Power System Analysis

Chapter 14 Semidefinite relaxations: BFM

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Outline

1. Single-phase OPF
2. Three-phase OPF

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1. Single-phase OPF
    - SOCP relaxation
    - Equivalence
    - Exactness condition: inactive injection lower bounds
    - Exactness condition: inactive voltage upper bounds
2. Three-phase OPF

Assumptions

Both single-phase & 3-phase OPF

Radial network

- BFM most useful for modeling distribution systems

$z^s_{jk} = z^s_{kj}$ or equivalently $y^s_{jk} = y^s_{kj}$

- Does not include 3-phase transformers in ΔY or $Y\Delta$ configuration (or single-phase transformers with complex gains)

$y^m_{jk} = y^m_{kj} = 0$

- Reasonable assumption for distribution line where $|y^m_{jk}|, |y^m_{kj}| \ll |y^s_{jk}|$

Includes only voltage sources and power sources

- Optimization variables are voltages (squared magnitudes) v_j and power injections s_j respectively
- A current source or an impedance will introduce additional var and constraint.

Single-phase OPF

Power flow equations

- All lines point towards bus 0 (root)

$$S_{jk} = \sum_{i:i\to j} \left(S_{ij} - z_{ij}\ell_{ij} \right) + s_j, \qquad j \in \overline{N}$$

$$v_j - v_k = 2\,\text{Re}\left(z_{jk}^{\mathsf{H}} S_{jk} \right) - |z_{jk}|^2 \ell_{jk}, \qquad j \to k \in E$$

$$v_j \ell_{jk} = |S_{jk}|^2, \qquad j \to k \in E$$

Operational constraints

$$s_j^{\min} \le s_j \le s_j^{\max}$$

$$v_j^{\min} \le v_j \le v_j^{\max}$$

$$\ell_{jk} \le \ell_{jk}^{\max}$$

nonconvex constraint
(other constraints are linear in x)

Single-phase OPF

Feasible set

$$\mathbb{T} := \left\{ x := (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies PF equations \& operational constraints} \right\}$$

OPF in BFM

$$\min_{x \in \mathbb{T}} \quad C(x)$$

SOCP relaxation

Power flow equations

- All lines point towards bus 0 (root)

$$S_{jk} = \sum_{i:i\to j} \left(S_{ij} - z_{ij}\ell_{ij} \right) + s_j, \qquad j \in \overline{N}$$

$$v_j - v_k = 2\,\mathrm{Re}\left(z_{jk}^{\mathsf{H}} S_{jk} \right) - |z_{jk}|^2 \ell_{jk}, \qquad j \to k \in E$$

$$v_j \ell_{jk} \geq |S_{jk}|^2, \qquad j \to k \in E$$

second-order cone

Operational constraints

$$s_j^{\min} \leq s_j \leq s_j^{\max}$$

$$v_j^{\min} \leq v_j \leq v_j^{\max}$$

$$\ell_{jk} \leq \ell_{jk}^{\max}$$

SOCP relaxation

Feasible set

$$\mathbb{T}^+ := \left\{ x := (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies } v_j \ell_{jk} \geq |S_{jk}|^2 \text{ \& operational constraints} \right\}$$

SOCP relaxation in BFM

$$\min_{x \in \mathbb{T}^+} \quad C(x)$$

Exactness

Definition (Strong exactness)

SOCP relaxation is exact if every optimal solution x^{opt} of SOCP relaxation attains equality:

$$v_j^{\text{opt}} \ell_{jk}^{\text{opt}} = \left| S_{jk}^{\text{opt}} \right|^2, \qquad j \to k \in E$$

- Convenient because any algorithm that solves an exact relaxation produces an optimal solution for original OFP
- Not necessary: under sufficient conditions for radial network, can always recover an optimal solution of OPF from any solution of SOCP relaxation, even when SOCP relaxation is not exact

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OPF in BIM

$$\min_{V \in \mathbb{C}^{N+1}} \quad V^{\mathsf{H}} C_0 V$$

$$\text{s.t.} \quad p_j^{\min} \leq \operatorname{tr}\left(\Phi_j V V^{\mathsf{H}}\right) \leq p_j^{\max}, \qquad j \in \overline{N}$$

$$q_j^{\min} \leq \operatorname{tr}\left(\Psi_j V V^{\mathsf{H}}\right) \leq q_j^{\max}, \qquad j \in \overline{N}$$

$$v_j^{\min} \leq \operatorname{tr}\left(J_j V V^{\mathsf{H}}\right) \leq v_j^{\max}, \qquad j \in \overline{N}$$

$$\operatorname{tr}\left(\hat{Y}_{jk} V V^{\mathsf{H}}\right) \leq \bar{I}_{jk}^{\max}, \qquad (j,k) \in E$$

$$\operatorname{tr}\left(\hat{Y}_{kj} V V^{\mathsf{H}}\right) \leq \bar{I}_{kj}^{\max}, \qquad (j,k) \in E$$

Feasible set in BIM

Given $V \in \mathbb{C}^{N+1|}$, define partial matrix W_G by

$$[W_G]_{jj} := |V_j|^2, \qquad j \in \overline{N}$$

$$\left[W_G\right]_{jk} := V_j V_k^{\mathsf{H}} =: [W_G]_{kj}^{\mathsf{H}}, \qquad (j,k) \in E$$

Constraints in terms of W_G

$$p_j^{\min} \le \text{tr}\left(\Phi_j W_G\right) \le p_j^{\max}$$

$$q_j^{\min} \le \text{tr}\left(\Psi_j W_G\right) \le q_j^{\max}$$

$$v_j^{\min} \le \text{tr}\left(J_j W_G\right) \le v_j^{\max}$$

$$\text{tr}\left(\hat{Y}_{jk} W_G\right) \le I_{jk}^{\max}$$

$$\text{tr}\left(\hat{Y}_{kj} W_G\right) \le I_{kj}^{\max}$$

$\mathbb{W}_G^+ := \{ W_G \mid W_G$ satisfies constraints$\}$

Equivalence

SOCP relaxation in BFM

$$\min_{x} \quad C(x) \quad \text{s.t.} \quad x \in \mathbb{T}^+ := \left\{ x \mid x \text{ satisfies } v_j \ell_{jk} \geq |S_{jk}|^2 \text{ \& operational constraints} \right\}$$

SOCP relaxation in BIM

$$\min_{W_G} \quad C_0(W_G) \quad \text{s.t.} \quad W_G \in \mathbb{W}_G^+ := \left\{ W_G \mid W_G \text{ satisfies constraints} \right\}$$

Theorem

$$\mathbb{T}^+ \equiv \mathbb{W}_G^+$$

Implication: The two problems are equivalent in the sense that ∃ bijection $g : \mathbb{W}_G^+ \longrightarrow \mathbb{T}^+$ s.t. W_G^{opt} is optimal in BIM iff $x^{\text{opt}} := g\left(W_G^{\text{opt}}\right)$ is optimal in BFM

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Exactness: injection lower bounds

Assume:

1. Cost function $C(x)$ is strictly increasing in ℓ, nondecreasing in s, and independent of S
2. No injection lower bounds: $s_j^{\min} = -\infty - i\infty$

Theorem

Suppose network graph G is tree and Assumptions 1 and 2 hold. Then SOCP relaxation is exact, i.e., every optimal solution x^{opt} of SOCP relaxation is optimal for OPF

Remark: Even when the SOCP relaxation is not exact, under these conditions, an optimal solution of OPF can always be recovered from any solution of SOCP relaxation

Exactness: voltage upper bounds

Example: geometric insight

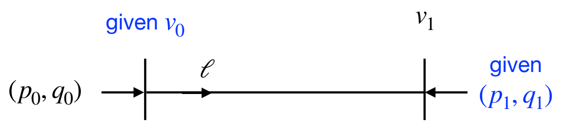
Power flow solution $x := (p_0, q_0, v_1, \ell)$ satisfies:

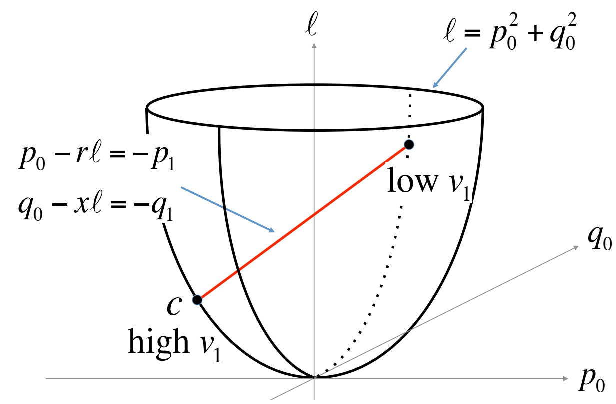
$$p_0 - r\ell = -p_1$$

$$q_0 - x\ell = -q_1$$

$$p_0^2 + q_0^2 = \ell$$

$$v_1 - v_0 = 2\left(rp_1 + xq_1\right) - (r^2 + x^2)\ell$$





power flow solutions (feasible set) : { 2 intersection points }

Exactness: voltage upper bounds

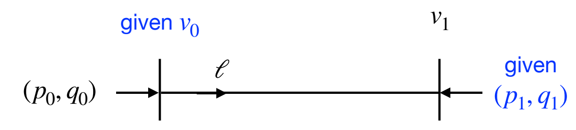
Example: geometric insight

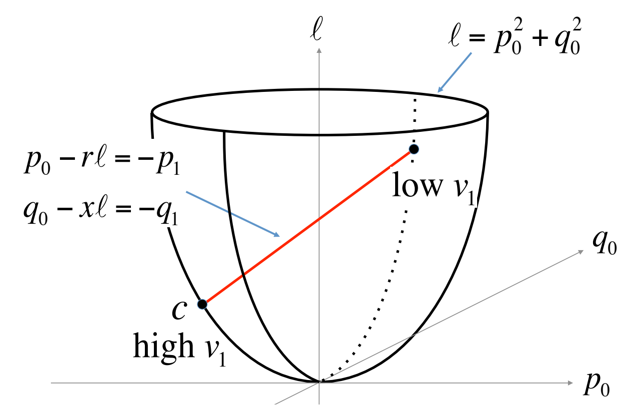
Feasible set (without voltage constraints)

- OPF : { 2 intersection points }, nonconvex
- SOCP relaxation : line segment, convex

Cost function $c(x)$ increasing in ℓ

- Optimal solution x^{opt} has high v_1
- SOCP relaxation is exact



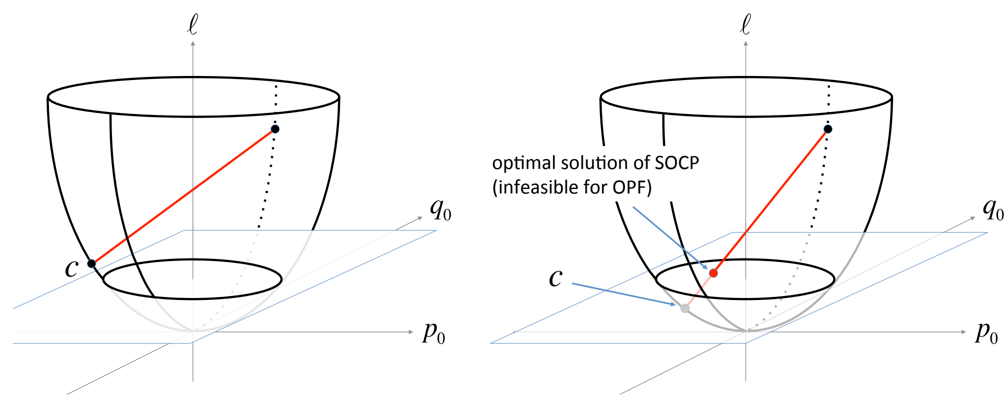


Exactness: voltage upper bounds

Example: geometric insight

Voltage constraints

- $\frac{1}{|z|^2}\left(a - v_1^{\max}\right) \;\le\; \ell \;\le\; \frac{1}{|z|^2}\left(a - v_1^{\min}\right)$
- $\therefore\ v_1^{\min}$ leads to upper bound on ℓ and will not affect exactness
- $v_1^{\max}$ leads to lower bound on ℓ and can affect exactness when it binds



(a) Voltage constraint not binding (b) Voltage constraint binding

Exactness: voltage upper bounds

Assume:

1. Cost function $C(x) := \sum_j C_j(p_j)$ with $C_0(p_0)$ strictly increasing in p_0. There is no constraint on s_0
2. $\hat{v}_j^{\text{lin}}(s) \leq v_j^{\max}, \quad j \in N$
3. Technical condition: small change in a line power affects all upstream line powers in the same direction

Theorem

Suppose network graph G is tree and Assumptions 1-3 hold. Then SOCP relaxation is exact, i.e., every optimal solution x^{opt} of SOCP relaxation is optimal for OPF

Remark: Even when the SOCP relaxation is not exact, under these conditions, an optimal solution of OPF can always be recovered from any solution of SOCP relaxation

Exactness implies uniqueness

Theorem

Suppose network graph G is tree, C_j are convex functions and injection regions are convex sets. If SOCP relaxation is exact, then its optimal solution is unique

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1. Single-phase OPF
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   - Reformulation
   - Semidefinite relaxation