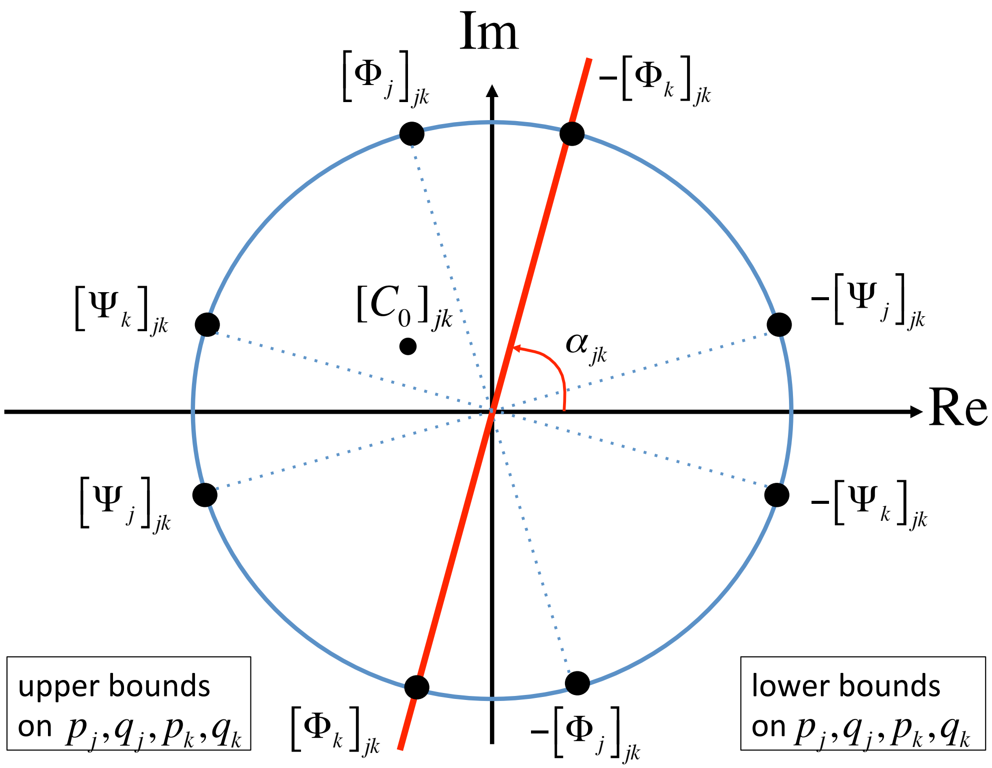
$$\min_{V \in \mathbb{C}^{N+1}} \quad V^{\mathsf{H}} C_0 V$$

$$\begin{aligned}
\text{s.t.} \quad & p_j^{\min} \ \le \ \text{tr}\left(\Phi_j V V^{\mathsf{H}}\right) \ \le \ p_j^{\max}, && j \in \overline{N} \\
& q_j^{\min} \ \le \ \text{tr}\left(\Psi_j V V^{\mathsf{H}}\right) \ \le \ q_j^{\max}, && j \in \overline{N} \\
& v_j^{\min} \ \le \ \text{tr}\left(J_j V V^{\mathsf{H}}\right) \ \le \ v_j^{\max}, && j \in \overline{N} \\
& \text{tr}\left(\hat{Y}_{jk} V V^{\mathsf{H}}\right) \ \le \ \bar{I}_{jk}^{\max}, && (j,k) \in E \\
& \text{tr}\left(\hat{Y}_{kj} V V^{\mathsf{H}}\right) \ \le \ \bar{I}_{kj}^{\max}, && (j,k) \in E
\end{aligned}$$

abbreviated as:
$\text{tr}\left(C_l V V^{\mathsf{H}}\right) \le b_l, \ l = 1, \ldots, L$

Application to OPF

Exactness condition



Corollary

Suppose G is a tree and both C13.1 and the diagram hold.

Then SOCP relaxation is exact

Outline

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2. Application to OPF
3. Exactness condition: linear separability
4. Exactness condition: small angle difference
    - Sufficient condition
    - 2-bus example
5. Condition for global optimality

Assumptions

Assume

1. Voltage magnitudes $|V_j|$ are fixed
2. Reactive powers are ignored
3. Shunt admittances are zero $y_{jk}^m = y_{kj}^m := 0$

OPF formulation

$$\min_{p,P,\theta} \quad C(p)$$

$$\text{s.t.} \quad p_j^{\min} \le p_j \le p_j^{\max}, \qquad j \in \overline{N}$$

$$\theta_{jk}^{\min} \le \theta_{jk} \le \theta_{jk}^{\max}, \quad (j,k) \in E$$

constraints on line flows, line losses, or stability

$$p_j = \sum_{k:k\sim j} P_{jk}, \qquad j \in \overline{N}$$

nodal power balance

$$P_{jk} = g_{jk} - g_{jk}\cos\theta_{jk} - b_{jk}\sin\theta_{jk}, \quad (j,k) \in E$$

power flow equation (polar form)

where $V_j = |V_j| e^{i\theta_j}$ with $|V_j| := 1$ and $\theta_{jk} := \theta_j - \theta_k$

Eliminate P_{jk} and θ_{jk}

OPF formulation

Define injection region

$$\mathbb{P}_\theta := \left\{ p \in \mathbb{R}^n \;\middle|\; p_j = \sum_{k:k\sim j} \left(g_{jk} - g_{jk}\cos\theta_{jk} - b_{jk}\sin\theta_{jk} \right), \quad \underline{\theta}_{jk} \le \theta_{jk} \le \overline{\theta}_{jk} \right\}$$

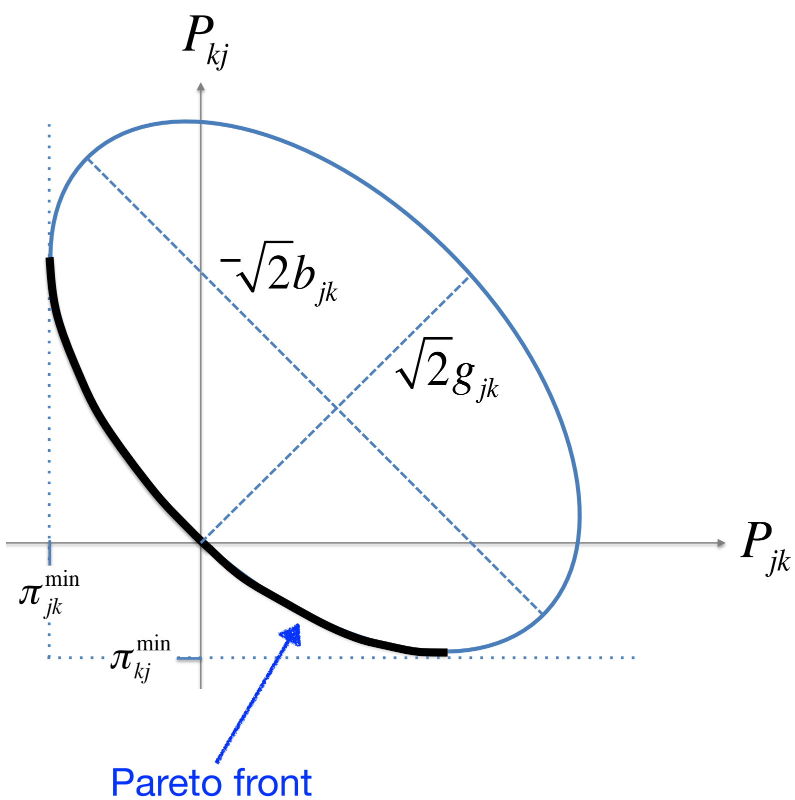
$$\mathbb{P}_p := \{ p \in \mathbb{R}^n \,|\, \underline{p}_j \le p_j \le \overline{p}_j, j \in N \}$$

OPF: $\min_p \; C(p) \quad$ s.t. $\quad p \in \mathbb{P}_\theta \cap \mathbb{P}_p$

SOCP relaxation: $\min_p \; C(p) \quad$ s.t. $\quad p \in \text{conv}\left(\mathbb{P}_\theta\right) \cap \mathbb{P}_p$

Definition: SOCP relaxation is exact if every optimal solution lies in $\mathbb{P}_\theta \cap \mathbb{P}_p$

Pareto front



Definitions

A point $x \in A \subseteq \mathbb{R}^n$ is a Pareto optimal point in A if there does not exist another $x' \in A$ such that

- $x' \leq x$, and
- $x'_j < x_j$ for at least one j

The Pareto front of A:

$\mathbb{O}(A) \quad := \quad \{$all Parento optimal points$\}$

Sufficient condition

C13.3: $C(p)$ is strictly increasing in each p_j

C13.4: for every $(j,k) \in E$, $\tan^{-1}\dfrac{b_{jk}}{g_{jk}} < \theta_{jk}^{\min} \leq \theta_{jk}^{\max} < \tan^{-1}\dfrac{-b_{jk}}{g_{jk}}$

Theorem

Suppose G is a tree and C13.3, C13.4 hold. Then

1. $\mathbb{P}_\theta \cap \mathbb{P}_p \;=\; \mathbb{O}(\text{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p)$ feasible set is Pareto front of its relaxation
2. SOCP relaxation is exact

Geometric insight

2-bus network

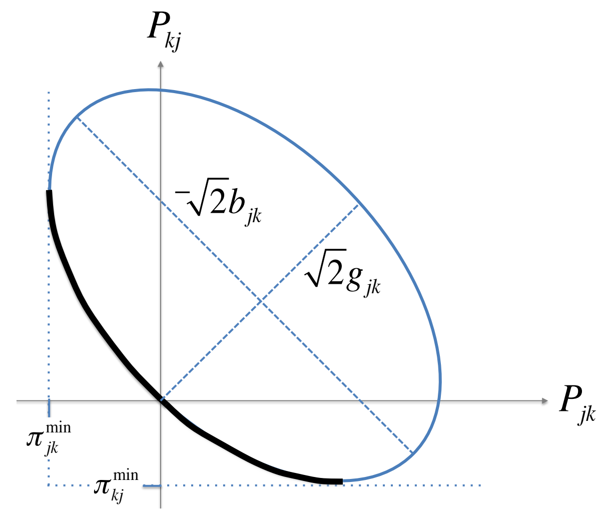
For each line $(j,k) \in E$, line flows $P := \left(P_{jk}, P_{kj}\right)$ and angle differences $\theta_{jk} := \theta_j - \theta_k$ satisfy

$$P - g_{jk}1 \;=\; A \begin{bmatrix} \cos\theta_{jk} \\ \sin\theta_{jk} \end{bmatrix} \qquad \text{where} \quad A \;:=\; \begin{bmatrix} -g_{jk} & -b_{jk} \\ -g_{jk} & b_{jk} \end{bmatrix}$$

1. P traces out an ellipse in $\mathbb{R}^2$ as θ_{jk} ranges over $[-\pi, \pi]$.

   Hence feasible set (subset of ellipse) is noncovex.

2. C13.4 restricts $\mathbb{P}_\theta$ to lower half of ellipse



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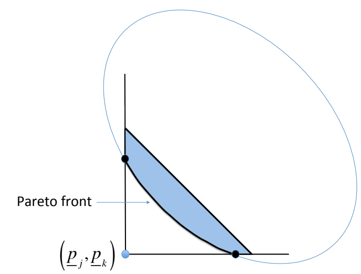
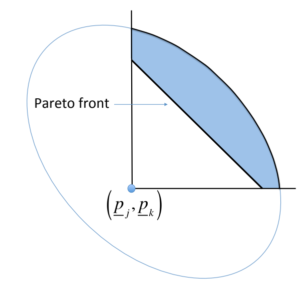
   Hence feasible set (subset of ellipse) is noncovex.
2. C13.4 restricts $\mathbb{P}_\theta$ to lower half of ellipse
3. C13.3 implies Pareto front of relaxed feasible set coincides

   with feasible set, i.e., relaxation is exact



(a) Exact relaxation with constraint



(b) Inexact relaxation with constraint

Outline

1. Relaxation of QCQP
2. Application to OPF
3. Exactness condition: linear separability
4. Exactness condition: small angle difference
5. Condition for global optimality
    - Sufficient condition
    - Application to OPF

No spurious local optima

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & x \in \mathcal{X} \end{array}$$

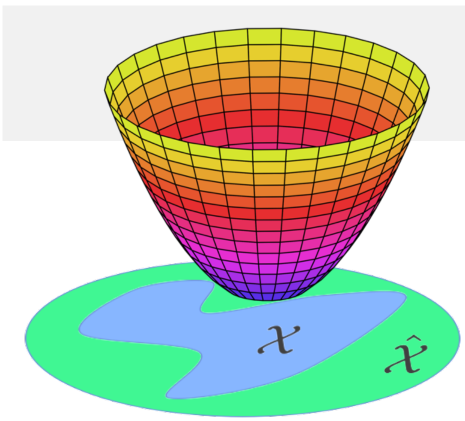
f : continuous, convex

X : compact, nonconvex

Convex relaxation:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & x \in \hat{\mathcal{X}}. \end{array}$$

$\hat{X}$: compact, convex, $X \subseteq \hat{X} \subseteq K^n$



No spurious local optima

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f(x) & f \text{ : continuous, convex} \\ \text{subject to} & x \in \mathcal{X} & X \text{ : compact, nonconvex} \end{array}$$

Convex relaxation:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f(x) & \\ \text{subject to} & x \in \hat{\mathcal{X}}. & \hat{X} \text{ : compact, convex, } X \subseteq \hat{X} \subseteq K^n \end{array}$$

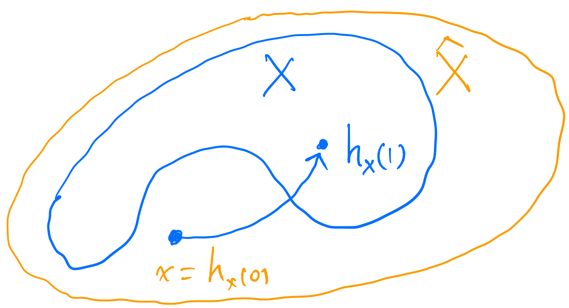
Relaxation (2) is exact if there exists optimal solution of (2) that is optimal for (1)

Key result [Zhou 2022]: Lyapunov-like conditions for

- Relaxation (2) is exact; and
- Any local optimum of (1) is globally optimal

No spurious local optima

Definition: A *path from* $x \in \hat{X} \setminus X$ *to* X is a continuous function $h_x : [0,1] \to \hat{X}$ such that $h_x(0) = x$ and $h_x(1) \in X$



Lemma [Zhou 2022]

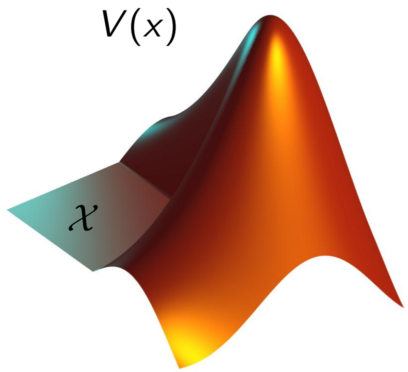
(2) is exact $\Leftrightarrow \forall x \in \hat{X} \setminus X$ there is a path h_x from x to X such that

- $f(h_x(t))$ nonincreasing in t
- $f(h_x(1)) < f(h_x(0))$

No spurious local optima

Definition: A *Lyapunov-like function* is a continuous function $V: \hat{X} \to \mathbb{R}_+$ such that

$$V(x) \begin{cases} = 0 & x \in X \\ > 0 & x \in \hat{X} \setminus X \end{cases}$$



No spurious local optima

Standard Lyapunov function

- Dynamical system: $\dot{y} = f(y(t))$
- Global asymptotic stability: $y(t) \to y^*$
- Stability certificate: Lyapunov function $V(y)$ s.t.
  1. $V(y) > 0$ if $y \neq y^*$, =0 if $y = y^*$
  2. $\dot{V}(y(t)) < 0$ along trajectory $y(t)$

No spurious local optima

Standard Lyapunov function

- Dynamical system: $\dot{y} = f(y(t))$
- Global asymptotic stability: $y(t) \rightarrow y^*$
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  1. $V(y) > 0$ if $y \neq y^*$, $=0$ if $y = y^*$
  2. $\dot{V}(y(t)) < 0$ along trajectory $y(t)$

Our case (dynamical system replaced by optimization)

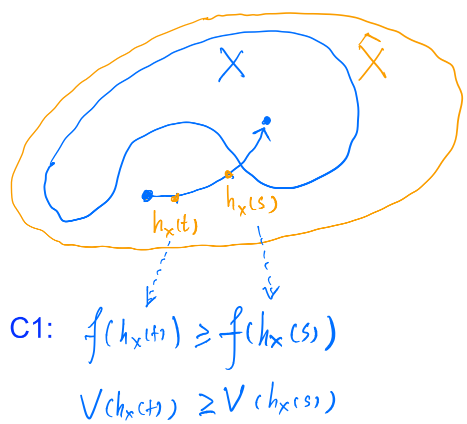
- Trajectory (path $y(t) = h_x(t)$) is not specified
- Goal is to enter X: $x = y(0) \rightarrow y(1) \in X$
- Lyapunov-like $V(y)$ s.t.
  1. $V(y) > 0$ if $y \neq y^*$, $=0$ if $y = y^*$
  2. C1: $V(y(t))$ non-increasing along trajectory $y(t)$
- Cost $f(y(t))$ must be non-increasing along $y(t)$ and

$$f\big(y(1)\big) < f\big(y(0)\big)$$

No spurious local optima

Conditions: ∃ paths $\{h_x : x \in \hat{X} \setminus X\}$ and a Lyapunov-like function V such that

- C1: both $f(h_x(t))$ and $V(h_x(t))$ are non-increasing for $t \in [0,1]$, and $f(h_x(0)) > f(h_x(1))$
- C2: $\{h_x : x \in \hat{X} \setminus X\}$ is uniformly bounded and uniformly equicontinuous



No spurious local optima

Conditions: $\exists$ paths $\{h_x : x \in \hat{X} \setminus X\}$ and a Lyapunov-like function V such that

- C1: both $f(h_x(t))$ and $V(h_x(t))$ are non-increasing for $t \in [0,1]$, and $f(h_x(0)) > f(h_x(1))$
- C2: $\{h_x : x \in \hat{X} \setminus X\}$ is uniformly bounded and uniformly equicontinuous

Theorem [Zhou 2022]

- C1, C2 $\Leftarrow$ all local optima of (1) globally optimal & (2) exact

Are C1, C2 sufficient ?

No spurious local optima

Conditions: $\exists$ paths $\{h_x : x \in \hat{X} \setminus X\}$ and a Lyapunov-like function V such that

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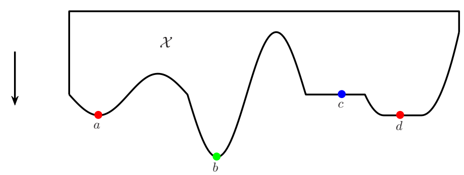
- C2: $\{h_x : x \in \hat{X} \setminus X\}$ is uniformly bounded and uniformly equicontinuous

Local algorithm may converge to any local optimum:

Examples
Global optimum (g.o.): *b*
Pseudo local optimum (p.l.o.): *c*
Genuine local optimum (g.l.o.): *a*, *d*



No spurious local optima

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- C2: $\{h_x : x \in \hat{X} \setminus X\}$ is uniformly bounded and uniformly equicontinuous
- C3: $\exists\, k > 0$ such that $f(h_x(t)) - f(h_x(s)) \geq k\|h_x(t) - h_x(s)\|$

Local algorithm may converge to any local optimum:

Examples

Global optimum (g.o.): *b*
Pseudo local optimum (p.l.o.): *c*
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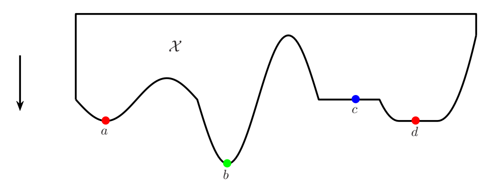

- C1, C2 eliminate genuine local optimal (a, d)
- C3 eliminates pseudo local optimum (c)

No spurious local optima

Conditions: $\exists$ paths $\{h_x: x \in \hat{X} \setminus X\}$ and a Lyapunov-like function V such that

- C1: both $f(h_x(t))$ and $V(h_x(t))$ are non-increasing for $t \in [0,1]$, and $f(h_x(0)) > f(h_x(1))$
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Theorem [Zhou 2022]

- C1, C2 $\Leftarrow$ all local optima of (1) globally optimal & (2) exact
- C1, C2, C3 $\Rightarrow$ all local optima of (1) globally optimal & (2) exact

Applications: OPF, low rank SDP, …
Suitable for problems with convex cost but nonconvex feasible set

No spurious local optima

Conditions: $\exists$ paths $\{h_x : x \in \hat{X} \setminus X\}$ and a Lyapunov-like function V such that

- C1: both $f(h_x(t))$ and $V(h_x(t))$ are non-increasing for $t \in [0,1]$, and $f(h_x(0)) > f(h_x(1))$
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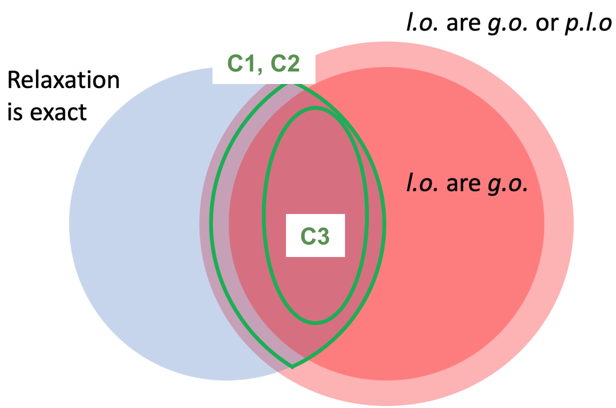

Application to OPF

Non-convex problem:

$$\begin{aligned} \min_{s,v,\ell,S} \quad & f(s) \\ \text{s.t.} \quad & \text{convex constr.} \\ & v_j \ell_{jk} = |S_{jk}|^2 \end{aligned}$$

Relaxed problem:

$$\begin{aligned} \min_{s,v,\ell,S} \quad & f(s) \\ \text{s.t.} \quad & \text{convex constr.} \\ & v_j \ell_{jk} \geq |S_{jk}|^2 \end{aligned}$$

Baran-Wu 1989 DistFlow model

Application to OPF

Non-convex problem:

$$\min_{s,v,\ell,S} \quad f(s)$$
$$\text{s.t.} \quad \text{convex constr.}$$
$$v_j \ell_{jk} = |S_{jk}|^2$$

Relaxed problem:

$$\min_{s,v,\ell,S} \quad f(s)$$
$$\text{s.t.} \quad \text{convex constr.}$$
$$v_j \ell_{jk} \geq |S_{jk}|^2$$

Construction

$$V := \sum_{jk} v_k \ell_{jk} - |S_{jk}|^2$$

h_x: linearly decrease ℓ_{jk} and linearly adjust s, S accordingly.

This construction satisfies C1, C2, C3

Theorem

If there are no lower bounds for s_j, i.e., bus injections, then any local optimum of the original non-convex OPF is also a global optimum.

First result on the local optimality for non-convex OPF problem. [Zhou, Low CDC2020]

Application to OPF

Non-convex problem:

$$\begin{aligned} \min_{s,v,\ell,S} \quad & f(s) \\ \text{s.t.} \quad & \text{convex constr.} \\ & v_j \ell_{jk} = |S_{jk}|^2 \end{aligned}$$

Relaxed problem:

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Construction (a 2-bus example)

- $V := v_1 \ell_{12} - |S_{12}|^2$
- For $x \in \hat{\mathcal{X}} \setminus \mathcal{X}$, we have $|S_{12}|^2 - v_1 \ell_{12} < 0$.
- Let Δ be the positive root of $\frac{|z_{12}|^2}{4} a^2 + \left(v_1 - \mathrm{Re}(z_{12} S_{12}^{\mathrm{H}})\right) a + |S_{12}|^2 - v_1 \ell_{12}$
- Consider the path:

$$\begin{aligned} \tilde{s}_j(t) &= s_j - \frac{t}{2} z_{12} \Delta - \frac{t}{2} z_{12} \Delta, \\ \tilde{v}_j(t) &= v_j, \\ \tilde{\ell}_{12}(t) &= \ell_{12} - t\Delta, \\ \tilde{S}_{12}(t) &= S_{12} - \frac{t}{2} z_{12} \Delta. \end{aligned}$$

Construction satisfies C1, C2, C3
- SOCP relaxation is exact
- Local optima are globally optimal

F. Zhou

Summary

OPF is nonconvex & NP hard

OPF is "easy" in practice

- Semidefinite relaxations often exact
- Local algorithms often globally optimal

Analytical properties

- Exact relaxation
- No spurious local optima