Power System Analysis

Chapter 8 Unbalanced network: component models

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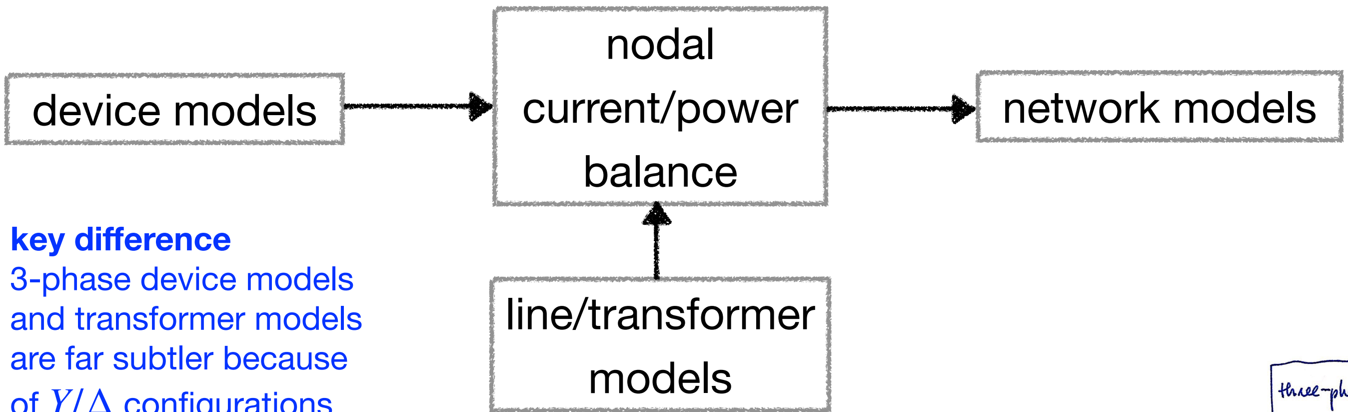
Outline

1. Overview
2. Mathematical properties
3. Three-phase device models
4. Three-phase line models
5. Three-phase transformer models

Outline

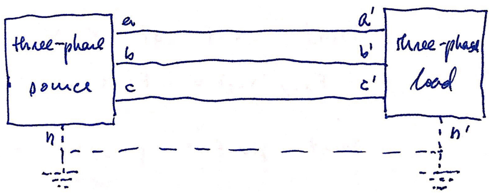
1. Overview
   - Internal & terminal variables
   - 3-phase device models
   - 3-phase line & transformer models
   - 3-phase network models
2. Mathematical properties
3. Three-phase device models
4. Three-phase line models
5. Three-phase transformer models

Overview



key difference
3-phase device models and transformer models are far subtler because of Y/Δ configurations

single-phase or 3-phase



Example

Single-phase system

System model = device model + network model

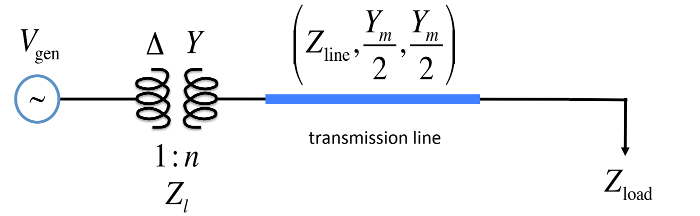
1. Device model: $V_1 = \dfrac{V_{\text{gen}}}{\sqrt{3}\, e^{i\pi/6}}$, $V_{\text{load}} = Z_{\text{load}} I_{\text{load}}$

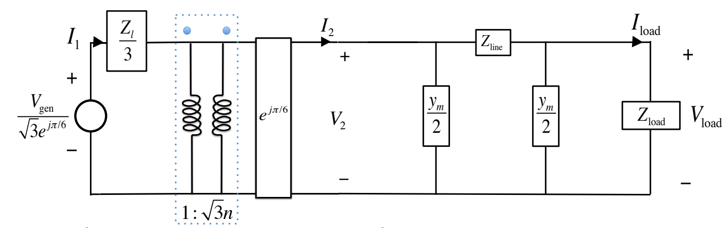
2. Transformer model: $\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = Y_{\text{transformer}} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

3. Line model: $\begin{bmatrix} I_2 \\ I_{\text{load}} \end{bmatrix} = Y_{\text{line}} \begin{bmatrix} V_2 \\ V_{\text{load}} \end{bmatrix}$

4. Nodal (current) balance are implicitly taken into account

5. 6 (linear) equations in 6 unknowns $\left(V_1, V_2, V_{\text{load}}\right), \left(I_1, I_2, I_{\text{load}}\right)$ each in $\mathbb{C}$



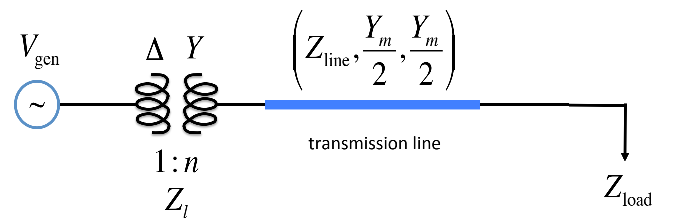


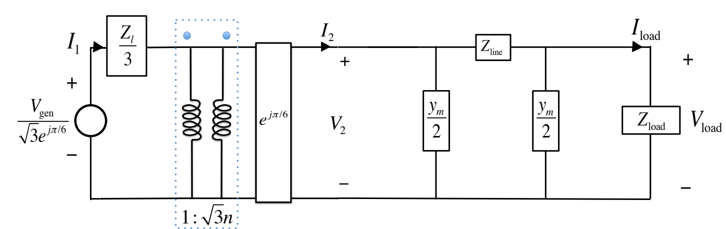
Example

Three-phase unbalanced system

System model = device model + network model

1. Device model: Y/Δ-configured devices are a key difference
2. Transformer model: Y/Δ-configured transformers are a key difference
3. Line model: 3-phase lines have straightforward extension
4. Nodal (current) balance are the same as for 1-phase network
5. 6 (linear) equations in 6 unknowns $\left(V_1, V_2, V_{\text{load}}\right), \left(I_1, I_2, I_{\text{load}}\right)$ each in $\mathbb{C}^3$





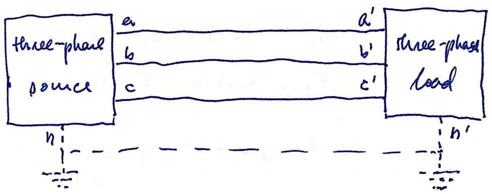
Overview

device models → nodal current/power balance → network models

line/transformer models → nodal current/power balance

key difference
3-phase device models and transformer models are far subtler because of Y/Δ configurations

single-phase or 3-phase



Key question

How to derive external models of 3-phase devices

1. Voltage/current/power sources, impedances (1-phase device: internal models)
2. … in Y/Δ configurations (conversion rules: int → ext)
3. … with or without neutral lines, grounded or ungrounded, zero or nonzero grounding impedances

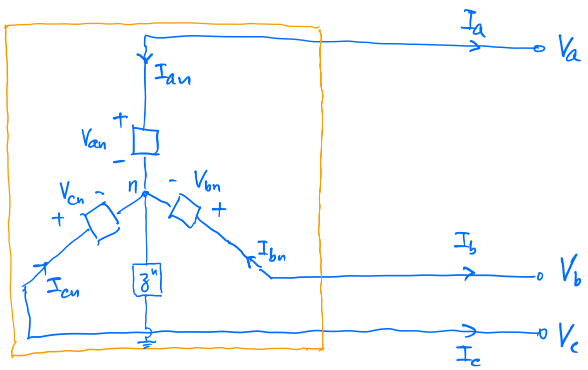
similar principle to derive external models of 3-phase transformers (but different details)

Internal variables

Y configuration

Internal voltage, current, power across single-phase devices:

$$V^Y := \begin{bmatrix} V^{an} \\ V^{bn} \\ V^{cn} \end{bmatrix}, \quad I^Y := \begin{bmatrix} I^{an} \\ I^{bn} \\ I^{cn} \end{bmatrix}, \quad s^Y := \begin{bmatrix} s^{an} \\ s^{bn} \\ s^{cn} \end{bmatrix} := \begin{bmatrix} V^{an}\bar{I}^{an} \\ V^{bn}\bar{I}^{bn} \\ V^{cn}\bar{I}^{cn} \end{bmatrix}$$



neutral voltage (wrt common reference pt) $V^n \in \mathbb{C}$

neutral current (away from neutral) $I^n \in \mathbb{C}$

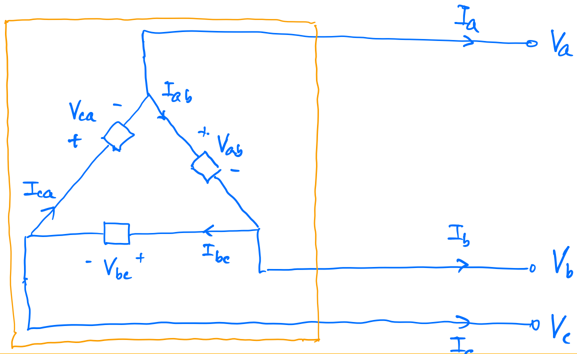
- Neutral line may or may not be present
- Device may or may not be grounded
- Neutral impedance z^n may or may not be zero

Internal variables

Δ configuration

Internal voltage, current, power across single-phase devices:

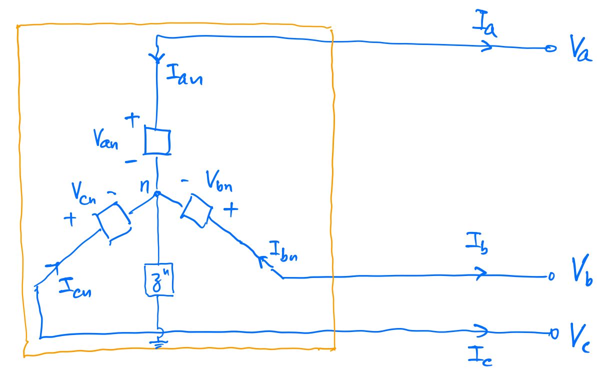
$$V^\Delta := \begin{bmatrix} V^{ab} \\ V^{bc} \\ V^{ca} \end{bmatrix}, \ I^\Delta := \begin{bmatrix} I^{ab} \\ I^{bc} \\ I^{ca} \end{bmatrix}, \ s^\Delta := \begin{bmatrix} s^{ab} \\ s^{bc} \\ s^{ca} \end{bmatrix} := \begin{bmatrix} V^{ab}\bar{I}^{ab} \\ V^{bc}\bar{I}^{bc} \\ V^{ca}\bar{I}^{ca} \end{bmatrix}$$



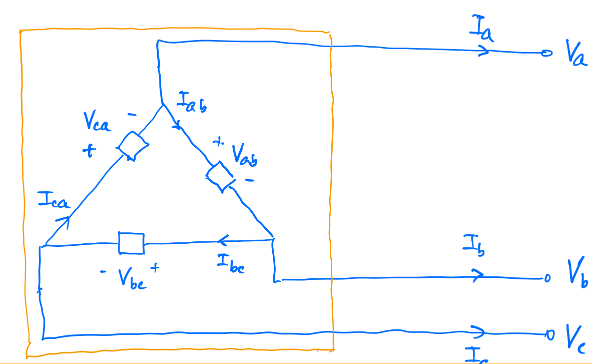
Terminal variables

Terminal voltage, current, power (for both Y and Δ) to reference:

$$V := \begin{bmatrix} V^a \\ V^b \\ V^c \end{bmatrix}, \; I := \begin{bmatrix} I^a \\ I^b \\ I^c \end{bmatrix}, \; s := \begin{bmatrix} s^a \\ s^b \\ s^c \end{bmatrix} := \begin{bmatrix} V^a \bar{I}^a \\ V^b \bar{I}^b \\ V^c \bar{I}^c \end{bmatrix}$$

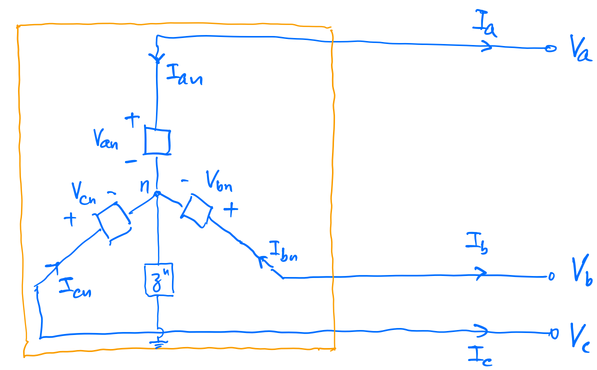


- V is with respect to an arbitrary common reference point, e.g. the ground
- I and s are in the direction out of the device

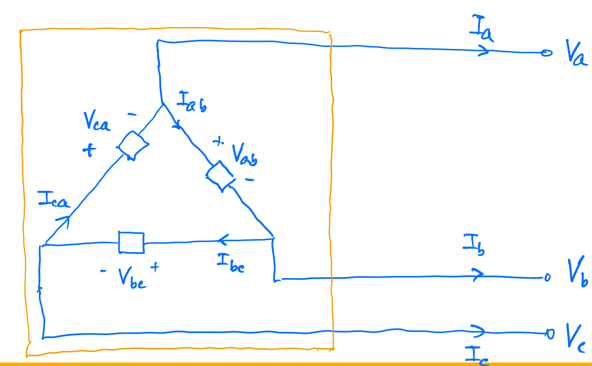


Internal vs terminal power

1. Internal power:
    - Across each single-phase device: $s^{Y/\Delta} := \text{diag}\left(V^{Y/\Delta} I^{Y/\Delta\mathsf{H}}\right)$
    - Across neutral conductor: $s^n := V^n \bar{I}^n$

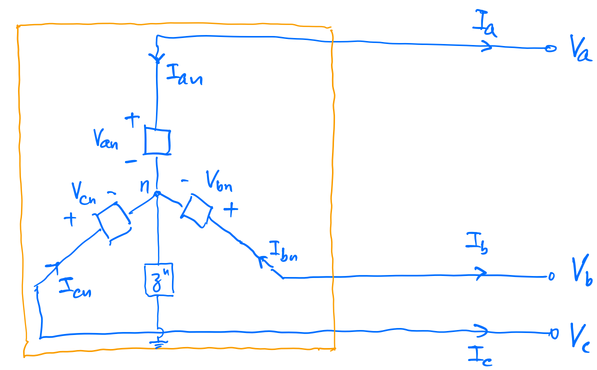


2. Terminal power:
    - Power injected from device to network: $s := \text{diag}\left(V I^{\mathsf{H}}\right)$

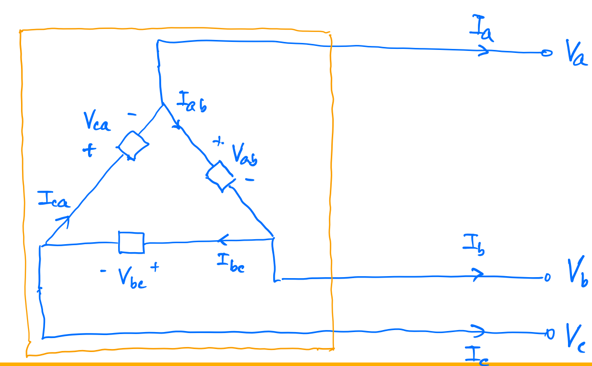


Summary: variables

	Voltage	Current	Power	Neutral line
Internal variable	$V^{Y/\Delta}$	$I^{Y/\Delta}$	$s^{Y/\Delta}$	(V^n, I^n, s^n)
External variable	V	I	s	$\left(V^{n'}, I^{n'}, s^{n'}\right)$



- Neutral line may or may not be present
- Device may or may not be grounded
- Neutral impedance z^n may or may not be zero



Device models

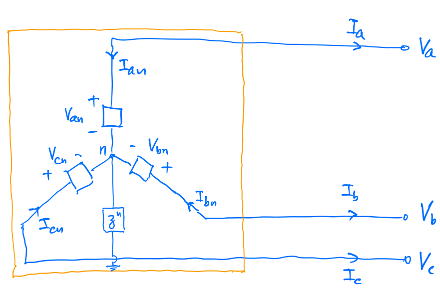
Internal model

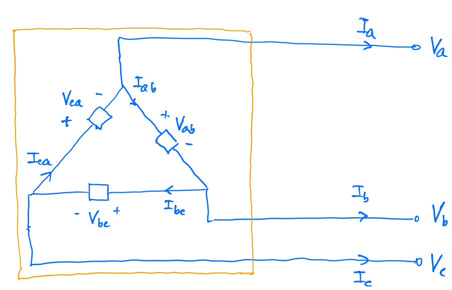
1. Relation between internal vars: $f^{\text{int}}\left(V^{Y/\Delta}, I^{Y/\Delta}\right) \;=\; 0, \qquad \text{diag}\left(V^{Y/\Delta} I^{Y/\Delta \mathsf{H}}\right) \;=\; s^{Y/\Delta}$

2. Examples

ideal voltage source: $\quad V^{Y/\Delta} \;=\; E^{Y/\Delta}, \qquad s^{Y/\Delta} \;=\; \text{diag}\left(E^{Y/\Delta}\left(I^{Y/\Delta}\right)^{\mathsf{H}}\right)$

impedance: $\quad V^{Y/\Delta} \;=\; z^{Y/\Delta}\, I^{Y/\Delta}, \qquad s^{Y/\Delta} \;=\; \text{diag}\left(V^{Y/\Delta}\left(I^{Y/\Delta}\right)^{\mathsf{H}}\right)$





Device models

Internal model

1. Relation between internal vars: $f^{\text{int}}\left(V^{Y/\Delta}, I^{Y/\Delta}\right) = 0, \qquad \text{diag}\left(V^{Y/\Delta} I^{Y/\Delta\mathsf{H}}\right) = s^{Y/\Delta}$

2. Examples

   ideal voltage source: $V^{Y/\Delta} = E^{Y/\Delta}, \qquad s^{Y/\Delta} = \text{diag}\left(E^{Y/\Delta}\left(I^{Y/\Delta}\right)^{\mathsf{H}}\right)$

   impedance: $V^{Y/\Delta} = z^{Y/\Delta} I^{Y/\Delta}, \qquad s^{Y/\Delta} = \text{diag}\left(V^{Y/\Delta}\left(I^{Y/\Delta}\right)^{\mathsf{H}}\right)$

3. Internal model
   - Independent of Y or Δ configuration
   - Depends only on behavior of single-phase devices
   - Voltage/current/power source, impedance

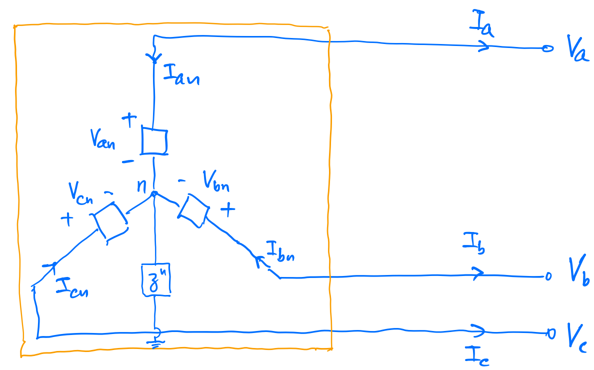
Device model

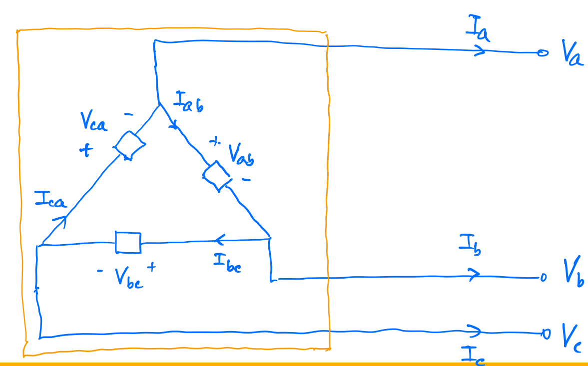
External model

1. External model = Internal model + Conversion rule
   - External model: relation between (V, I, s)

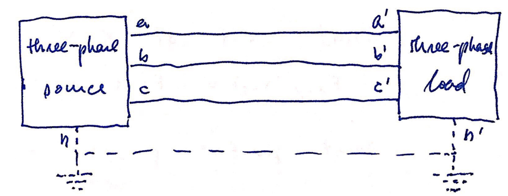
$$f^{\text{ext}}(V, I) = 0, \qquad s = \text{diag}\left(VI^{\mathsf{H}}\right)$$

   - Devices interact over network only through their terminal vars
2. Internal model : relation between $\left(V^{Y/\Delta}, I^{Y/\Delta}, s^{Y/\Delta}\right)$
   - Independent of Y or Δ configuration
   - Depends only on behavior of single-phase devices
3. Conversion rule : converts between internal and terminal vars
   - Depends only on Y or Δ configuration
   - Independent of type of single-phase devices



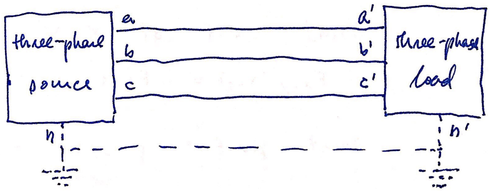


Line or transformer model



1. A line or transformer has two terminals j and k
    - Each terminal may have 3 wires (ports) or 4 wires (ports) if neutral line present
2. Terminal variables (3-wired)
    - Terminal voltages: $V_j := \left(V_j^a, V_j^b, V_j^c\right) \in \mathbb{C}^3, \ \ V_k := \left(V_k^a, V_k^b, V_k^c\right) \in \mathbb{C}^3$
    - Sending-end currents: $I_{jk} := \left(I_{jk}^a, I_{jk}^b, I_{jk}^c\right) \in \mathbb{C}^3, \ \ I_{kj} := \left(I_{kj}^a, I_{kj}^b, I_{kj}^c\right) \in \mathbb{C}^3$
    - Sending-end powers: $S_{jk} := \left(S_{jk}^a, S_{jk}^b, S_{jk}^c\right) \in \mathbb{C}^3, \ \ S_{kj} := \left(S_{kj}^a, S_{kj}^b, S_{kj}^c\right) \in \mathbb{C}^3$
3. Model in terms of 3×3 admittance matrices:
    - IV relation: $g\left(V_j, V_k, I_{jk}, I_{kj}\right) \ = \ 0$
    - sV relation: $S_{jk}^\phi \ := \ V_j^\phi \left(I_{jk}^\phi\right)^{\mathsf{H}}$ or in vector form $S_{jk} \ := \ \text{diag}\left(V_j I_{jk}^{\mathsf{H}}\right), \ \ S_{kj} \ := \ \text{diag}\left(V_k I_{kj}^{\mathsf{H}}\right)$

Network model



Network balance equations relate terminal vars

- Nodal current balance: $I_j \;=\; \sum_{k:j\sim k} I_{jk}$
- Nodal power balance: $s_j \;=\; \sum_{k:j\sim k} S_{jk}$

Overall model

Device + network

1. Device model for each 3-phase device
    - Internal model on $\left(V_j^{Y/\Delta}, I_j^{Y/\Delta}, s_j^{Y/\Delta}\right)$ + conversion rules
    - External model on $\left(V_j, I_j, s_j\right)$
    - Either can be used
    - Power source models are nonlinear; other devices are linear

2. Network model relates terminal vars (V, I, s)
    - Nodal current balance equation: linear
    - Nodal power balance equation: nonlinear
    - Either can be used

Overall model will be linear if and only if only voltage/current sources and impedances are present (but no power sources)

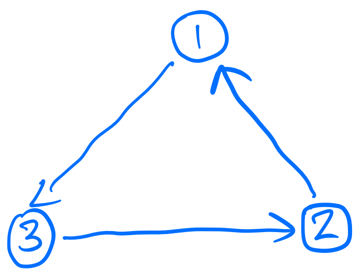
Outline

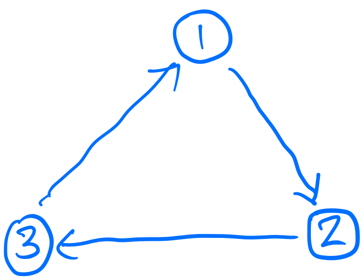
1. Overview
2. Mathematical properties
    - Conversion matrices $\Gamma, \Gamma^\mathsf{T}$
    - Sequence variables
3. Three-phase device models
4. Three-phase line models
5. Three-phase transformer models

Conversion matrices

$$\Gamma := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \qquad \Gamma^{\mathsf{T}} := \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Incidence matrices for:





Conversion matrices

Convert between internal vars and external vars

$$\begin{bmatrix} V_{ab} \\ V_{bc} \\ V_{ca} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}}_{\Gamma} \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix}, \qquad \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = - \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{\Gamma^{\mathsf{T}}} \begin{bmatrix} I_{ab} \\ I_{bc} \\ I_{ca} \end{bmatrix}$$

Conversion matrices

Convert between internal vars and external vars

$$\begin{bmatrix} V_{ab}\\ V_{bc}\\ V_{ca}\end{bmatrix} = \underbrace{\begin{bmatrix}1&-1&0\\0&1&-1\\-1&0&1\end{bmatrix}}_{\Gamma}\begin{bmatrix}V_a\\V_b\\V_c\end{bmatrix}, \qquad \begin{bmatrix}I_a\\I_b\\I_c\end{bmatrix} = -\underbrace{\begin{bmatrix}1&0&-1\\-1&1&0\\0&-1&1\end{bmatrix}}_{\Gamma^{\mathsf{T}}}\begin{bmatrix}I_{ab}\\I_{bc}\\I_{ca}\end{bmatrix}$$

In vector form

$$V^{\Delta} \; = \; \Gamma V, \qquad\qquad I \; = \; -\Gamma^{\mathsf{T}} I^{\Delta}$$

internal voltage | terminal voltage | terminal current | internal current

Conversion matrices

Lemma

Let $M \in \mathbb{C}^{n\times n}$ be a normal matrix, i.e., $MM^{\mathsf{H}} = M^{\mathsf{H}}M$.

1. *Decomposition*: $M = U\Lambda U^{\mathsf{H}}$ where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ are eigenvalues and columns of U are eigenvectors of M.
2. *Pseudo-inverse*: $M^\dagger = U\Lambda^\dagger U^{\mathsf{H}}$ where $\Lambda^\dagger := \text{diag}\left(\lambda_1^{-1}, \ldots, \lambda_n^{-1}\right)$ with $\lambda_j^{-1} := 0$ if $\lambda_j = 0$.
3. *Solution of $Mx = b$*: A solution x exists if and only if b is orthogonal to $\text{null}\left(M^{\mathsf{H}}\right)$ in which case

$$x = M^\dagger b + w, \qquad w \in \text{null}(M)$$

Conversion matrices

Spectral decomposition

Spectral decomposition:

$$\Gamma \;=\; F\Lambda\overline{F}, \qquad \Gamma^{\mathsf{T}} \;=\; \overline{F}\Lambda F$$

where

$$\Lambda := \begin{bmatrix} 0 & & \\ & 1-\alpha & \\ & & 1-\alpha^2 \end{bmatrix},$$

eigenvectors of Γ, Γ^{T}

$$F := \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix}$$

positive-seq balanced vector α_+

negative-seq balanced vector α_-

and $\alpha := e^{-\mathrm{i}2\pi/3}$

Conversion matrices

Theorem

1. The null spaces of Γ and Γ^{T} are both span(1).
2. Γ is normal. Moreover, $\Gamma\Gamma^{\dagger} = \Gamma^{\dagger}\Gamma = \frac{1}{3}\Gamma\Gamma^{\mathsf{T}} = \frac{1}{3}\Gamma^{\mathsf{T}}\Gamma = \mathbb{I} - \frac{1}{3}11^{\mathsf{T}}$
3. Their pseudo-inverses are: $\Gamma^{\dagger} = \dfrac{1}{3}\Gamma^{\mathsf{T}}, \quad \Gamma^{\mathsf{T}\dagger} = \dfrac{1}{3}\Gamma$
4. Consider $\Gamma x = b$. Solutions x exist if and only if $1^{\mathsf{T}}b = 0$, in which case
$$x = \frac{1}{3}\Gamma^{\mathsf{T}}b + \gamma 1, \qquad \gamma \in \mathbb{C}$$
5. Consider $\Gamma^{\mathsf{T}}x = b$. Solutions x exist if and only if $1^{\mathsf{T}}b = 0$, in which case
$$x = \frac{1}{3}\Gamma b + \beta 1, \qquad \beta \in \mathbb{C}$$

Sequence variables

Fortescue matrix F

1. F is unitary and complex symmetric (recall $\Gamma = F\Lambda\overline{F}$)
2. Its inverse is:

$$F^{-1} \;=\; F^{\mathsf{H}} \;=\; \overline{F} \;=\; \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & \overline{\alpha}_+ & \overline{\alpha}_- \end{bmatrix}$$

3. F defines a similarity transformation:

$$x \;=\; F\tilde{x}, \qquad \tilde{x} \;:=\; F^{-1}x \;=\; \overline{F}x$$

4. $\tilde{x}$ is called the sequence variable of x. Its components are

$$\tilde{x}_0 \;:=\; \frac{1}{\sqrt{3}}1^{\mathsf{H}}x, \qquad \tilde{x}_+ \;:=\; \frac{1}{\sqrt{3}}\alpha_+^{\mathsf{H}}x, \qquad \tilde{x}_- \;:=\; \frac{1}{\sqrt{3}}\alpha_-^{\mathsf{H}}x$$

zero-sequence   positive-sequence   negative-sequence

Sequence variables

Sequence voltage, current, power

1. Sequence voltage and current:

$$\tilde{V} \;=\; \overline{F}V, \qquad \tilde{I} \;=\; \overline{F}I$$

2. Powers in phase and sequence coordinates:

$$s := \text{diag}\left(VI^{\mathsf{H}}\right), \qquad \tilde{s} := \text{diag}\left(\tilde{V}\tilde{I}^{\mathsf{H}}\right)$$

3. The total powers are equal $1^{\mathsf{T}}\tilde{s} = 1^{\mathsf{T}}s$:

$$1^{\mathsf{T}}\tilde{s} \;=\; \tilde{I}^{\mathsf{H}}\tilde{V} \;=\; \left(I^{\mathsf{H}}\overline{F}^{\mathsf{H}}\right)\left(\overline{F}V\right) \;=\; I^{\mathsf{H}}V \;=\; 1^{\mathsf{T}}s$$

since $\overline{F}^{\mathsf{H}}\overline{F} = F\overline{F} = \mathbb{I}$

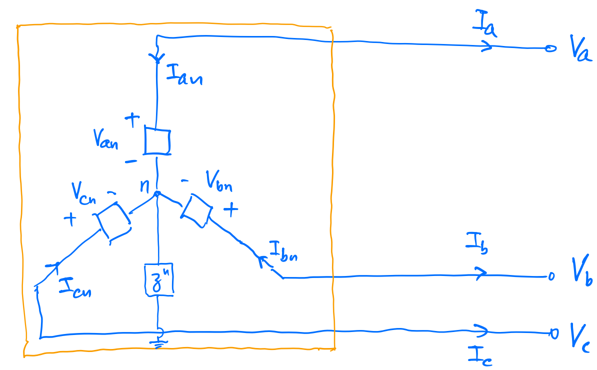
Outline

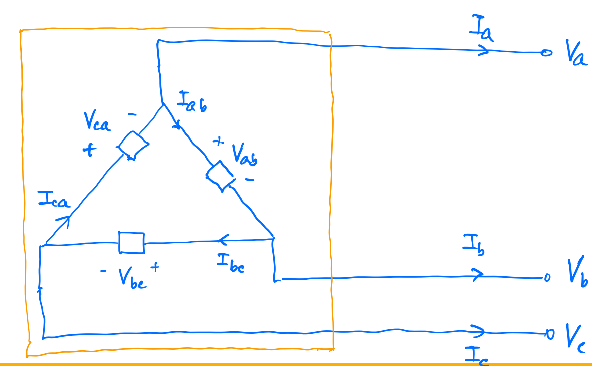
1. Overview
2. Mathematical properties
3. Three-phase device models
    - Conversion rules
    - Devices in Y configuration
    - Devices in Δ configuration
    - Y-Δ transformation (ideal devices)
4. Three-phase line models
5. Three-phase transformer models

How to derive external models

Recall

1. External model = Internal model + Conversion rule
    - External model: relation between (V, I, s)
    - Devices interact over network only through their terminal vars
2. Internal model : relation between $\left(V^{Y/\Delta}, I^{Y/\Delta}, s^{Y/\Delta}\right)$
    - Independent of Y or Δ configuration
    - Depends only on behavior of single-phase devices
    - Voltage/current/power source, impedance
3. Conversion rule : converts between internal and terminal vars
    - Depends only on Y or Δ configuration
    - Independent of type of single-phase devices





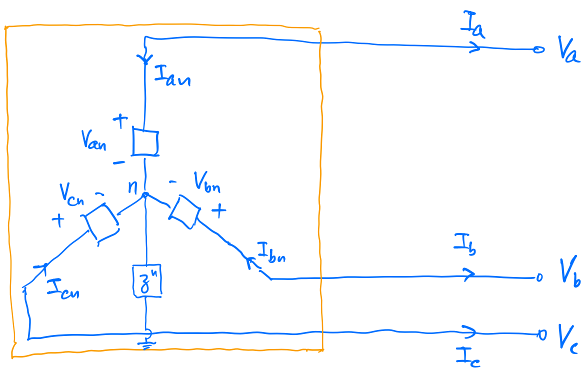
Conversion rule

Y configuration

1. Converts between internal and terminal variables

$$\boxed{V \;=\; V^Y + V^n \mathbf{1}, \qquad I \;=\; -I^Y,} \qquad s \;=\; -\left(s^Y + V^n \bar{I}^Y\right)$$

$$\mathbf{1}^\mathsf{T} I \;=\; -\mathbf{1}^\mathsf{T} I^Y \;=\; -I^n$$

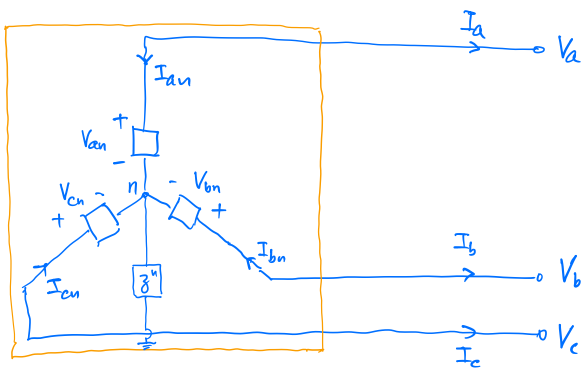


2. Negative signs in I, s due to directions of currents and powers
   - (I, s) : current & power injection from 3-phase device to rest of network
   - $\left(I^Y, s^Y\right)$: current & power delivered to the single-phase devices
3. If there is no neutral line, then $z^n := \infty, \quad I^n := 0$
   - $\mathbf{1}^\mathsf{T} I \;=\; -\mathbf{1}^\mathsf{T} I^Y \;=\; 0,$ V^n determined by network interaction

Conversion rule

Y configuration: assumption C8.1

1. Assumption C8.1
    - All voltages are defined wrt the ground
    - All neutrals are grounded through z^n (which may be zero)
2. If Assumption C8.1 holds
    - $V^n = -z^n \left(1^\top I\right)$
    - $V^n = 0$ if $z^n = 0$
3. If neutrals are ungrounded but connected to neutrals of other devices through 4-wire lines
    - $\left(V^n, I^n\right)$ determined by network interaction



C8.1 often assumed sometimes implicitly in literature

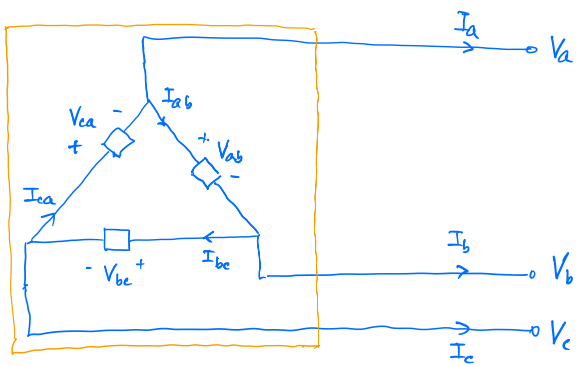
Conversion rule

Δ configuration: voltage conversion

1. Converts between internal and terminal voltages & currents

$$\boxed{V^\Delta \;=\; \Gamma V,} \qquad\qquad I \;=\; -\Gamma^{\mathsf{T}} I^\Delta$$

2. Given V^Δ, solution V exists iff $1^{\mathsf{T}} V^\Delta = 0$, i.e.
   - $V^{ab} + V^{bc} + V^{ca} \;=\; 0$ (Kirchhoff's Voltage Law)
3. Solution: terminal voltage $V \;=\; \dfrac{1}{3}\Gamma^{\mathsf{T}} V^\Delta \;+\; \gamma\, 1, \qquad \gamma \in \mathbb{C}$
4. $\gamma := \frac{1}{3} 1^T V$: (scaled) zero-sequence terminal voltage
   - A given reference voltage, e.g., $V_0 := \alpha_+$, fixes γ for every Δ-configured device



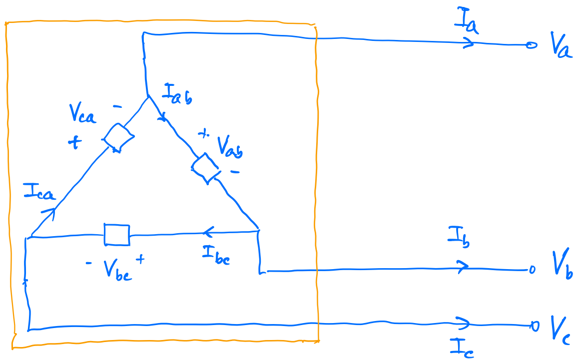
Conversion rule

Δ configuration: current conversion

1. Converts between internal and terminal voltages & currents

$$V^\Delta = \Gamma V, \qquad \boxed{I = -\Gamma^\mathsf{T} I^\Delta}$$

2. Given I, solution I^Δ exists iff $1^\mathsf{T} I = 0$, i.e.
   - $I^a + I^b + I^c = 0$ (Kirchhoff's Current Law)
3. Solution: internal current $I^\Delta = -\frac{1}{3}\Gamma I + \beta 1, \qquad \beta \in \mathbb{C}$
4. $\beta := \frac{1}{3} 1^T I^\Delta$: (scaled) zero-sequence internal current
   - Zero-sequence internal current does not affect terminal current I



Conversion rule

Δ configuration: power conversion

1. Relation between s and s^Δ is indirect, through $\left(V^\Delta, I^\Delta\right)$, through (V, I), or through $\left(V, I^\Delta\right)$
   - Follows from voltage and current conversions
2. Given $\left(V^\Delta, I^\Delta\right)$ with $1^\mathsf{T} V^\Delta = 0$, $s^\Delta :=$ diag $\left(V^\Delta I^{\Delta \mathsf{H}}\right)$ and terminal power is

$$s \quad := \quad \text{diag}\left(VI^\mathsf{H}\right) \quad = \quad -\,\text{diag}\left(\Gamma^\dagger\left(V^\Delta I^{\Delta\mathsf{H}}\right)\Gamma\right) \;+\; \gamma \bar{I}$$

3. Given (V, I) with $1^\mathsf{T} I = 0$, $s :=$ diag $\left(VI^\mathsf{H}\right)$ and internal power is

$$s^\Delta \quad := \quad \text{diag}\left(V^\Delta I^{\Delta\mathsf{H}}\right) \quad = \quad -\,\text{diag}\left(\Gamma\left(VI^\mathsf{H}\right)\Gamma^\dagger\right) \;+\; \bar{\beta} V^\Delta$$

4. Zero-sequence voltage γ and current β may be determined by spec or network interaction
5. Total powers $1^\mathsf{T} s$ and $1^\mathsf{T} s^\Delta$ are independent of (γ, β)
   - Because $1^\mathsf{T} I = 0$ and $1^\mathsf{T} V^\Delta = 0$

Conversion rule

Δ configuration: power conversion

6. Relation between s and s^Δ through $\left(V, I^\Delta\right)$:

$$s = -\,\text{diag}\left(VI^{\Delta \mathsf{H}}\Gamma\right), \quad s^\Delta \;=\; \text{diag}\left(\Gamma V I^{\Delta \mathsf{H}}\right)$$

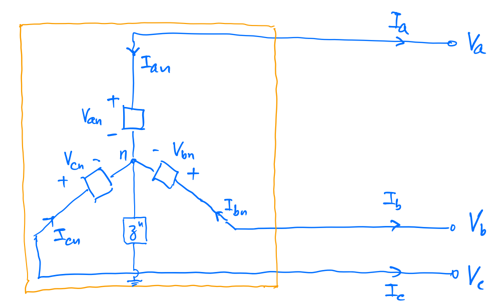
- no direct relation between s and s^Δ
- follows from voltage & current conversions

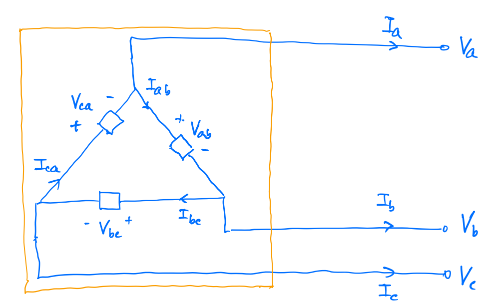
- The parameterization $\left(V, I^\Delta\right)$ implicitly contains $\gamma := \frac{1}{3}1^{\mathsf{T}}V$ and $\beta := \frac{1}{3}1^{\mathsf{T}}I^\Delta$ and is more convenient computationally

Three-phase devices

We next specify internal models and derive external models of 3-phase devices:

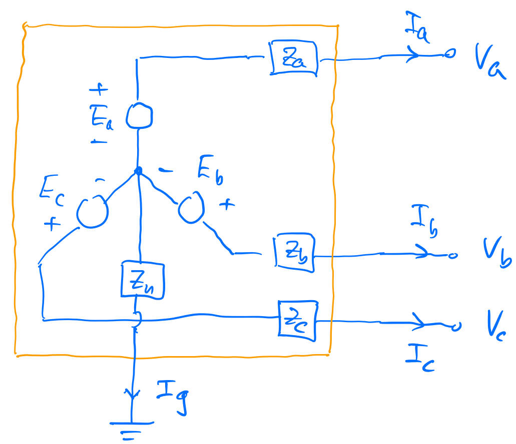
1. External model = Internal model + Conversion rule
    - Internal model: relation between $\left(V^{Y/\Delta}, I^{Y/\Delta}, s^{Y/\Delta}\right)$
    - External model: relation between (V, I, s)
2. … for devices
    - Voltage source
    - Current source
    - Power source
    - Impedance
3. … in Y and Δ configurations





Voltage source $\left(E^Y, z^Y, z^n\right)$: Y configuration

Internal model



1. Internal voltages and currents

$$V^Y = E^Y + z^Y I^Y, \qquad I^n = 1^\mathsf{T} I^Y, \qquad V^n = z^n \left(1^\mathsf{T} I^Y\right)$$

2. Internal powers:

- Across each single-phase device: $s^Y := \text{diag}\left(V^Y I^{Y\mathsf{H}}\right)$
- Across neutral conductor: $s^n := V^n \bar{I}^n$

$$s^Y = \text{diag}\left(E^Y I^{Y\mathsf{H}}\right) + \text{diag}\left(z^Y I^Y I^{Y\mathsf{H}}\right) = \underbrace{\begin{bmatrix} E^{an} I^{an\mathsf{H}} \\ E^{bn} I^{bn\mathsf{H}} \\ E^{cn} I^{cn\mathsf{H}} \end{bmatrix}}_{s^Y_{\text{ideal}}} + \underbrace{\begin{bmatrix} z^{an} \left|I^{an}\right|^2 \\ z^{bn} \left|I^{bn}\right|^2 \\ z^{cn} \left|I^{cn}\right|^2 \end{bmatrix}}_{s_{\text{imp}}}, \qquad s^n = z^n \left|1^\mathsf{T} I^Y\right|^2$$

Voltage source $\left(E^Y, z^Y, z^n\right)$: Y configuration

External model

1. Internal model

$$V^Y = E^Y + z^Y I^Y$$

2. Conversion rule for Y configuration

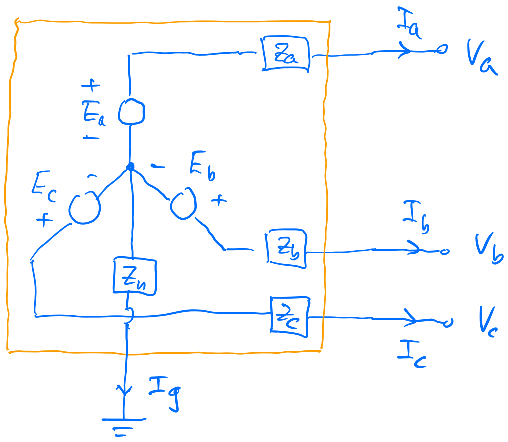
$$V = V^Y + V^n 1, \qquad I = -I^Y$$

3. $\Longrightarrow$ External model (under Assumption C8.1$\Rightarrow V^n = -z^n\left(1^{\mathsf{T}} I\right)$)

$$V = E^Y - \underbrace{\left(z^Y + z^n 1 1^{\mathsf{T}}\right)}_{Z^Y} I$$

neutral conductor z^n couples the phases

$$s = \operatorname{diag}\left(V\left(E^Y - V\right)^{\mathsf{H}}\left((Z^Y)^{-1}\right)^{\mathsf{H}}\right)$$



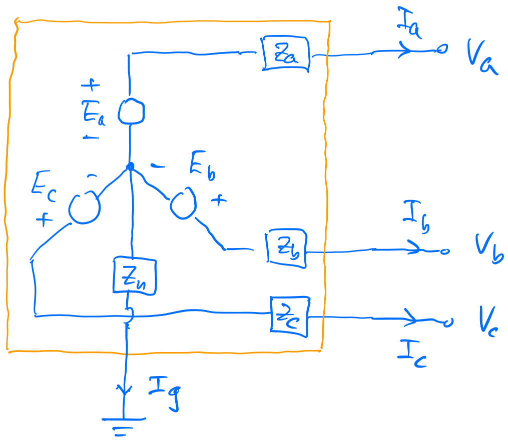
Voltage source $\left(E^Y, z^Y, z^n\right)$: Y configuration
External model

4. Comparison

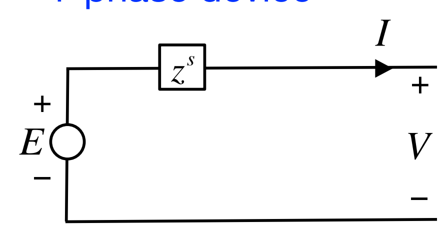
Single-phase : $V = E - zI \quad \in \quad \mathbb{C}$

Three-phase : $V \;=\; E^Y \;-\; Z^Y I \quad \in \quad \mathbb{C}^3$

$$Z^Y \;:=\; \begin{bmatrix} z^{an}+z^n & z^n & z^n \\ z^n & z^{bn}+z^n & z^n \\ z^n & z^n & z^{cn}+z^n \end{bmatrix}$$

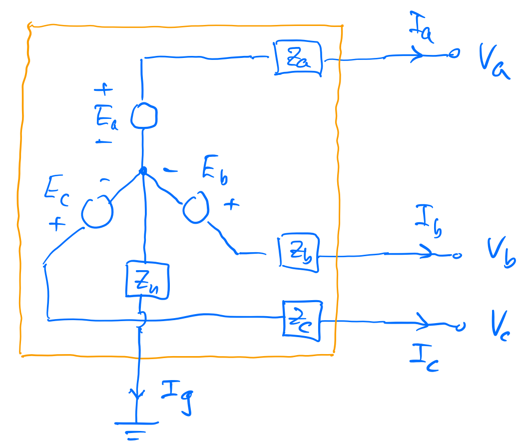


1-phase device



Voltage source $\left(E^Y, z^Y, z^n\right)$: Y configuration

Ideal source



1. Assumptions
   - $z^Y = 0$
   - Assumption C8.1 with $z^n = 0 : V^n = 0$
2. Internal model

$$V^Y = E^Y$$

3. Conversion rule for Y configuration

$$V = V^Y, \qquad I = -I^Y$$

4. $\Longrightarrow$ External model

$$V = E^Y$$
$$s = \text{diag}\left(E^Y I^{\mathsf{H}}\right)$$

Current source $\left(J^Y, y^Y, z^n\right)$: Y configuration

Internal model

1. Internal voltages and currents

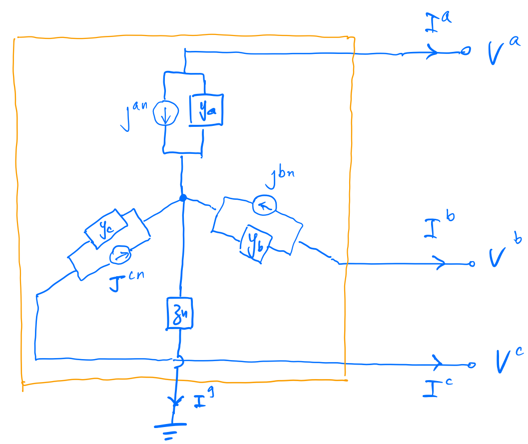
$$I^Y \;=\; J^Y \;+\; y^Y V^Y$$

2. Internal powers:

$$s^Y \;:=\; \text{diag}\left(V^Y I^{Y\mathsf{H}}\right) \;=\; \text{diag}\left(V^Y J^{Y\mathsf{H}}\right) + \text{diag}\left(V^Y V^{Y\mathsf{H}} y^{Y\mathsf{H}}\right)$$

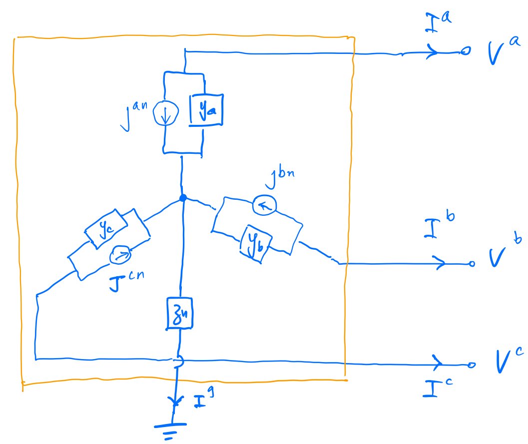
$$= \underbrace{\begin{bmatrix} V^{an} J^{an\mathsf{H}} \\ V^{bn} J^{bn\mathsf{H}} \\ V^{cn} J^{cn\mathsf{H}} \end{bmatrix}}_{s^Y_{\text{ideal}}} + \underbrace{\begin{bmatrix} y^{an\mathsf{H}} \left| V^{an} \right|^2 \\ y^{bn\mathsf{H}} \left| V^{bn} \right|^2 \\ y^{cn\mathsf{H}} \left| V^{cn} \right|^2 \end{bmatrix}}_{s_{\text{adm}}}$$

$$s^n \;:=\; V^n I^{n\mathsf{H}} \;=\; z^n \left| \mathsf{1}^\mathsf{T} J^Y \;+\; \text{diag}\left(y^Y\right)^\mathsf{T} V^Y \right|^2$$



Current source (J^Y, y^Y, z^n): Y configuration

External model



1. Internal model

$$I^Y \;=\; J^Y \;+\; y^Y V^Y$$

2. Conversion rule

$$V \;=\; V^Y \;+\; V^n \mathbf{1}, \qquad I \;=\; -I^Y$$

3. $\Longrightarrow$ External model (under Assumption C8.1$\Rightarrow V^n = -\, z^n \left(\mathbf{1}^\mathsf{T} I\right)$)

$$I \;=\; -A\left(J^Y + y^Y V\right) \qquad \text{where} \quad A \;:=\; \mathbb{I} - \frac{z^n}{1 + z^n\left(\mathbf{1}^\mathsf{T} y^Y \mathbf{1}\right)}\, y^Y \mathbf{1}\mathbf{1}^\mathsf{T}$$

$$s \;=\; -\,\mathrm{diag}\left(V\left(J^{Y\mathsf{H}} + V^\mathsf{H} y^{Y\mathsf{H}}\right) A^\mathsf{H}\right)$$

Current source (J^Y, y^Y, z^n): Y configuration

External model

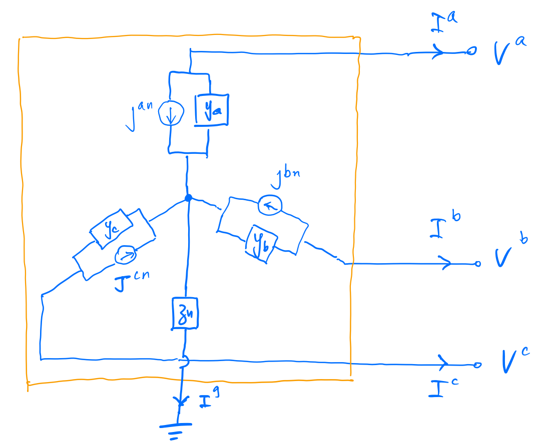
4. Comparison

Single-phase : $I \;=\; J - yV$

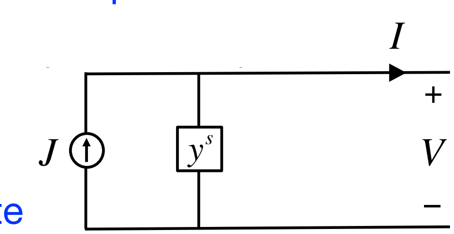
Three-phase : $I \;=\; A\left(-J^Y - y^Y V\right)$

$$A \;:=\; \mathbb{I} - \frac{z^n}{1 + z^n\left(1^{\mathsf{T}} y^Y 1\right)}\, y^Y 1 1^{\mathsf{T}}$$

$A = \mathbb{I}$ if $z^n = 0$

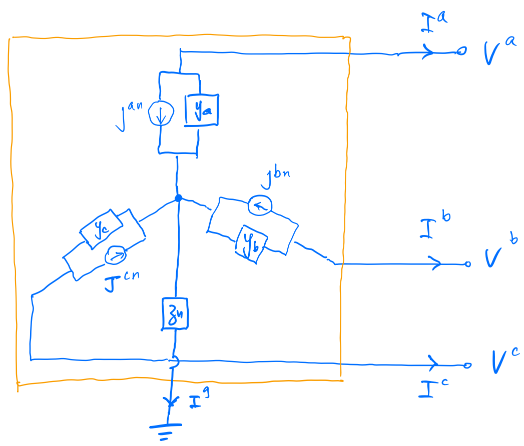


1-phase device



Note: directions of J are opposite

Current source $\left(J^Y, y^Y, z^n\right)$: Y configuration
Ideal source



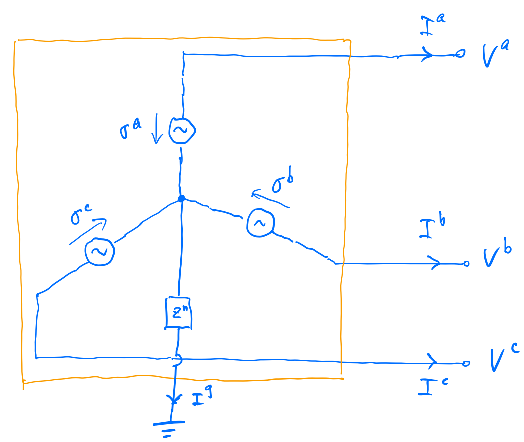
1. Assumptions
   - $y^Y = 0$
   - Assumption C8.1 with $z^n = 0 : V^n = 0$
2. $\Longrightarrow$ External model

$$I \;=\; -J^Y$$
$$s \;=\; -\,\text{diag}\left(VJ^{Y\mathsf{H}}\right)$$

Power source (σ^Y, z^n): Y configuration
Internal model

1. Internal powers

$$s^Y \;=\; \sigma^Y, \qquad s^n \;:=\; V^n I^{n\mathsf{H}} \;=\; z^n \left| \mathbf{1}^\mathsf{T} I^Y \right|^2$$



Power source (σ^Y, z^n): Y configuration
External model

1. Internal model

$$s^Y \; = \; \sigma^Y$$

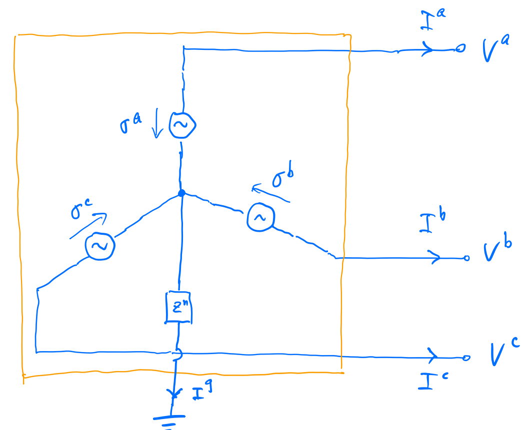
2. Conversion rule

$$V \; = \; V^Y \, + \, V^n 1, \qquad I \; = \; -I^Y$$

3. $\Longrightarrow$ External model (under Assumption C8.1$\Rightarrow V^n = -\, z^n \left(1^\mathsf{T} I\right)$)

IV relation: $V \; = -\,\mathrm{diag}\left(I^\mathsf{H}\right)^{-1} \sigma^Y \, - \, z^n \left(1 1^\mathsf{T}\right) I$

Is relation: $s \; = -\left(\sigma^Y \, + \, z^n \left(\bar{I} I^\mathsf{T}\right) 1\right)$



Power source $\left(\sigma^Y, z^n\right)$: Y configuration
External model

4. Comparison

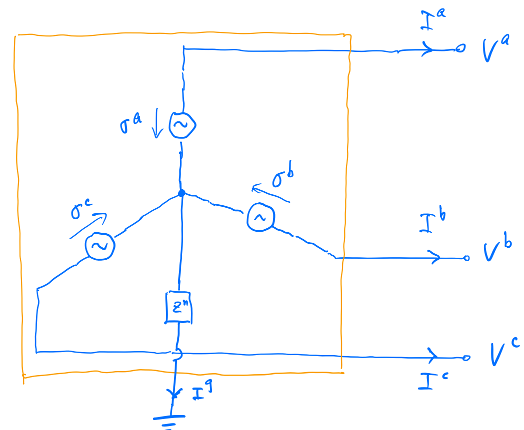
Single-phase : $s \;=\; \sigma$

Three-phase : $s \;=\; -\left(\sigma^Y + z^n\left(\bar{I}I^{\mathsf{T}}\right)1\right)$

power delivered to z^n

Total power (3-phase) :

$$-1^{\mathsf{T}}\sigma^Y \;=\; 1^{\mathsf{T}}s + \underbrace{z^n\left(1^{\mathsf{T}}I^Y\right)}_{-V^n}\underbrace{\left(1^{\mathsf{T}}\bar{I}^Y\right)}_{-I^{nH}} \;=\; 1^{\mathsf{T}}s + s^n$$



1-phase device

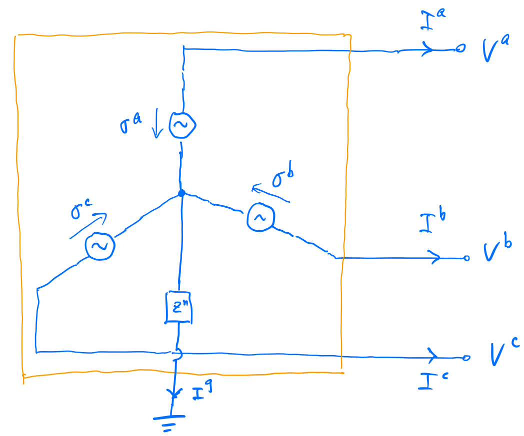
Note: directions of σ are opposite

Power source (σ^Y, z^n): Y configuration

Ideal source

1. Assumption
   - Assumption C8.1 with $z^n = 0 : V^n = 0$
2. $\Longrightarrow$ External model

$$s \;=\; -\,\sigma^Y$$



Impedance $\left(z^Y, z^n\right)$: Y configuration

Internal model

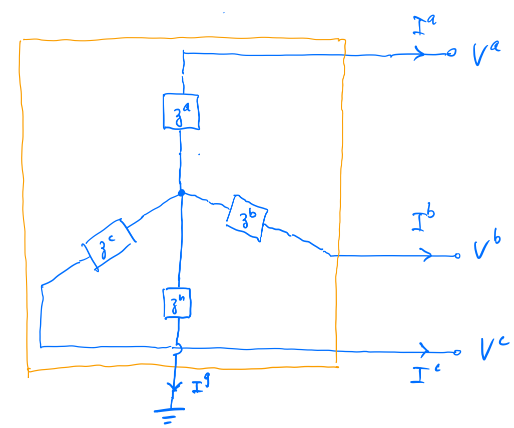
1. Internal voltage and current:

$$V^Y \;=\; z^Y I^Y$$

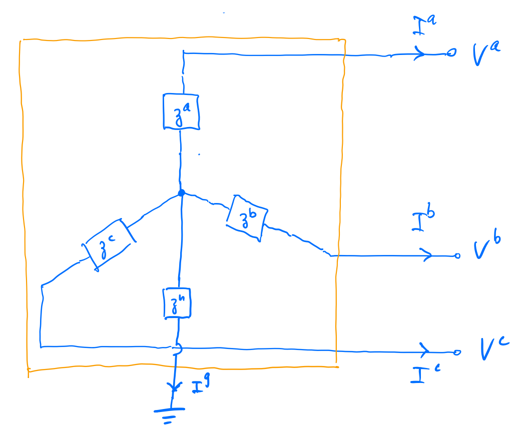
2. Internal power:

$$s^Y \;:=\; \text{diag}\left(V^Y I^{Y\mathsf{H}}\right) \;=\; \text{diag}\left(V^Y V^{Y\mathsf{H}}\left(y^Y\right)^{\mathsf{H}}\right)$$

$$s^n \;:=\; V^n I^{n\mathsf{H}} \;=\; z^n \left|1^{\mathsf{T}} I^Y\right|^2$$



Impedance (z^Y, z^n): Y configuration
External model



1. Internal model

$$V^Y \;=\; z^Y I^Y$$

2. Conversion rule for Y configuration

$$V \;=\; V^Y + V^n \mathbf{1}, \qquad I \;=\; -I^Y$$

3. $\Longrightarrow$ External model (under Assumption C8.1$\Rightarrow V^n = -z^n\left(\mathbf{1}^{\mathsf{T}} I\right)$)

$$V \;=\; -Z^Y I \;=\; \left(z^Y + z^n \mathbf{1}\mathbf{1}^{\mathsf{T}}\right) I$$

$$s \;=\; -\,\mathrm{diag}\left(V V^{\mathsf{H}} \left((Z^Y)^{-1}\right)^{\mathsf{H}}\right)$$

neutral conductor z^n couples the phases

Impedance (z^Y, z^n): Y configuration

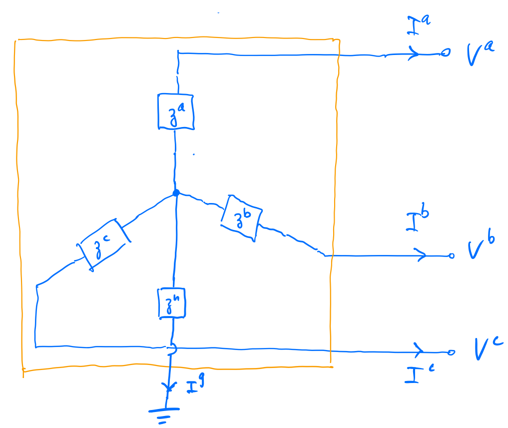
External model

4. Comparison

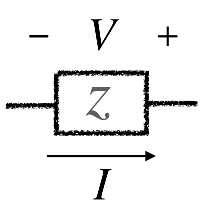
Single-phase : $V = -\,zI \quad \in \quad \mathbb{C}$

Three-phase : $V \;=\; -\,Z^Y I \quad \in \quad \mathbb{C}^3$

$$Z^Y \;:=\; \begin{bmatrix} z^{an}+z^n & z^n & z^n \\ z^n & z^{bn}+z^n & z^n \\ z^n & z^n & z^{cn}+z^n \end{bmatrix}$$



1-phase device

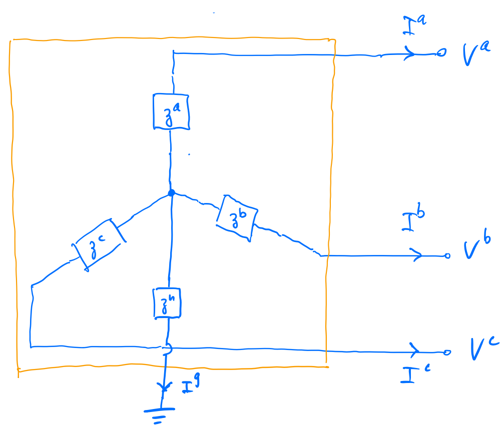


Impedance (z^Y, z^n): Y configuration
Ideal impedance

1. Assumption
   - Assumption C8.1 with $z^n = 0 : V^n = 0$
2. $\Longrightarrow$ External model

$$Z^Y \;=\; z^Y, \qquad V \;=\; z^Y I$$

phases are decoupled



Balanced impedance
When $z^n \neq 0$ but z^Y is balanced, i.e., $z^{an} = z^{bn} = z^{cn}$, then similarity transformation using F produces a sequence impedance that is decoupled in the sequence coordinate

$$\tilde{Z}^Y \;=\; \begin{bmatrix} z^{an} + 3z^n & 0 & 0 \\ 0 & z^{an} & 0 \\ 0 & 0 & z^{an} \end{bmatrix}$$

Recap: external models

Y-configured devices (ideal)

Device	Y configuration	
Voltage source	$V = E^Y + \gamma \mathbf{1}$	$s = \text{diag}\left(E^Y I^{\mathsf{H}}\right) + \gamma \bar{I}$
Current source	$I = -J^Y$	$s = -\text{diag}\left(V J^{Y\mathsf{H}}\right)$
Power source	$\text{diag}\left(I^{\mathsf{H}}\right)(V - \gamma \mathbf{1}) = -\sigma$	$s = -\sigma^Y + \gamma \bar{I}$
Impedance	$V = -z^Y I + \gamma \mathbf{1}$	$s = -\text{diag}\left(V (V - \gamma \mathbf{1})^{\mathsf{H}} y^{Y\mathsf{H}}\right)$

1. $\gamma := V^n$ is neutral voltage
2. Negative signs are only due to directions of I and s (out of device)
3. total terminal power $1^{\mathsf{T}} s$ = total internal power $1^{\mathsf{T}} s^Y$ + power delivered across neutral

Outline

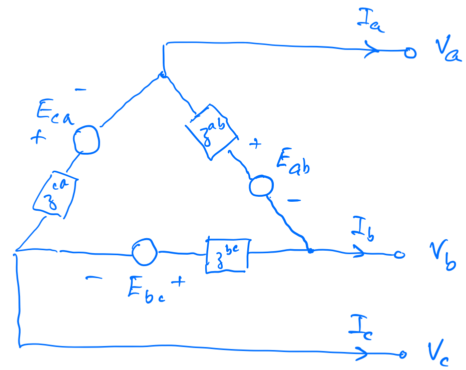
1. Overview
2. Mathematical properties
3. Three-phase device models
    - Conversion rules
    - Devices in Y configuration
    - Devices in Δ configuration
    - Y-Δ transformation
4. Three-phase line models
5. Three-phase transformer models

Voltage source $\left(E^{\Delta}, z^{\Delta}\right)$: Δ configuration
Internal model

1. Internal voltages and currents

$$V^{\Delta} \;=\; E^{\Delta} \;+\; z^{\Delta} I^{\Delta}$$

independent of Y/Δ config



2. Internal powers:

$$s^{\Delta} \;:=\; \text{diag}\left(V^{\Delta} I^{\Delta H}\right) \;=\; \text{diag}\left(E^{\Delta} I^{\Delta \mathrm{H}}\right) + \text{diag}\left(z^{\Delta}\, I^{\Delta} I^{\Delta \mathrm{H}}\right)$$

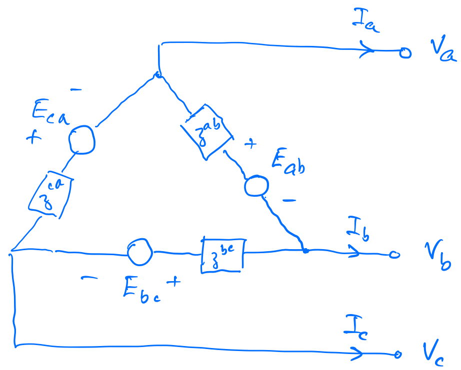
Voltage source (E^{Δ}, z^{Δ}): Δ configuration
External model

1. Internal model

$$V^{\Delta} \;=\; E^{\Delta} \,+\, z^{\Delta} I^{\Delta}$$

2. Conversion rule for Δ configuration

$$V^{\Delta} \;=\; \Gamma V, \qquad\qquad I \;=\; -\,\Gamma^{\mathsf{T}} I^{\Delta}$$



Voltage source $\left(E^\Delta, z^\Delta\right)$: Δ configuration

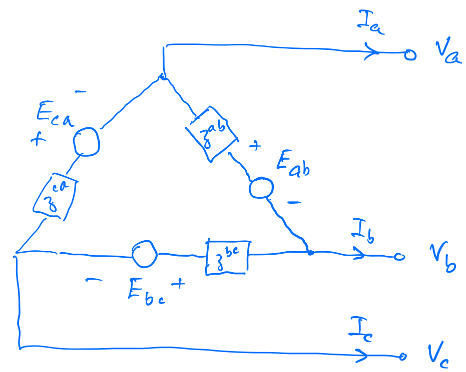
External model

1. Internal model

$$V^\Delta \;=\; E^\Delta \;+\; z^\Delta I^\Delta$$

2. Conversion rule for Δ configuration

$$V^\Delta \;=\; \Gamma V, \qquad\qquad I \;=\; -\,\Gamma^{\mathsf{T}} I^\Delta$$



3. Two (asymmetric) relations between terminal vars (V, I)
   - Given V, 1st relation uniquely determines I (hence $\left(V^\Delta, I^\Delta\right)$ as well)
   - Given I, 2nd relation determines V up to zero-sequence voltage γ

Asymmetry is because V contains more info (γ) than I does (which contains no info about zero-sequence current $\beta := \frac{1}{3}1^{\mathsf{T}} I^\Delta$)

Voltage source $\left(E^{\Delta}, z^{\Delta}\right)$: Δ configuration
External model

4. Given V,

$$\boxed{I \;=\; \left(\Gamma^{\mathsf{T}} y^{\Delta}\right) E^{\Delta} \;-\; Y^{\Delta}\, V}$$

$$Y^{\Delta} \;:=\; \Gamma^{\mathsf{T}} y^{\Delta}\, \Gamma \;=\; \begin{bmatrix} y^{ab} + y^{ca} & -y^{ab} & -y^{ca} \\ -y^{ab} & y^{ab} + y^{bc} & -y^{bc} \\ -y^{ca} & -y^{bc} & y^{ca} + y^{bc} \end{bmatrix}, \quad y^{\Delta} := \left(z^{\Delta}\right)^{-1}$$

Voltage source (E^{Δ}, z^{Δ}): Δ configuration

External model

4. Given V,

$$\boxed{I \;=\; \left(\Gamma^{\mathsf{T}} y^{\Delta}\right) E^{\Delta} \;-\; Y^{\Delta} V}$$

$$Y^{\Delta} \;:=\; \Gamma^{\mathsf{T}} y^{\Delta} \Gamma \;=\; \begin{bmatrix} y^{ab} + y^{ca} & -y^{ab} & -y^{ca} \\ -y^{ab} & y^{ab} + y^{bc} & -y^{bc} \\ -y^{ca} & -y^{bc} & y^{ca} + y^{bc} \end{bmatrix}, \quad y^{\Delta} := \left(z^{\Delta}\right)^{-1}$$

5. Given I with $1^{\mathsf{T}} I = 0$,

$$\boxed{V \;=\; \hat{\Gamma} E^{\Delta} \;-\; Z^{\Delta} I \;+\; \gamma 1,} \qquad 1^{\mathsf{T}} I \;=\; 0$$

$$\hat{\Gamma} \;:=\; \frac{1}{3} \Gamma^{\mathsf{T}} \left(\mathbb{I} - \frac{1}{\zeta}\, \tilde{z}^{\Delta}\, 1^{\mathsf{T}} \right), \qquad Z^{\Delta} \;:=\; \frac{1}{9} \Gamma^{\mathsf{T}} z^{\Delta} \left(\mathbb{I} - \frac{1}{\zeta}\, 1\, \tilde{z}^{\Delta\mathsf{T}} \right) \Gamma$$

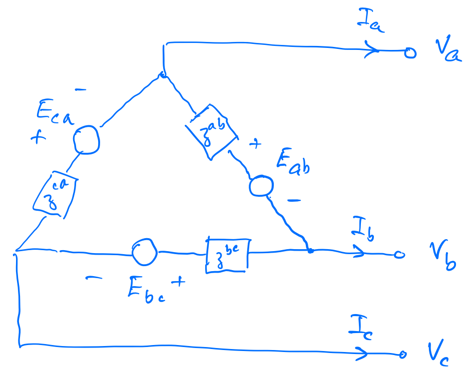
Voltage source $\left(E^{\Delta}, z^{\Delta}\right)$: Δ configuration

External model

6. Terminal power in terms of V or I:

$$s \;=\; \text{diag}\left(VI^{\mathsf{H}}\right) \;=\; \text{diag}\left(V\left(\Gamma^{\mathsf{T}} y^{\Delta} E^{\Delta} \;-\; Y^{\Delta}\, V\right)^{\mathsf{H}}\right)$$

$$s \;=\; \text{diag}\left(VI^{\mathsf{H}}\right) \;=\; \text{diag}\left(\left(\hat{\Gamma} E^{\Delta} \;-\; Z^{\Delta} I\right) I^{\mathsf{H}}\right) \;+\; \gamma \bar{I}$$



Power due to zero-sequence voltage γ

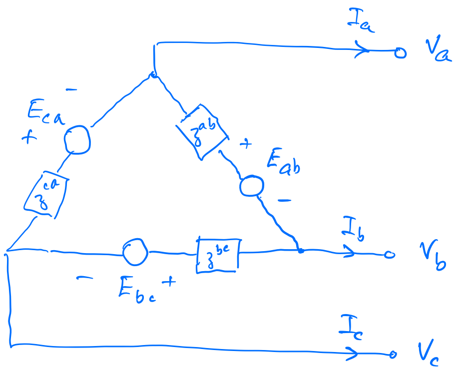
Total power $1^{\mathsf{T}} s$ is independent of γ because $\gamma 1^{\mathsf{T}} \bar{I} = 0$

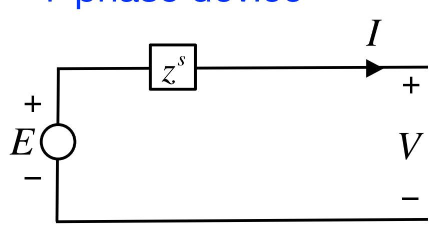
Voltage source $\left(E^{\Delta}, z^{\Delta}\right)$: Δ configuration
External model

7. Comparison

$$\text{Single-phase : } V = E - zI$$

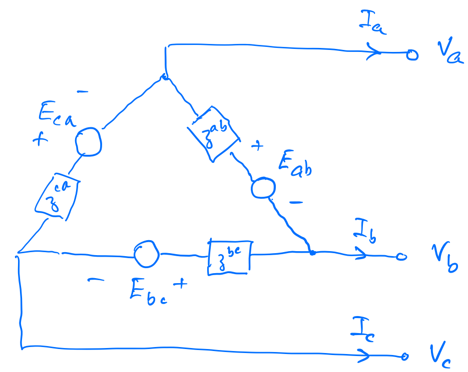
$$\text{Three-phase : } V = \hat{\Gamma} E^{\Delta} - Z^{\Delta} I + \gamma 1, \qquad 1^{\mathsf{T}} I = 0$$





Voltage source $\left(E^{\Delta}, z^{\Delta}\right)$: Δ configuration
Ideal source

1. Assumption
   - $z^{\Delta} = 0$
2. $\Longrightarrow \quad \hat{\Gamma} = \frac{1}{3}\Gamma^{\mathsf{T}}, \qquad Z^{\Delta} = 0$
3. $\Longrightarrow$ External model



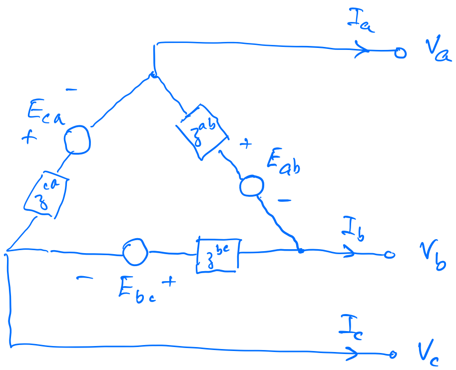
Zero-sequence voltage γ

$$V = \frac{1}{3}\Gamma^{\mathsf{T}}E^{\Delta} + \gamma \mathbf{1}, \qquad \mathbf{1}^{\mathsf{T}}I = 0$$

$$s = \frac{1}{3}\text{diag}\left(\Gamma^{\mathsf{T}}E^{\Delta}I^{\mathsf{H}}\right) + \gamma \bar{I}$$

Voltage source (E^{Δ}, z^{Δ}): Δ configuration

Voltage source specifies E^{Δ} which does not uniquely determine terminal voltage V



- Because the zero-sequence voltage $\gamma := \frac{1}{3} 1^{\mathsf{T}} V$ is arbitrary
- γ needs to be specified, e.g., fixed by a reference voltage or grounding
- … for both ideal or non-ideal voltage sources

Current source (J^Δ, y^Δ): Δ configuration

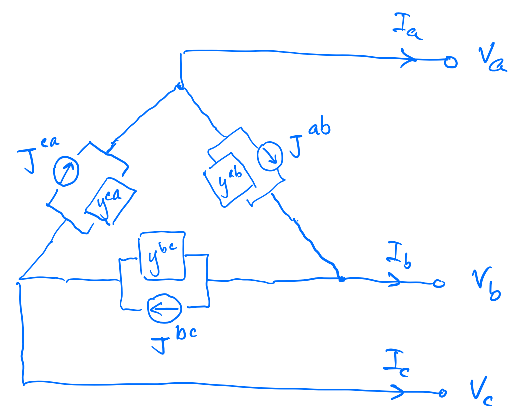
Internal model

1. Internal voltages and currents

$$I^\Delta \;=\; J^\Delta \,+\, y^\Delta V^\Delta$$

2. Internal powers:

$$\begin{aligned} s^\Delta \;&:=\; \text{diag}\left(V^\Delta I^{\Delta H}\right) \\ &=\; \text{diag}\left(V^\Delta J^{\Delta \text{H}}\right) \,+\, \text{diag}\left(V^\Delta V^{\Delta \text{H}} y^{\Delta \text{H}}\right) \end{aligned}$$



Current source (J^Δ, y^Δ): Δ configuration
External model

1. Internal model

$$I^\Delta \ = \ J^\Delta \ + \ y^\Delta V^\Delta$$

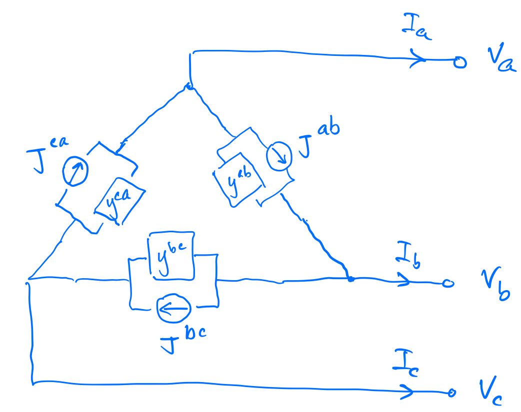
2. Conversion rule

$$V^\Delta \ = \ \Gamma V, \qquad\qquad I \ = \ -\Gamma^\mathsf{T} I^\Delta$$

3. $\implies$ External model

$$I \ = \ -\left(\Gamma^\mathsf{T} J^\Delta \ + \ Y^\Delta V\right)$$

where (as before): $Y^\Delta \ := \ \Gamma^\mathsf{T} y^\Delta \Gamma \ = \ \begin{bmatrix} y^{ab}+y^{ca} & -y^{ab} & -y^{ca} \\ -y^{ab} & y^{ab}+y^{bc} & -y^{bc} \\ -y^{ca} & -y^{bc} & y^{ca}+y^{bc} \end{bmatrix}$



Current source (J^Δ, y^Δ): Δ configuration
External model

1. Internal model

$$I^\Delta \;=\; J^\Delta \;+\; y^\Delta\, V^\Delta$$

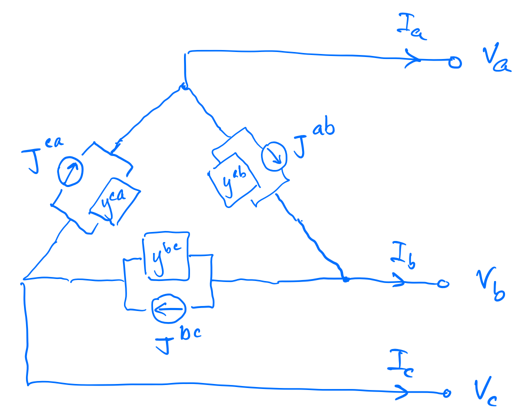
2. Conversion rule

$$V^\Delta \;=\; \Gamma V, \qquad\qquad I \;=\; -\,\Gamma^{\mathsf{T}} I^\Delta$$

3. $\Longrightarrow$ External model

$$I \;=\; -\left(\Gamma^{\mathsf{T}} J^\Delta \;+\; Y^\Delta\, V\right)$$

$$s \;=\; \mathrm{diag}\left(V I^{\mathsf{H}}\right) \;=\; -\,\mathrm{diag}\left(V J^{\Delta\mathsf{H}}\,\Gamma \;+\; V V^{\mathsf{H}}\, Y^{\Delta\mathsf{H}}\right)$$



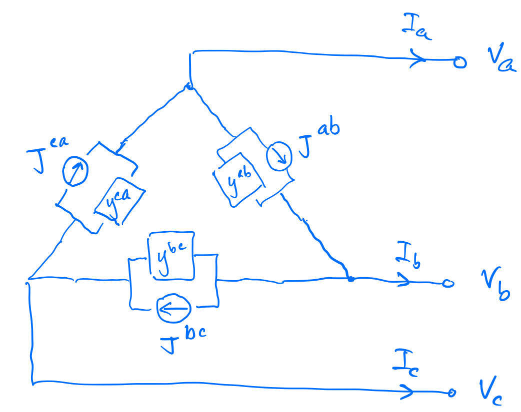
Current source $\left(J^{\Delta}, y^{\Delta}\right)$: Δ configuration
External model

4. Comparison

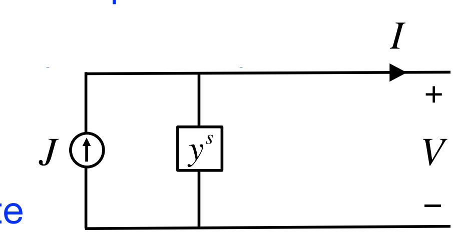
$$\text{Single-phase}: \quad I \;=\; J - yV$$

$$\text{Three-phase}: \quad I \;=\; -\Gamma^{\mathsf{T}} J^{\Delta} \;-\; Y^{\Delta}\, V$$

$$Y^{\Delta} \;:=\; \Gamma^{\mathsf{T}} y^{\Delta}\, \Gamma$$



1-phase device



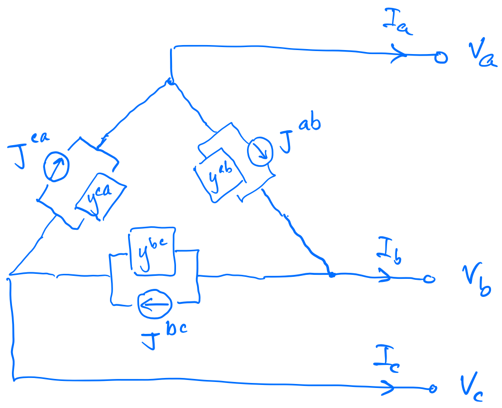
Note: directions of J are opposite

Current source (J^Δ, y^Δ): Δ configuration

Ideal source

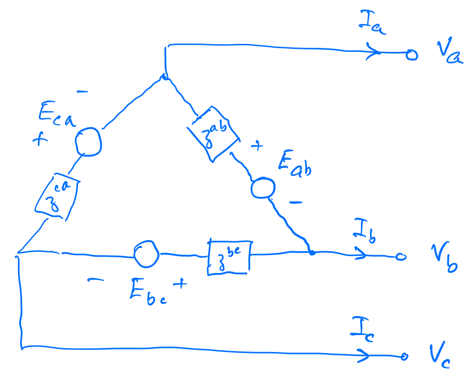
1. Assumption
   - $y^\Delta = 0$
2. $\implies$ External model

$$I = -\Gamma^{\mathsf{T}} J^\Delta$$
$$s = -\,\mathrm{diag}(V J^{\Delta \mathsf{H}} \Gamma)$$

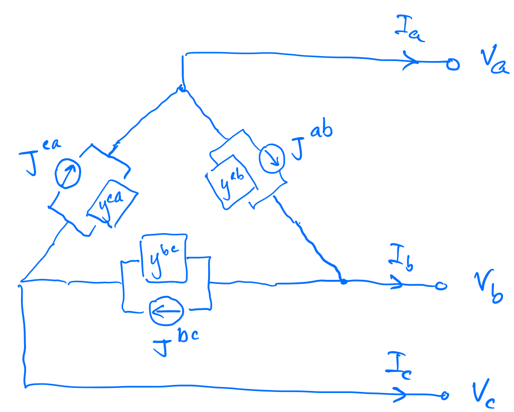


Voltage & current sources: comparison

1. Voltage source specifies E^Δ which does not uniquely determine terminal voltage V
   - $V \;=\; \hat{\Gamma} E^\Delta \;-\; Z^\Delta I \;+\; \gamma 1, \qquad 1^\mathsf{T} I \;=\; 0$
   - due to arbitrary zero-sequence voltage $\gamma := \frac{1}{3} 1^\mathsf{T} V$



2. Current source specifies J^Δ which uniquely determines terminal current I
   - $I \;=\; -\left(\Gamma^\mathsf{T} J^\Delta \;+\; Y^\Delta\, V\right)$
   - J^Δ contains its zero-sequence current $\beta := \frac{1}{3} 1^\mathsf{T} J^\Delta$

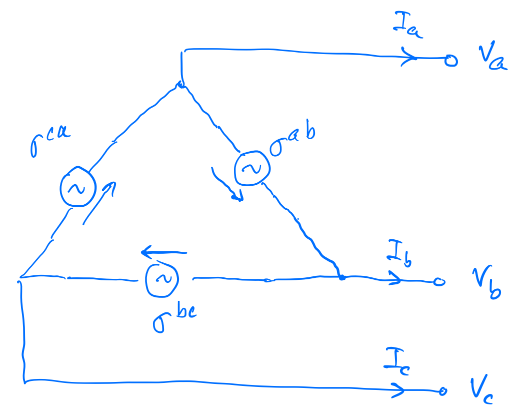


Power source σ^{Δ}: Δ configuration

Internal model

1. Internal powers

$$s^{\Delta} \quad := \quad \text{diag}\left(V^{\Delta} I^{\Delta \mathsf{H}}\right) \;=\; \sigma^{\Delta}$$



Power source σ^Δ: Δ configuration
External model

1. Internal model

$$s^\Delta \; = \; \sigma^\Delta$$

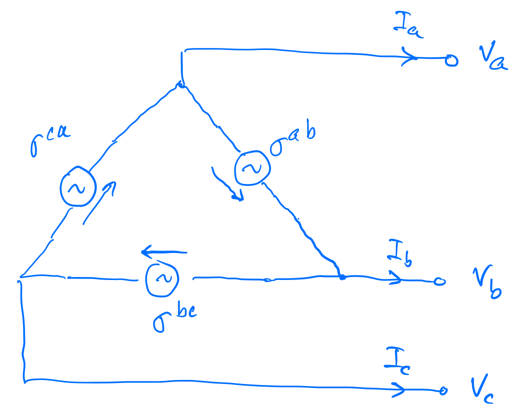
2. Conversion rule

$$V^\Delta \; = \; \Gamma V, \qquad\qquad I \; = \; -\Gamma^\mathsf{T} I^\Delta$$

3. $\Longrightarrow$ External model

$$IV \text{ relation: } \sigma^\Delta \; = \; -\frac{1}{3}\,\text{diag}\left(\Gamma\left(VI^\mathsf{H}\right)\Gamma^\mathsf{T}\right) \; + \; \bar{\beta}\Gamma V, \qquad 1^\mathsf{T} I \; = \; 0$$

7 complex vars (V, I, β), 4 quadratic equations



Power source σ^{Δ}: Δ configuration
External model

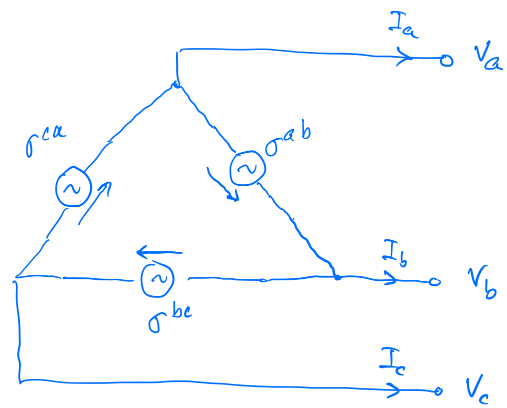
1. Internal model

$$s^{\Delta} \;=\; \sigma^{\Delta}$$

2. Conversion rule

$$V^{\Delta} \;=\; \Gamma V, \qquad\qquad I \;=\; -\Gamma^{\mathsf{T}} I^{\Delta}$$

3. $\Longrightarrow$ External model



IV relation: $\sigma^{\Delta} \;=\; -\frac{1}{3}\,\mathrm{diag}\left(\Gamma\left(VI^{\mathsf{H}}\right)\Gamma^{\mathsf{T}}\right) \;+\; \bar{\beta}\Gamma V, \qquad 1^{\mathsf{T}} I \;=\; 0$

Equivalent model: $\sigma^{\Delta} \;=\; \mathrm{diag}\left(\Gamma V I^{\Delta \mathsf{H}}\right)$

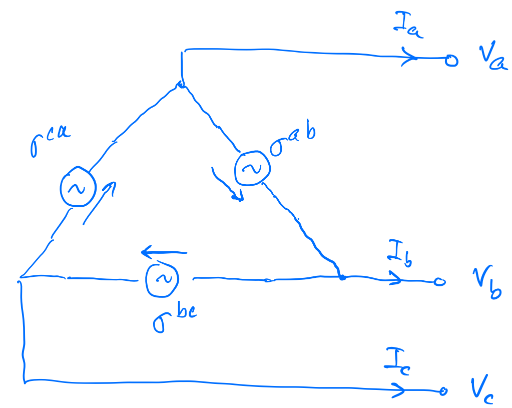
Power source σ^Δ: Δ configuration
External model

4. Comparison

$$\text{Single-phase}: \quad s \;=\; \sigma$$

$$\text{Three-phase}: \quad s \;=\; -\,\text{diag}\left(VI^{\Delta\mathsf{H}}\Gamma\right)$$

$$\sigma^\Delta \;=\; \text{diag}\left(\Gamma V I^{\Delta\mathsf{H}}\right) \;=\; \begin{bmatrix} \left(V_a - V_b\right)\bar{I}^{ab} \\ \left(V_b - V_c\right)\bar{I}^{bc} \\ \left(V_c - V_a\right)\bar{I}^{ca} \end{bmatrix}$$



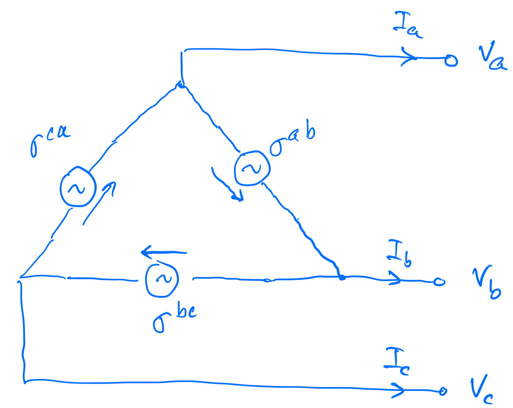
1-phase device

Given V (and σ^Δ), I^Δ and hence s are uniquely determined
Given I^Δ (and σ^Δ), only ΓV is uniquely determined, not V nor s

Power source σ^Δ: Δ configuration
Ideal source

1. Assumption
   - Assumption C8.1 with $z^n = 0 : V^n = 0$
2. $\Longrightarrow$ External model

$$s \;=\; -\,\sigma^Y$$



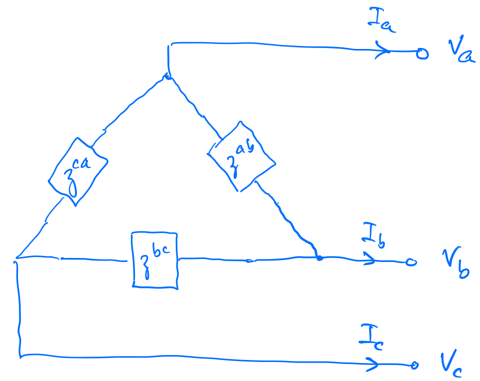
Impedance z^Δ: Δ configuration
Internal model

1. Internal voltage and current:

$$V^\Delta \;=\; z^\Delta I^\Delta$$

2. Internal power:

$$s^\Delta \;=\; \text{diag}\left(V^\Delta I^{\Delta \mathsf{H}}\right) \;:=\; \text{diag}\left(z^\Delta I^\Delta I^{\Delta \mathsf{H}}\right)$$



Impedance z^Δ: Δ configuration

External model

1. Internal model

$$V^\Delta \;=\; z^\Delta I^\Delta$$

2. Conversion rule

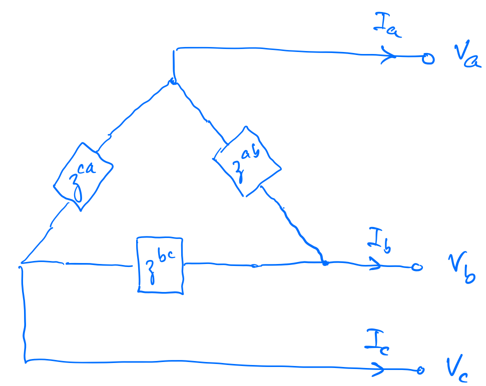
$$V^\Delta \;=\; \Gamma V, \qquad\qquad I \;=\; -\Gamma^{\mathsf{T}} I^\Delta$$

3. $\Longrightarrow$ External model

$$\text{Given } V,\; I \;=\; -Y^\Delta V \;:=\; -\left(\Gamma^{\mathsf{T}} y^\Delta \Gamma\right) V$$

$$\text{Given } I,\; V \;=\; -Z^\Delta I \;+\; \gamma 1, \qquad 1^{\mathsf{T}} I \;=\; 0$$

$$Z^\Delta \;:=\; \frac{1}{9}\Gamma^{\mathsf{T}} z^\Delta \left(\mathbb{I} - \frac{1}{\zeta} 1\, \tilde{z}^{\Delta \mathsf{T}}\right)\Gamma$$



As for voltage source, the asymmetry is because V contains more info (γ) than I does

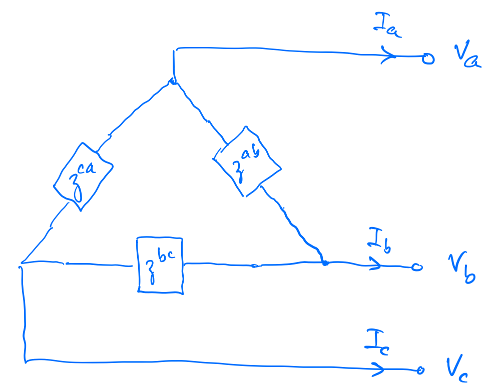
Impedance z^Δ: Δ configuration

External model

4. Terminal power s can be related to V or to I:

$$\text{Given } V, \quad s \;=\; \text{diag}\left(VI^{\mathsf{H}}\right) \;=\; -\,\text{diag}\left(VV^{\mathsf{H}}Y^{\Delta\mathsf{H}}\right)$$

$$\text{Given } I, \quad s \;=\; \text{diag}\left(VI^{\mathsf{H}}\right) \;=\; -\,\text{diag}\left(Z^{\Delta}II^{\mathsf{H}}\right) \;+\; \gamma\bar{I}$$



As for voltage source, the asymmetry is because V contains more info (γ) than I does

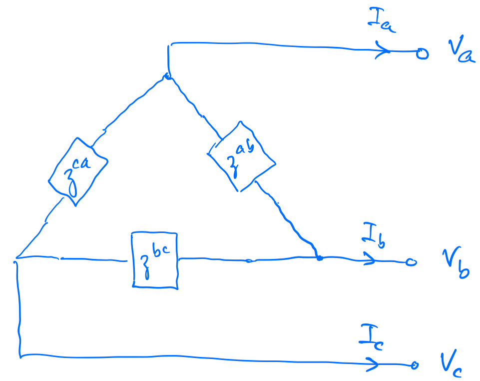
Impedance z^{Δ}: Δ configuration
External model

5. Comparison

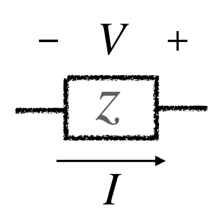
Single-phase : $I = -yZ$ or $V = -zI \quad \in \quad \mathbb{C}$

Three-phase :

$$I \;=\; -Y^{\Delta}V \quad \in \quad \mathbb{C}^3$$
$$V \;=\; -Z^{\Delta}I \;+\; \gamma 1, \qquad\qquad 1^{\mathsf{T}} I \;=\; 0$$



1-phase device



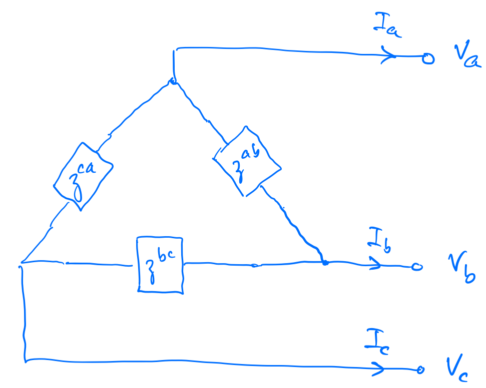
Impedance z^Δ: Δ configuration
Balanced impedance

1. Assumption
   - $z^{ab} = z^{bc} = z^{ca}$
2. External model

$$V \;=\; -Z^\Delta I \;+\; \gamma 1, \qquad 1^{\mathsf{T}} I \;=\; 0$$

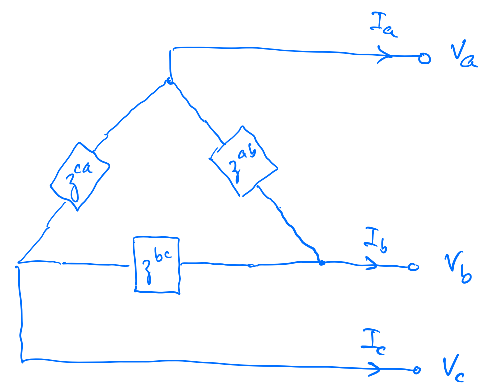
$$Z^\Delta \;=\; \frac{z^{ab}}{3}\left(\mathbb{I} - \frac{1}{3}11^{\mathsf{T}}\right)$$

phases are coupled (Z^Δ is not diagonal)



Impedance z^Δ: Δ configuration
Balanced impedance

1. Assumption
   - $z^{ab} = z^{bc} = z^{ca}$



2. External model

$$V \;=\; -Z^\Delta I \;+\; \gamma 1, \qquad 1^\mathsf{T} I \;=\; 0$$

$$Z^\Delta \;=\; \frac{z^{ab}}{3}\left(\mathbb{I} - \frac{1}{3} 1 1^\mathsf{T}\right)$$

phases are coupled (Z^Δ is not diagonal)

3. Sequence impedance $\tilde{Z}^\Delta$ is decoupled in sequence coordinate

$$\tilde{Z}^\Delta \;=\; \frac{z^{ab}}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

zero-sequence component (first row & col) is zero because $I^a + I^b + I^c = 0$

Recap: external models

Δ-configured devices (ideal)

Device	Δ configuration	
Voltage source	$V = \frac{1}{3}\Gamma^{\mathsf{T}} E^{\Delta} + \gamma \mathbf{1},\ \mathbf{1}^{\mathsf{T}} I = 0$	$s = \frac{1}{3}\mathrm{diag}\left(\Gamma^{\mathsf{T}} E^{\Delta} I^{\mathsf{H}}\right) + \gamma \bar{I}$
Current source	$I = -\Gamma^{\mathsf{T}} J^{\Delta}$	$s = -\mathrm{diag}\left(V J^{\Delta\mathsf{H}} \Gamma\right)$
Power source	$\sigma^{\Delta} = \mathrm{diag}\left(\Gamma V I^{\Delta\mathsf{H}}\right)$	
Impedance	$I = -Y^{\Delta} V$	$s = -\mathrm{diag}\left(V V^{\mathsf{H}} Y^{\Delta\mathsf{H}}\right)$

1. $\gamma := \frac{1}{3} 1^{\mathsf{T}} V$ is zero-seq terminal voltage
2. total terminal power $1^{\mathsf{T}} s$ is independent of γ because $1^{\mathsf{T}} \bar{I} = 0$

Outline

1. Overview
2. Mathematical properties
3. Three-phase device models
    - Conversion rules
    - Devices in Y configuration
    - Devices in Δ configuration
    - Y-Δ transformation
4. Three-phase line models
5. Three-phase transformer models

Δ-Y transformation

Ideal voltage source (E^Δ, γ)

1. External model

$$V \ = \ \frac{1}{3}\Gamma^\mathsf{T} E^\Delta \ + \ \gamma 1, \qquad 1^\mathsf{T} I \ = \ 0$$

2. Y equivalent

   - Ideal voltage source $V = \ E^Y + V^n 1, \ \ 1^\mathsf{T} I \ = -\, I^n$ with

$$E^Y \ := \ \frac{1}{3}\Gamma^\mathsf{T} E^\Delta, \qquad V^n \ := \ \gamma, \qquad \text{no neutral line so that } \ I^n = 0$$

   - Not necessarily balanced

Δ-Y transformation
Ideal voltage source (E^Δ, γ)

3. If E^Δ is balanced then

$$\Gamma^{\mathsf{T}} E^\Delta = (1-\alpha^2)E^\Delta = \sqrt{3}\, e^{-\mathrm{i}\pi/6} E^\Delta$$

$$V \;=\; \frac{1}{\sqrt{3}} e^{-\mathrm{i}\pi/6} E^\Delta \;+\; \gamma 1, \qquad 1^{\mathsf{T}} I \;=\; 0$$

Y equivalent:

$$E^Y \;=\; \frac{1}{\sqrt{3}\, e^{\mathrm{i}\pi/6}} E^\Delta, \qquad V^n \;:=\; \gamma, \qquad \text{no neutral line so that } \; I^n = 0$$

Δ-Y transformation

Non-ideal voltage source $(E^\Delta, z^\Delta, \gamma)$

1. External model

$$V \;=\; \hat{\Gamma} E^\Delta \;-\; Z^\Delta I \;+\; \gamma \mathbf{1}, \qquad \mathbf{1}^{\mathsf{T}} I \;=\; 0$$

$$\text{where } \hat{\Gamma} \;:=\; \frac{1}{3}\Gamma^{\mathsf{T}}\left(\mathbb{I} - \frac{1}{\zeta}\,\tilde{z}^\Delta\,\mathbf{1}^{\mathsf{T}}\right), \qquad Z^\Delta \;:=\; \frac{1}{9}\Gamma^{\mathsf{T}} z^\Delta \left(\mathbb{I} - \frac{1}{\zeta}\,\mathbf{1}\,\tilde{z}^{\Delta\mathsf{T}}\right)\Gamma$$

2. There is no Y equivalent

- Y equivalent has no neutral line so that $\mathbf{1}^{\mathsf{T}} I = 0$
- External model: $V \;=\; E^Y - z^Y I + V^n \mathbf{1}$
- Z^Δ is generally not diagonal (even if $z^\Delta = z^{ab}\mathbb{I}$), but z^Y is diagonal

Δ-Y transformation

Ideal current source J^Δ

1. External model

$$I \;=\; -\Gamma^{\mathsf T} J^\Delta$$

2. Y equivalent

   - Ideal current source $I = -J^Y$, $1^{\mathsf T} I = -I^n$ with

$$J^Y \;:=\; \Gamma^{\mathsf T} J^\Delta, \quad \text{no neutral line } (1^{\mathsf T} I = 0)$$

3. If J^Δ is balanced then

$$J^Y \;=\; (1-\alpha^2) J^\Delta \;=\; \frac{\sqrt{3}}{e^{\mathrm{i}\pi/6}} J^\Delta$$

Outline

1. Overview
2. Mathematical properties
3. Three-phase device models
4. Three-phase line models
    - 4-wire model
    - 3-wire model
5. Three-phase transformer models

4-wire line model

Series impedance matrix $\hat{z}^s_{jk}$

1. Single-phase line: $V_j - V_k = z^s_{jk}\, I_{jk}$

2. Three-phase line: $\hat{V}_j - \hat{V}_k = \hat{z}^s_{jk}\, I_{jk}$

$$\begin{bmatrix} V_j^a \\ V_j^b \\ V_j^c \\ V_j^n \end{bmatrix} - \begin{bmatrix} V_k^a \\ V_k^b \\ V_k^c \\ V_k^n \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{z}_{jk}^{aa} & \hat{z}_{jk}^{ab} & \hat{z}_{jk}^{ac} & \hat{z}_{jk}^{an} \\ \hat{z}_{jk}^{ba} & \hat{z}_{jk}^{bb} & \hat{z}_{jk}^{bc} & \hat{z}_{jk}^{bn} \\ \hat{z}_{jk}^{ca} & \hat{z}_{jk}^{cb} & \hat{z}_{jk}^{cc} & \hat{z}_{jk}^{cn} \\ \hat{z}_{jk}^{na} & \hat{z}_{jk}^{nb} & \hat{z}_{jk}^{nc} & \hat{z}_{jk}^{nn} \end{bmatrix}}_{\text{impedance matrix } \hat{z}^s_{jk}} \begin{bmatrix} I_{jk}^a \\ I_{jk}^b \\ I_{jk}^c \\ I_{jk}^n \end{bmatrix}$$

3. Impedance matrix $\hat{z}^s_{jk}$ depends on
   - wire materials, lengths, distances between wires, frequency, earth resistivity

4-wire line model

Interpretation

Complete circuit a only

$$\begin{bmatrix} V_j^a \\ V_j^b \\ V_j^c \\ V_j^n \end{bmatrix} - \begin{bmatrix} V_k^a \\ V_k^b \\ V_k^c \\ V_k^n \end{bmatrix} = \begin{bmatrix} \hat{z}_{jk}^{aa} & \hat{z}_{jk}^{ab} & \hat{z}_{jk}^{ac} & \hat{z}_{jk}^{an} \\ \hat{z}_{jk}^{ba} & \hat{z}_{jk}^{bb} & \hat{z}_{jk}^{bc} & \hat{z}_{jk}^{bn} \\ \hat{z}_{jk}^{ca} & \hat{z}_{jk}^{cb} & \hat{z}_{jk}^{cc} & \hat{z}_{jk}^{cn} \\ \hat{z}_{jk}^{na} & \hat{z}_{jk}^{nb} & \hat{z}_{jk}^{nc} & \hat{z}_{jk}^{nn} \end{bmatrix} \begin{bmatrix} I_{jk}^a \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

self impedance:

$$\hat{z}_{jk}^{aa} = \frac{V_j^a - V_k^a}{I_{jk}^a}$$

mutual impedance:

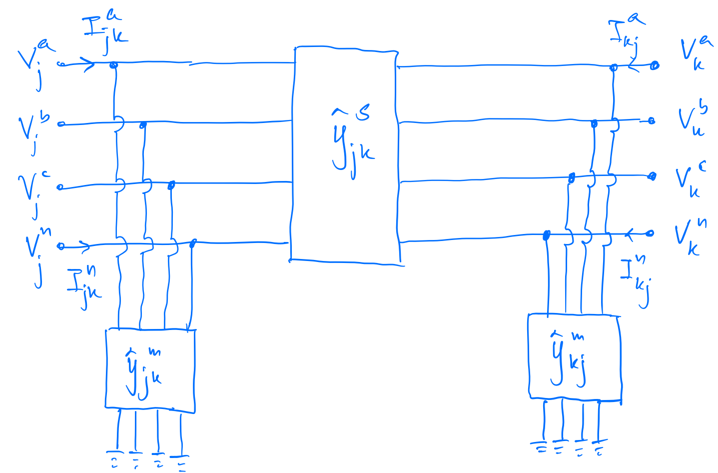
$$\hat{z}_{jk}^{ba} = \frac{V_j^b - V_k^b}{I_{jk}^a}$$

4-wire line model

With shunt admittances

Each line is characterized by

- Series admittance $\hat{y}^s_{jk} := \left(\hat{z}^s_{jk}\right)^{-1}$
- Shunt admittances $\left(\hat{y}^m_{jk}, \hat{y}^m_{kj}\right)$



Terminal voltages $\left(\hat{V}_j, \hat{V}_k\right)$ and terminal currents $\left(\hat{I}_{jk}, \hat{I}_{kj}\right)$ satisfy

$$\hat{I}_{jk} = \hat{y}^s_{jk}\left(\hat{V}_j - \hat{V}_k\right) + \hat{y}^m_{jk}\hat{V}_j$$

$$\hat{I}_{kj} = \hat{y}^s_{jk}\left(\hat{V}_k - \hat{V}_j\right) + \hat{y}^m_{kj}\hat{V}_k$$

3-wire line model

Series impedance matrix z^s_{jk}

$$\begin{bmatrix} V_j^a \\ V_j^b \\ V_j^c \\ V_j^n \end{bmatrix} - \begin{bmatrix} V_k^a \\ V_k^b \\ V_k^c \\ V_k^n \end{bmatrix} = \begin{bmatrix} \hat{z}_{jk}^{aa} & \hat{z}_{jk}^{ab} & \hat{z}_{jk}^{ac} & \hat{z}_{jk}^{an} \\ \hat{z}_{jk}^{ba} & \hat{z}_{jk}^{bb} & \hat{z}_{jk}^{bc} & \hat{z}_{jk}^{bn} \\ \hat{z}_{jk}^{ca} & \hat{z}_{jk}^{cb} & \hat{z}_{jk}^{cc} & \hat{z}_{jk}^{cn} \\ \hat{z}_{jk}^{na} & \hat{z}_{jk}^{nb} & \hat{z}_{jk}^{nc} & \hat{z}_{jk}^{nn} \end{bmatrix} \begin{bmatrix} I_{jk}^a \\ I_{jk}^b \\ I_{jk}^c \\ I_{jk}^n \end{bmatrix}$$

1. $I^n_{jk} = 0$: can eliminate last column and row of $\hat{z}^s_{jk}$
   - There is no neutral line, e.g., Δ-configured device
2. $V^n_j = V^n_k$: can eliminate $I^n_{jk} = -\dfrac{1}{\hat{z}^{nn}_{jk}} \left(\hat{z}^{na}_{jk} I^a_{jk} + \hat{z}^{nb}_{jk} I^b_{jk} + \hat{z}^{nc}_{jk} I^c_{jk} \right)$
   - Neutrals at both ends are grounded with $z^n_j = z^n_k = 0$

3-wire line model

Series impedance matrix z_{jk}^s

Both cases can be modeled by 3×3 impedance matrix

$$\begin{bmatrix} V_j^a \\ V_j^b \\ V_j^c \end{bmatrix} - \begin{bmatrix} V_k^a \\ V_k^b \\ V_k^c \end{bmatrix} = \underbrace{\begin{bmatrix} z_{jk}^{aa} & z_{jk}^{ab} & z_{jk}^{ac} \\ z_{jk}^{ba} & z_{jk}^{bb} & z_{jk}^{bc} \\ z_{jk}^{ca} & z_{jk}^{cb} & z_{jk}^{cc} \end{bmatrix}}_{z_{jk}} \begin{bmatrix} I_{jk}^a \\ I_{jk}^b \\ I_{jk}^c \end{bmatrix}$$

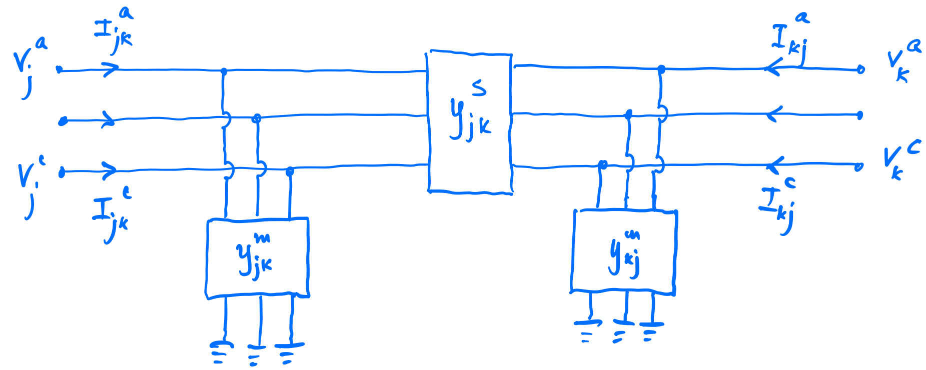
Three-phase line: $V_j - V_k = z_{jk}^s I_{jk}$

3-wire line model

With shunt admittances

Each line is characterized by

- Series admittance $y_{jk}^s := \left(z_{jk}^s\right)^{-1}$
- Shunt admittances $\left(y_{jk}^m, y_{kj}^m\right)$



Terminal voltages $\left(V_j, V_k\right)$ and terminal currents $\left(I_{jk}, I_{kj}\right)$ satisfy

$$I_{jk} = y_{jk}^s\left(V_j - V_k\right) + y_{jk}^m V_j$$

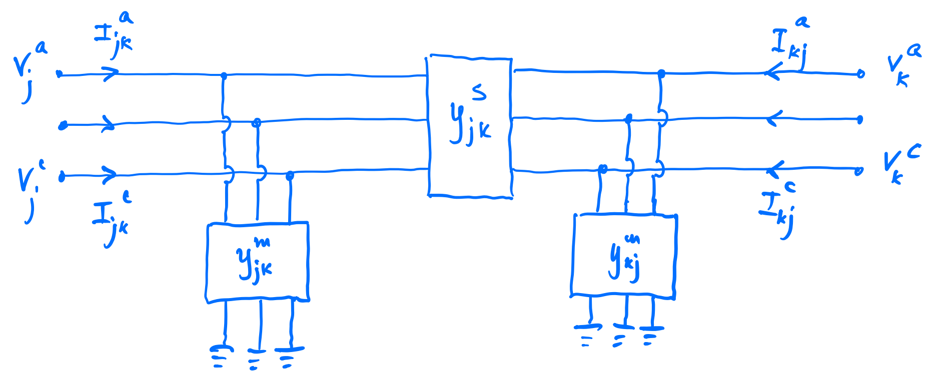
$$I_{kj} = y_{jk}^s\left(V_k - V_j\right) + y_{kj}^m V_k$$

3-wire line model

With shunt admittances

Each line is characterized by

- Series admittance $y_{jk}^s := \left(z_{jk}^s\right)^{-1}$
- Shunt admittances $\left(y_{jk}^m, y_{kj}^m\right)$



Terminal voltages $\left(V_j, V_k\right)$ and line power matrices $\left(S_{jk}, S_{kj}\right) \in \mathbb{C}^{6\times 6}$ satisfy

$$S_{jk} := V_j \left(I_{jk}\right)^{\mathsf{H}} = V_j \left(V_j - V_k\right)^{\mathsf{H}} \left(y_{jk}^s\right)^{\mathsf{H}} + V_j V_j^{\mathsf{H}} \left(y_{jk}^m\right)^{\mathsf{H}}$$

$$S_{kj} := V_k \left(I_{kj}\right)^{\mathsf{H}} = V_k \left(V_k - V_j\right)^{\mathsf{H}} \left(y_{jk}^s\right)^{\mathsf{H}} + V_k V_k^{\mathsf{H}} \left(y_{kj}^m\right)^{\mathsf{H}}$$

line flows are diag(S_{jk}), diag(S_{kj})

Comparison

IV relation

$$I_{jk}\left(V_j, V_k\right) = y_{jk}^s\left(V_j - V_k\right) + y_{jk}^m V_j$$

$$I_{kj}\left(V_j, V_k\right) = y_{jk}^s\left(V_k - V_j\right) + y_{kj}^m V_k$$

SV relation

$$S_{jk}\left(V_j, V_k\right) = V_j\left(V_j - V_k\right)^{\mathsf{H}}\left(y_{jk}^s\right)^{\mathsf{H}} + V_j V_j^{\mathsf{H}}\left(y_{jk}^m\right)^{\mathsf{H}}$$

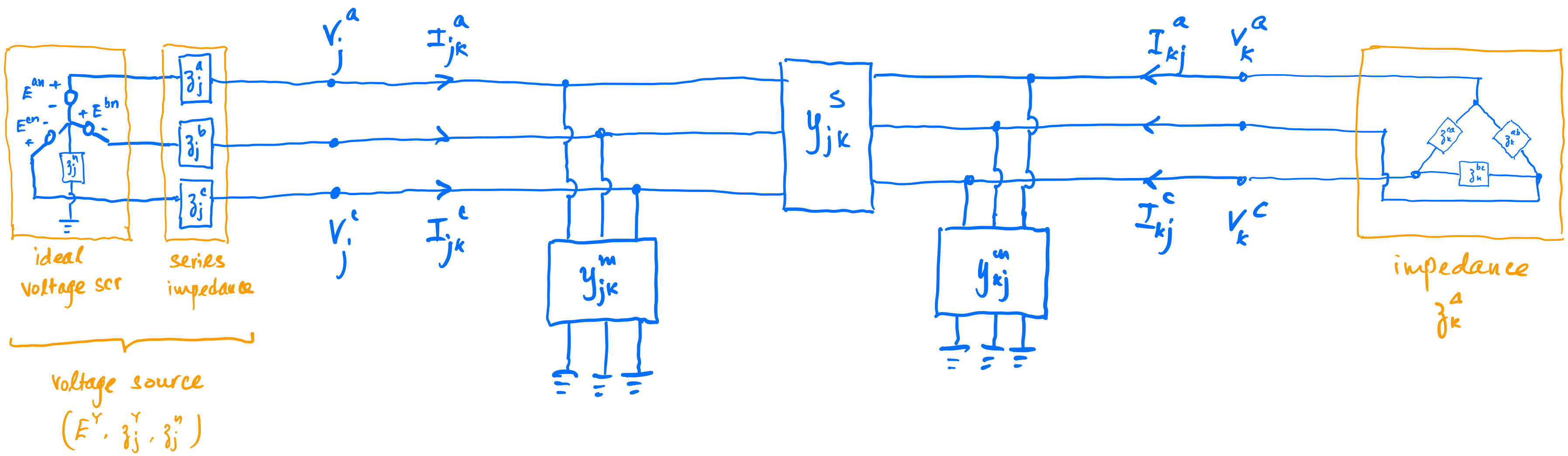
$$S_{kj}\left(V_j, V_k\right) = V_k\left(V_k - V_j\right)^{\mathsf{H}}\left(y_{jk}^s\right)^{\mathsf{H}} + V_k V_k^{\mathsf{H}}\left(y_{kj}^m\right)^{\mathsf{H}}$$

same expressions for 1 or 3 phases !

	1-phase	**3-phase (4-wire)**	**3-phase (3-wire)**
admittances $y_{jk}^s, y_{jk}^m, y_{kj}^m$	$\mathbb{C}$	$\mathbb{C}^{4\times 4}$	$\mathbb{C}^{3\times 3}$
voltages V_j, V_k	$\mathbb{C}$	$\mathbb{C}^4$	$\mathbb{C}^3$
currents I_{jk}, I_{kj}	$\mathbb{C}$	$\mathbb{C}^4$	$\mathbb{C}^3$
line powers S_{jk}, S_{kj}	$\mathbb{C}$	$\mathbb{C}^{4\times 4}$	$\mathbb{C}^{3\times 3}$

3-wire line model

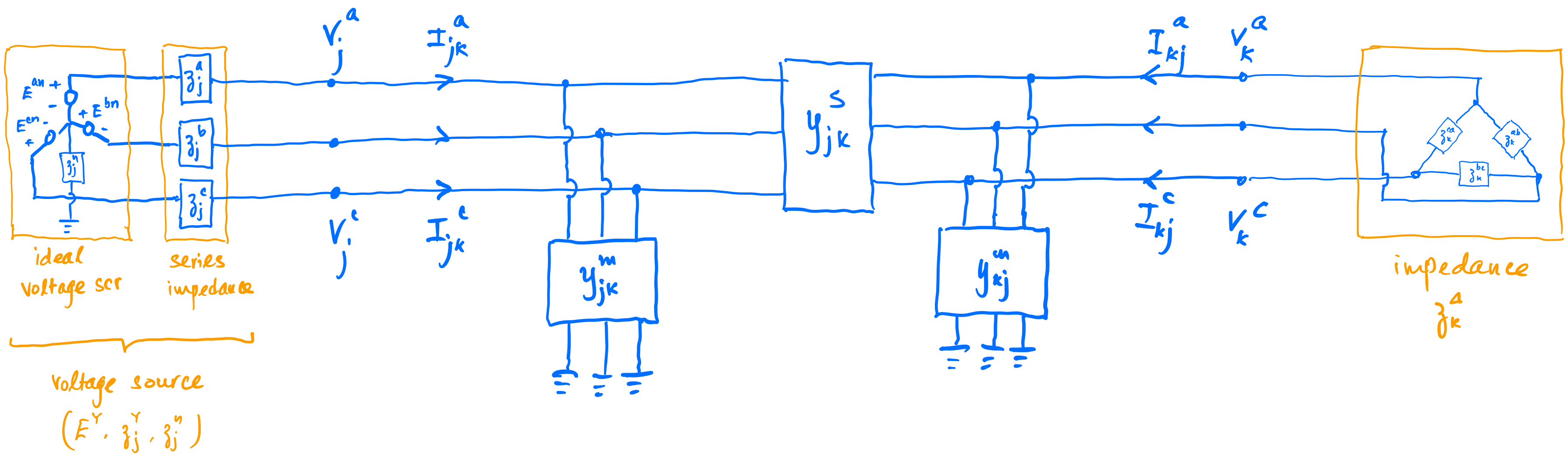
Example



- Line model relates terminal voltages and currents at both ends of the line, regardless of device Y/Δ configuration

3-wire line model

Example



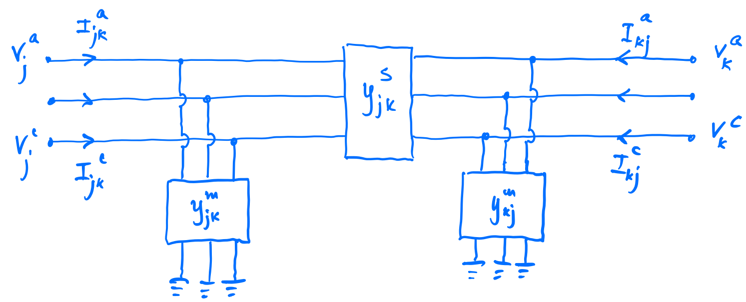
Terminal vars $\left(V_j, I_j, s_j\right)$ at bus j satisfy external device model and line model (that relate $\left(V_j, I_j, s_j\right)$ to V_k)

- Device j model: $0 = f_j^{\text{ext}}\left(V_j, I_j\right), \quad s_j = \text{diag}\left(V_j I_j^{\text{H}}\right)$
- Line (j,k) model: $I_j = I_{jk}\left(V_j, V_k\right), \quad s_j = \text{diag}\left(S_{jk}\left(V_j, V_k\right)\right)$

3-wire line model

Properties

1. Properties of admittance matrices $\left(y_{jk}^s, y_{jk}^m, y_{kj}^m\right)$
   - They are typically complex symmetric (not Hermitian)
   - y_{jk}^s is typically invertible

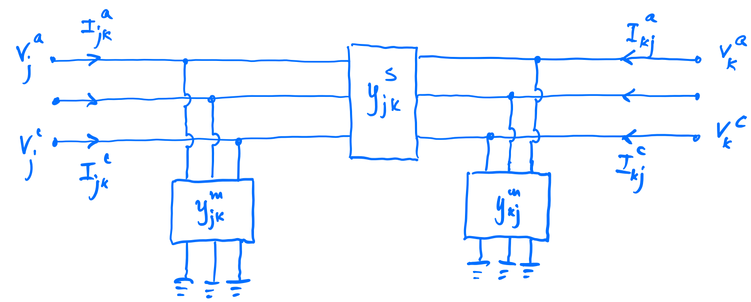


Complex symmetry of y_{jk}^s leads to single-phase equivalent of 3-phase networks (see later)

3-wire line model

Properties

1. Properties of admittance matrices $\left(y_{jk}^s, y_{jk}^m, y_{kj}^m\right)$
    - They are typically complex symmetric (not Hermitian)
    - y_{jk}^s is typically invertible
2. Symmetric line, e.g., through transpose and symmetric line geometry

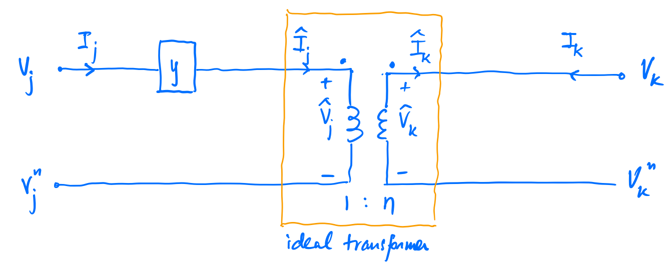


$$z_{jk}^s = \begin{bmatrix} z_{jk}^1 & z_{jk}^2 & z_{jk}^2 \\ z_{jk}^2 & z_{jk}^1 & z_{jk}^2 \\ z_{jk}^2 & z_{jk}^2 & z_{jk}^1 \end{bmatrix}, \qquad y_{jk}^s = \begin{bmatrix} y_{jk}^1 & y_{jk}^2 & y_{jk}^2 \\ y_{jk}^2 & y_{jk}^1 & y_{jk}^2 \\ y_{jk}^2 & y_{jk}^2 & y_{jk}^1 \end{bmatrix}$$

Outline

1. Overview
2. Mathematical properties
3. Three-phase device models
4. Three-phase line models
5. Three-phase transformer models
    - General derivation method
    - $YY, \Delta\Delta, \Delta Y, Y\Delta$ configurations
    - UVN-based model

Review: single-phase transformer



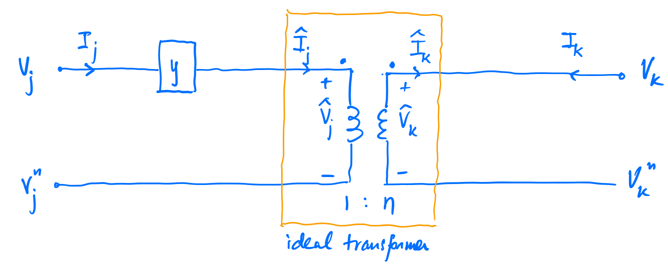
1. Internal and terminal vars
   - Internal vars: $\left(\hat{V}_j, \hat{I}_j\right)$ and $\left(\hat{V}_k, \hat{I}_k\right)$
   - Terminal vars: $\left(V_j, V_j^n, I_j\right)$ and $\left(V_k, V_k^n, I_k\right)$
2. Internal model on internal vars between primary & secondary sides
   - (Ideal) transformer gains: $\hat{V}_k \ = \ n\hat{V}_j, \quad \hat{I}_k \ = \ a\hat{I}_j$
3. Conversion between internal & terminal vars on each side

$$V_j = \ y^{-1}\hat{I}_j \ + \ \hat{V}_j \ + \ V_j^n, \qquad I_j = \ \hat{I}_j$$

$$V_k = \ \hat{V}_k \ + \ V_k^n, \qquad I_k = -\hat{I}_k$$

Review: single-phase transformer

4. External model on external vars across pri & sec sides
   - Eliminate internal vars from internal model and conversion

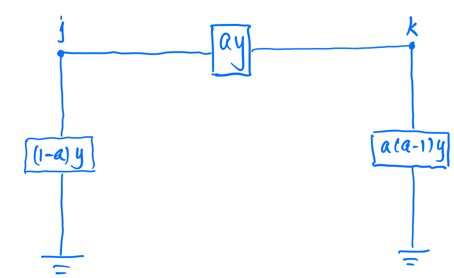


$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} = \begin{bmatrix} y & -ay \\ -ay & a^2 y \end{bmatrix} \left(\begin{bmatrix} V_j \\ V_k \end{bmatrix} - \begin{bmatrix} V_j^n \\ V_k^n \end{bmatrix} \right)$$

5. If neutrals are grounded with zero grounding impedance so that $V_j^n = V_k^n = 0$ (often assumed)

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} = \begin{bmatrix} y & -ay \\ -ay & a^2 y \end{bmatrix} \begin{bmatrix} V_j \\ V_k \end{bmatrix}$$

   - Reduces to a Π circuit



Three-phase transformers

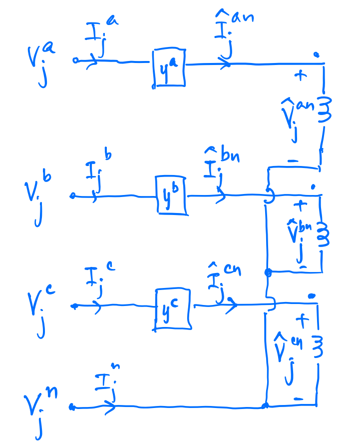
1. Three-phase transformers consists of 3 single-phase transformers in Y/Δ configuration
2. External models can be derived following the same procedure

General method

Primary side

1. Internal vars (defined across individual windings)

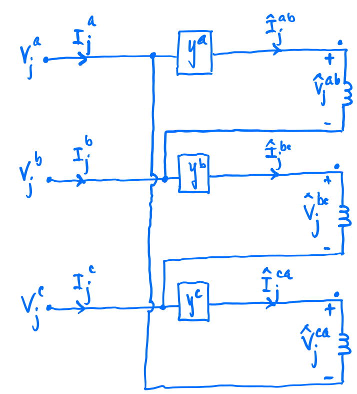
$$\hat V_j^Y := \begin{bmatrix} \hat V_j^{an} \\ \hat V_j^{bn} \\ \hat V_j^{cn} \end{bmatrix}, \quad \hat I_j^Y := \begin{bmatrix} \hat I_j^{an} \\ \hat I_j^{bn} \\ \hat I_j^{cn} \end{bmatrix}, \quad \hat V_j^\Delta := \begin{bmatrix} \hat V_j^{ab} \\ \hat V_j^{bc} \\ \hat V_j^{ca} \end{bmatrix}, \quad \hat I_j^\Delta := \begin{bmatrix} \hat I_j^{ab} \\ \hat I_j^{bc} \\ \hat I_j^{ca} \end{bmatrix}$$



Y

2. Terminal vars (voltages wrt common reference, e.g., ground)

$$V_j := \begin{bmatrix} V_j^a \\ V_j^b \\ V_j^c \end{bmatrix}, \quad I_j := \begin{bmatrix} I_j^a \\ I_j^b \\ \hat I_j^c \end{bmatrix}, \quad \text{for } Y \text{ configuration: } \left(V_j^n, I_j^n\right)$$



Δ

3. Leakage admittance matrix $y := \text{diag}\left(y^a, y^b, y^c\right)$

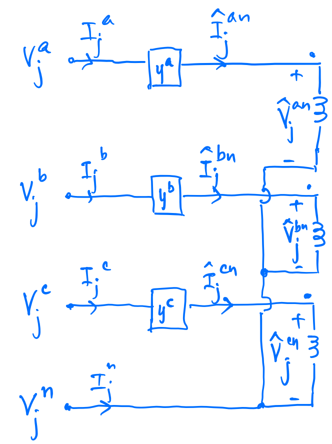
General method

Primary side

4. Conversion between internal and terminal vars

- Y configuration

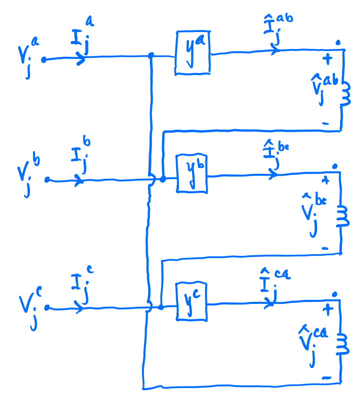
$$I_j = y\left(V_j - V_j^n 1 - \hat{V}_j^Y\right), \qquad I_j = \hat{I}_j^Y, \qquad I_j^n = -1^\mathsf{T} \hat{I}_j^Y$$



Y

- Δ configuration

$$\hat{I}_j^\Delta = y\left(\Gamma V_j - \hat{V}_j^\Delta\right), \qquad I_j = \Gamma^\mathsf{T} \hat{I}_j^\Delta$$



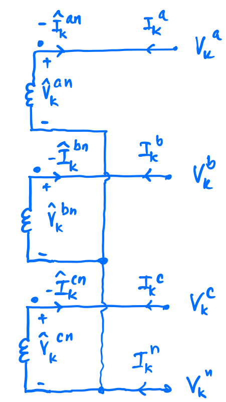
Δ

General method

Secondary side

1. Internal vars (defined across individual windings)

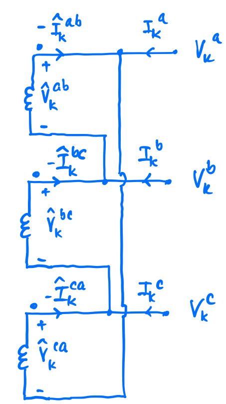
$$\hat{V}_k^Y := \begin{bmatrix} \hat{V}_k^{an} \\ \hat{V}_k^{bn} \\ \hat{V}_k^{cn} \end{bmatrix}, \quad \hat{I}_k^Y := -\begin{bmatrix} \hat{I}_k^{an} \\ \hat{I}_k^{bn} \\ \hat{I}_k^{cn} \end{bmatrix}, \quad \hat{V}_k^\Delta := \begin{bmatrix} \hat{V}_k^{ab} \\ \hat{V}_k^{bc} \\ \hat{V}_k^{ca} \end{bmatrix}, \quad \hat{I}_k^\Delta := -\begin{bmatrix} \hat{I}_k^{ab} \\ \hat{I}_k^{bc} \\ \hat{I}_k^{ca} \end{bmatrix}$$



Y

2. Terminal vars (voltages defined wrt common reference, e.g., ground)

$$V_k := \begin{bmatrix} V_k^a \\ V_k^b \\ \hat{V}_k^c \end{bmatrix}, \quad I_k := \begin{bmatrix} I_k^a \\ I_k^b \\ \hat{I}_k^c \end{bmatrix}, \quad \text{for } Y \text{ configuration: } \left(V_k^n, I_k^n\right)$$



Δ

3. Admittances in secondary side assumed to have been referred to primary

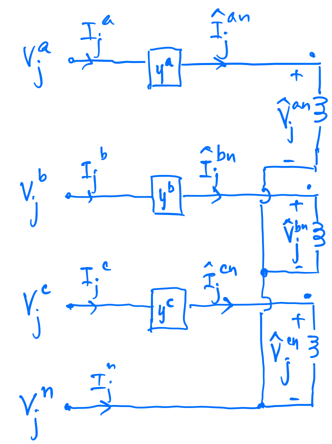
General method

Secondary side

4. Conversion between internal and terminal vars

- Y configuration

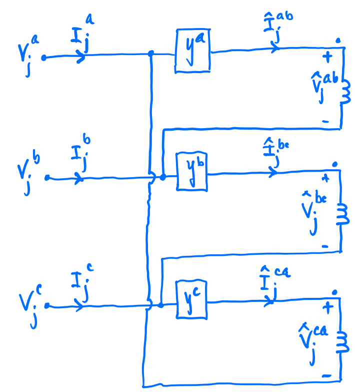
$$V_k = \hat{V}_k^Y + V_k^n 1, \qquad I_k = -\hat{I}_k^Y, \qquad I_k^n = 1^{\mathsf{T}} \hat{I}_k^Y$$



Y

- Δ configuration

$$\hat{V}_k^\Delta = \Gamma V_k, \qquad I_k = -\Gamma^{\mathsf{T}} \hat{I}_k^\Delta$$



Δ

General method

Internal model

1. Voltage gain (real) $n := \text{diag}\left(n^a, n^b, n^c\right) \in \mathbb{R}^{3\times3}$, turns ratio $a := n^{-1} \in \mathbb{R}^{3\times3}$
   - Voltage gains (or turns ratios) may be different across phases a, b, c

2. Transformer gains on internal vars across primary and secondary sides

$$
\begin{array}{llll}
YY \text{ configuration:} & \hat{V}_k^Y = n\,\hat{V}_j^Y, & \hat{I}_k^Y = a\,\hat{I}_j^Y \\
\Delta\Delta \text{ configuration:} & \hat{V}_k^\Delta = n\,\hat{V}_j^\Delta, & \hat{I}_k^\Delta = a\,\hat{I}_j^\Delta \\
\Delta Y \text{ configuration:} & \hat{V}_k^Y = n\,\hat{V}_j^\Delta, & \hat{I}_k^Y = a\,\hat{I}_j^\Delta \\
Y\Delta \text{ configuration:} & \hat{V}_k^\Delta = n\,\hat{V}_j^Y, & \hat{I}_k^\Delta = a\,\hat{I}_j^Y
\end{array}
$$

Voltage and current gains follow the same gains as those for single-phase transformers, regardless of 3-phase configuration

General method

External model: summary

1. Couple internal vars $\left(\hat{V}_j^{Y/\Delta}, \hat{I}_j^{Y/\Delta}\right), \left(\hat{V}_k^{Y/\Delta}, \hat{I}_k^{Y/\Delta}\right)$ across pri and sec sides through transformer gains, the same way as in single-phase transformers
2. Relate terminal vars $\left(V_j, V_j^n, I_j\right), \left(V_k, V_k^n, I_k\right)$ to internal vars $\left(\hat{V}_j^{Y/\Delta}, \hat{I}_j^{Y/\Delta}\right), \left(\hat{V}_k^{Y/\Delta}, \hat{I}_k^{Y/\Delta}\right)$ on each of primary and secondary sides
3. Eliminate internal vars from equations in Steps 1 and 2 (in previous slides) to obtain an external model relating only terminal vars $\left(V_j, V_j^n, I_j\right), \left(V_k, V_k^n, I_k\right)$

The method is modular with respect to $YY, \Delta\Delta, \Delta Y, Y\Delta$ configurations, as we will see

3-phase transformers

Overview

- Let $V := \left(V_j, V_k\right) \in \mathbb{C}^6$ and $I := \left(I_j, I_k\right) \in \mathbb{C}^6$
- Define 6×6 admittance matrix Y_{YY} and column vector

$$Y_{YY} \ := \ \begin{bmatrix} y & -ay \\ -ay & a^2 y \end{bmatrix}, \qquad \gamma \ := \ \left(V_j^n \mathbf{1}, V_k^n \mathbf{1}\right)$$

where $a \ := \ \text{diag}\left(a^a, a^b, a^c\right), \qquad y \ := \ \text{diag}\left(y^a, y^b, y^c\right)$

External models: $I \ = \ D^{\mathsf{T}} Y_{YY} D \left(V - \gamma\right)$ where

$$YY: \ D := \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{bmatrix}, \ \Delta\Delta: \ D := \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix}, \ \Delta Y: \ D := \begin{bmatrix} \Gamma & 0 \\ 0 & \mathbb{I} \end{bmatrix}, \ Y\Delta: \ D := \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \Gamma \end{bmatrix}$$

3-phase transformers

Overview

External models: $I \;=\; D^{\mathsf{T}} Y_{YY} D \left(V - \gamma \right)$ where

$$YY: \;\; D := \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{bmatrix}, \;\; \Delta\Delta: \;\; D := \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix}, \;\; \Delta Y: \;\; D := \begin{bmatrix} \Gamma & 0 \\ 0 & \mathbb{I} \end{bmatrix}, \;\; Y\Delta: \;\; D := \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \Gamma \end{bmatrix}$$

- $YY, \Delta\Delta$: $D^{\mathsf{T}} Y_{YY} D$ is block symmetric and has 3-phase Π circuit representation
- $\Delta Y, Y\Delta$: Not

Next: derive external models for each configuration in detail

YY configuration
Internal and terminal vars

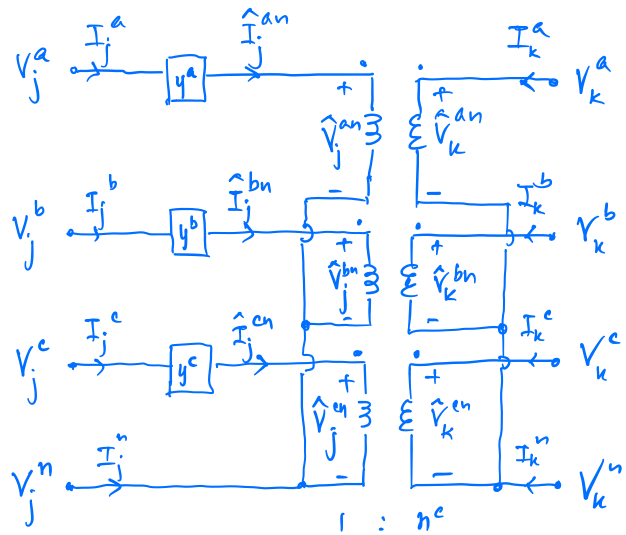
1. Internal vars (defined across individual windings)

$$\hat V_j^Y := \begin{bmatrix} \hat V_j^{an} \\ \hat V_j^{bn} \\ \hat V_j^{cn} \end{bmatrix}, \quad \hat I_j^Y := \begin{bmatrix} \hat I_j^{an} \\ \hat I_j^{bn} \\ \hat I_j^{cn} \end{bmatrix}, \quad \hat V_k^Y := \begin{bmatrix} \hat V_k^{an} \\ \hat V_k^{bn} \\ \hat V_k^{cn} \end{bmatrix}, \quad \hat I_k^Y := - \begin{bmatrix} \hat I_k^{an} \\ \hat I_k^{bn} \\ \hat I_k^{cn} \end{bmatrix},$$

2. Terminal vars (voltages wrt common reference, e.g., ground)

$$V_j := \begin{bmatrix} V_j^a \\ V_j^b \\ \hat V_j^c \end{bmatrix}, \quad I_j := \begin{bmatrix} I_j^a \\ I_j^b \\ \hat I_j^c \end{bmatrix}, \quad V_k := \begin{bmatrix} V_k^a \\ V_k^b \\ \hat V_k^c \end{bmatrix}, \quad I_k := \begin{bmatrix} I_k^a \\ I_k^b \\ \hat I_k^c \end{bmatrix}$$

$$\left(V_j^n, I_j^n\right), \quad \left(V_k^n, I_k^n\right)$$

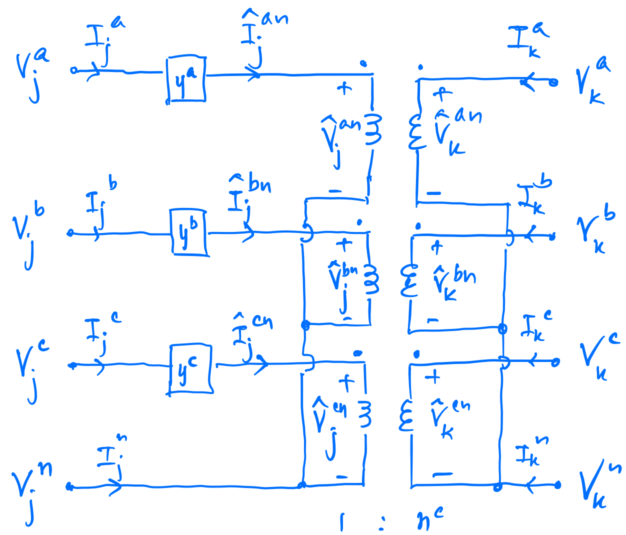


YY configuration
External model

1. External model

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} = \underbrace{\begin{bmatrix} y & -ay \\ -ay & a^2 y \end{bmatrix}}_{Y_{YY}} \left(\begin{bmatrix} V_j \\ V_k \end{bmatrix} - \begin{bmatrix} V_j^n 1 \\ V_k^n 1 \end{bmatrix} \right)$$

$$I_j^n = -1^\mathsf{T} I_j, \quad I_k^n = -1^\mathsf{T} I_k$$



2. If both neutrals are grounded with zero impedance and voltages are defined wrt ground

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} = \begin{bmatrix} y & -ay \\ -ay & a^2 y \end{bmatrix} \begin{bmatrix} V_j \\ V_k \end{bmatrix}$$

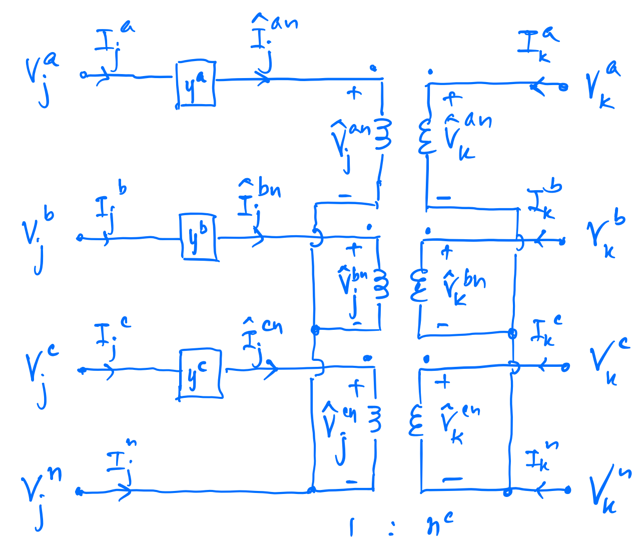
which can be represented as a Π circuit

Comparison

With single-phase transformer

External models: exactly the same expression

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} = \underbrace{\begin{bmatrix} y & -ay \\ -ay & a^2 y \end{bmatrix}}_{Y_{YY}} \left(\begin{bmatrix} V_j \\ V_k \end{bmatrix} - \begin{bmatrix} V_j^n 1 \\ V_k^n 1 \end{bmatrix} \right)$$



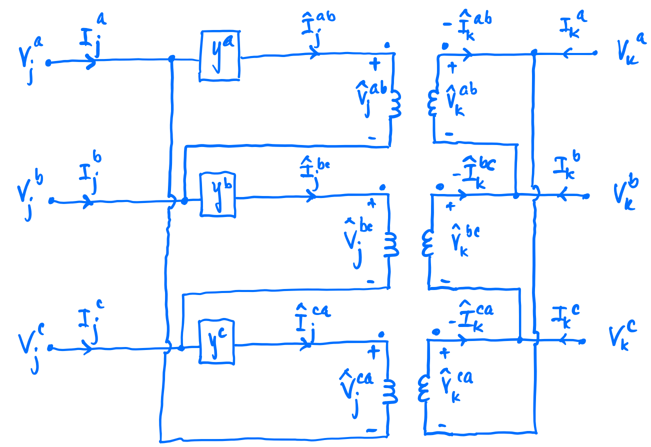
- Single-phase: $Y_{YY} \in \mathbb{C}^{2\times 2}$
- Three-phase: $Y_{YY} \in \mathbb{C}^{6\times 6}$

$\Delta\Delta$ configuration

Internal and terminal vars

1. Internal vars (defined across individual windings)

$$\hat{V}_j^\Delta := \begin{bmatrix} \hat{V}_j^{ab} \\ \hat{V}_j^{bc} \\ \hat{V}_j^{ca} \end{bmatrix}, \quad \hat{I}_j^\Delta := \begin{bmatrix} \hat{I}_j^{ab} \\ \hat{I}_j^{bc} \\ \hat{I}_j^{ca} \end{bmatrix}, \quad \hat{V}_k^\Delta := \begin{bmatrix} \hat{V}_k^{ab} \\ \hat{V}_k^{bc} \\ \hat{V}_k^{ca} \end{bmatrix}, \quad \hat{I}_k^\Delta := -\begin{bmatrix} \hat{I}_k^{ab} \\ \hat{I}_k^{bc} \\ \hat{I}_k^{ca} \end{bmatrix}$$

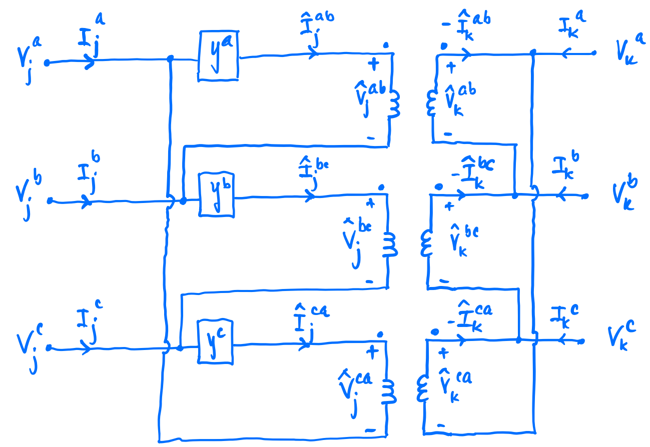


2. Terminal vars $\left(V_j, I_j\right), \left(V_k, I_k\right)$ same as for YY config
   - without neutral vars

$\Delta\Delta$ configuration
External model

External model

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} = \begin{bmatrix} \Gamma^\mathsf{T} & 0 \\ 0 & \Gamma^\mathsf{T} \end{bmatrix} \underbrace{\begin{bmatrix} y & -ay \\ -ay & a^2 y \end{bmatrix}}_{Y_{YY}} \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix} \begin{bmatrix} V_j \\ V_k \end{bmatrix}$$



- Can be represented as Π circuit
- Conversion matrices due to Δ configurations

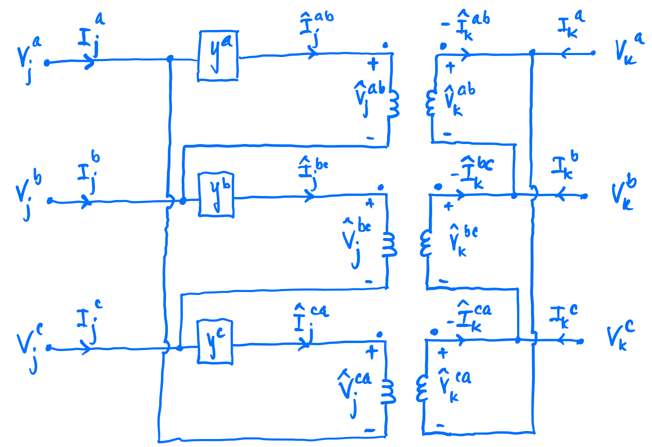
Comparison

With single-phase transformer

Single-phase: $Y_{YY} \;\in\; \mathbb{C}^{2\times 2}$

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} y & -ay \\ -ay & a^2 y \end{bmatrix}}_{Y_{YY}} \begin{bmatrix} V_j \\ V_k \end{bmatrix}$$

- No neutral lines



Three-phase: $Y_{YY} \;\in\; \mathbb{C}^{6\times 6}$

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} \;=\; \begin{bmatrix} \Gamma^{\mathsf{T}} & 0 \\ 0 & \Gamma^{\mathsf{T}} \end{bmatrix} \underbrace{\begin{bmatrix} y & -ay \\ -ay & a^2 y \end{bmatrix}}_{Y_{YY}} \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix} \begin{bmatrix} V_j \\ V_k \end{bmatrix}$$

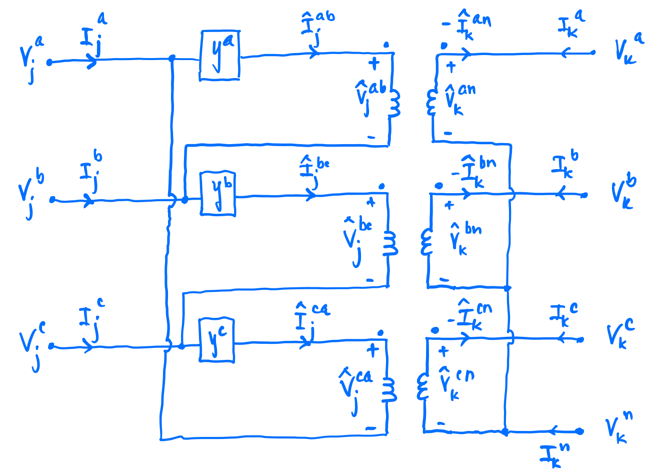
- Conversion matrices due to Δ configurations

ΔY configuration

Internal and terminal vars

1. Internal vars (defined across individual windings)

$$\hat{V}_j^{\Delta} := \begin{bmatrix} \hat{V}_j^{ab} \\ \hat{V}_j^{bc} \\ \hat{V}_j^{ca} \end{bmatrix}, \quad \hat{I}_j^{\Delta} := \begin{bmatrix} \hat{I}_j^{ab} \\ \hat{I}_j^{bc} \\ \hat{I}_j^{ca} \end{bmatrix}, \quad \hat{V}_k^{Y} := \begin{bmatrix} \hat{V}_k^{an} \\ \hat{V}_k^{bn} \\ \hat{V}_k^{cn} \end{bmatrix}, \quad \hat{I}_k^{Y} := -\begin{bmatrix} \hat{I}_k^{an} \\ \hat{I}_k^{bn} \\ \hat{I}_k^{cn} \end{bmatrix},$$



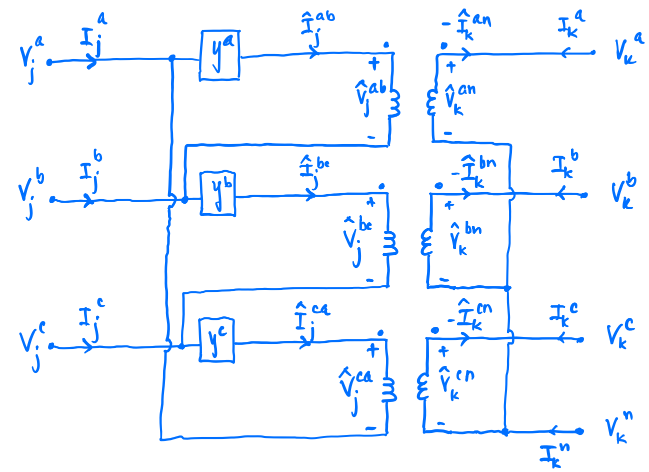
2. Terminal vars $\left(V_j, I_j\right), \ \left(V_k, I_k\right)$ same as before

ΔY configuration

External model

1. External model

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} = \begin{bmatrix} \Gamma^{\mathsf{T}} & 0 \\ 0 & \mathbb{I} \end{bmatrix} \underbrace{\begin{bmatrix} y & -ay \\ -ay & a^2 y \end{bmatrix}}_{Y_{YY}} \begin{bmatrix} \Gamma & 0 \\ 0 & \mathbb{I} \end{bmatrix} \begin{bmatrix} V_j \\ V_k \end{bmatrix} - \begin{bmatrix} -\Gamma^{\mathsf{T}} ay \\ a^2 y \end{bmatrix} V_k^n 1$$



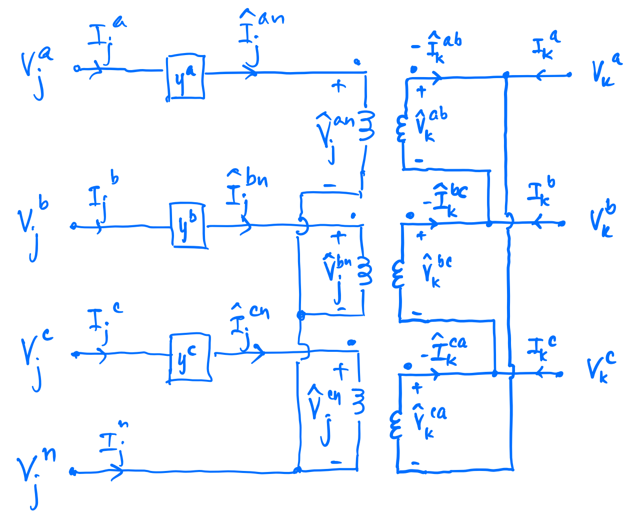
2. Comparison with YY and $\Delta\Delta$ configurations (modular)
   - I_j depends on (V_j, V_k) similarly to $\Delta\Delta$ config
   - I_k depends on (V_j, V_k) similarly to YY config
   - Even though there is no neutral line on primary side, I_j depends on V_k^n on secondary side
   - If $a = a^a \mathbb{I}$, $y = y^a \mathbb{I}$, i.e., identical single-phase transformers, then I_j becomes independent of V_k^n (because $\Gamma^{\mathsf{T}} 1 = 0$)

$Y\Delta$ configuration
External model

1. External model

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \Gamma^\mathsf{T} \end{bmatrix} \underbrace{\begin{bmatrix} y & -ay \\ -ay & a^2 y \end{bmatrix}}_{Y_{YY}} \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \Gamma \end{bmatrix} \begin{bmatrix} V_j \\ V_k \end{bmatrix} - \begin{bmatrix} y \\ -\Gamma^\mathsf{T} a y \end{bmatrix} V_j^n 1$$

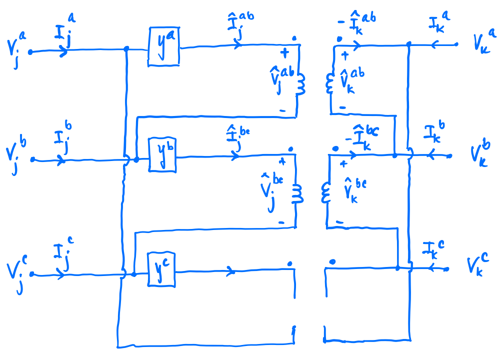
2. Same modular structure as for ΔY configuration
   - I_j depends on (V_j, V_k) similarly to YY config
   - I_k depends on (V_j, V_k) similarly to $\Delta\Delta$ config
   - Even though there is no neutral line on secondary side, I_k depends on V_j^n on primary side
   - If $a = a^a \mathbb{I}$, $y = y^a \mathbb{I}$, i.e., identical single-phase transformers, then I_k becomes independent of V_j^n (because $\Gamma^\mathsf{T} 1 = 0$)



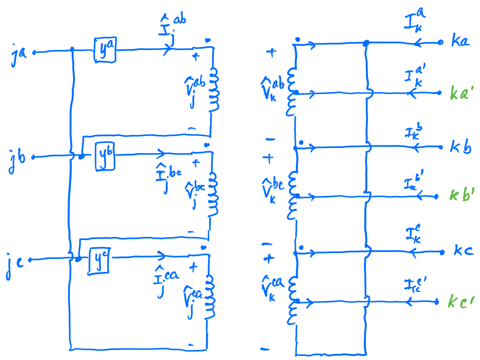
Other transformers

Same method can be applied to derive external models for other transformers

- Open transformer
- Split-phase transformer
- See textbook



Open ΔΔ transformer



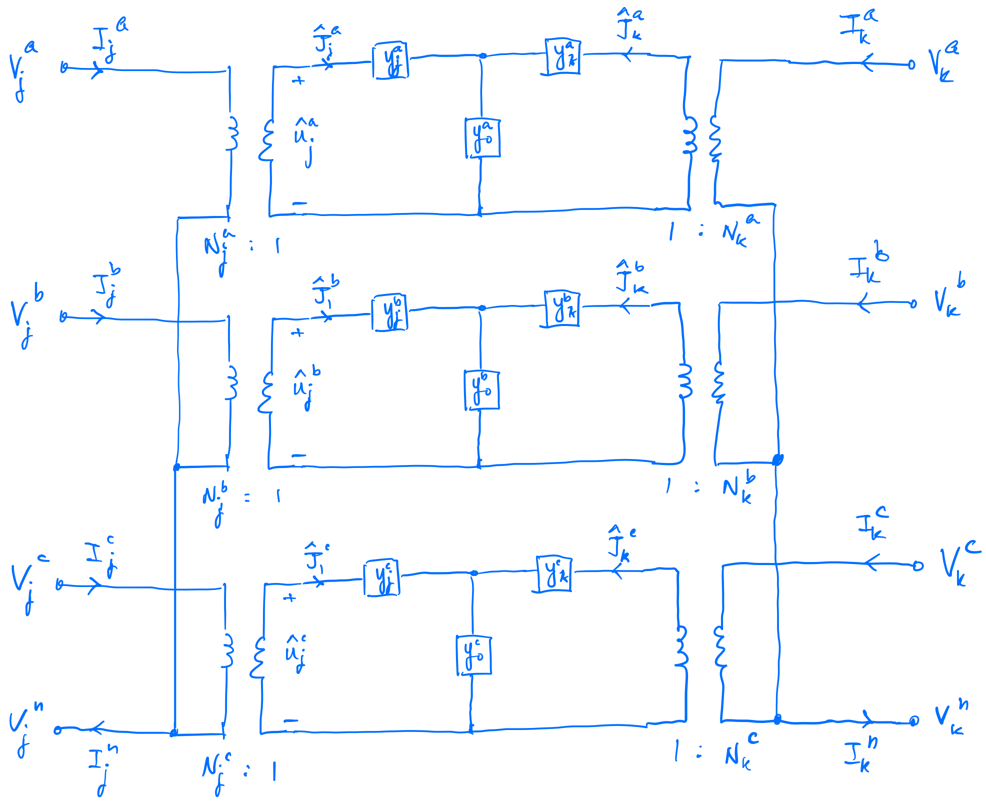
Split-phase transformer

Outline

1. Overview
2. Mathematical properties
3. Three-phase device models
4. Three-phase line models
5. Three-phase transformer models
    - General derivation method
    - $YY, \Delta\Delta, \Delta Y, Y\Delta$ configurations
    - UVN-based model

Three-phase transformers

Example: YY configuration



External vars:

$$I \ := \begin{bmatrix} I_j \\ I_k \end{bmatrix} \in \mathbb{C}^6, \quad V \ := \begin{bmatrix} V_j \\ V_k \end{bmatrix} \in \mathbb{C}^6$$

Transformer parameters:

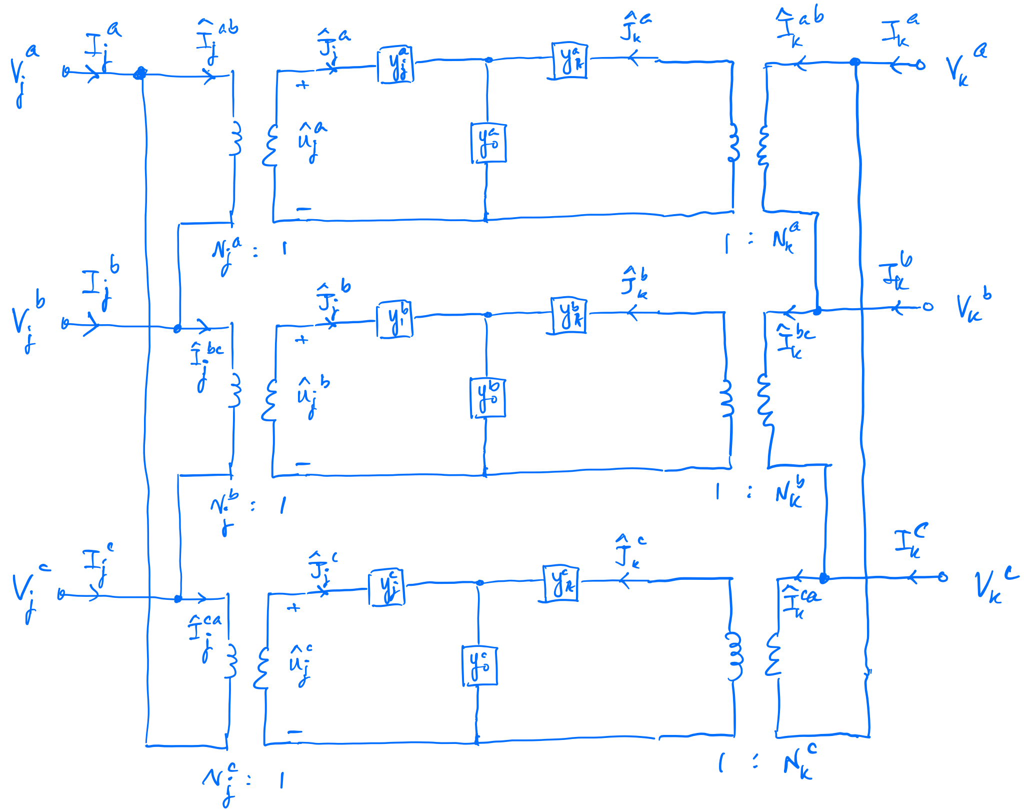
$$M := \begin{bmatrix} \text{diag}(1/N_j^{abc}) & 0 \\ 0 & \text{diag}(1/N_k^{abc}) \end{bmatrix} \in \mathbb{C}^{6\times6}$$

$$y_i := \text{diag}\left(y_i^a, y_i^b, y_i^c\right) \in \mathbb{C}^{3\times3}, \quad i = 0, j, k$$

Unitary voltage network **per phase**

Three-phase transformers

Example: ΔΔ configuration



External vars:

$$I \;:=\; \begin{bmatrix} I_j \\ I_k \end{bmatrix} \in \mathbb{C}^6, \quad V \;:=\; \begin{bmatrix} V_j \\ V_k \end{bmatrix} \in \mathbb{C}^6$$

Transformer parameters:

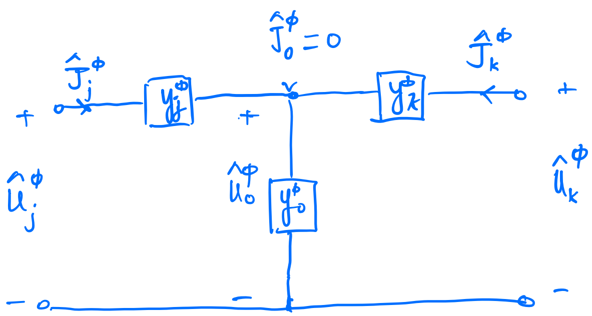
$$M := \begin{bmatrix} \text{diag}(1/N_j^{abc}) & 0 \\ 0 & \text{diag}(1/N_k^{abc}) \end{bmatrix} \in \mathbb{C}^{6\times 6}$$

$$y_i := \text{diag}\left(y_i^a, y_i^b, y_i^c\right) \in \mathbb{C}^{3\times 3}, \quad i = 0, j, k$$

Unitary voltage network **per phase**

Three-phase transformers

Unitary voltage network per phase



$$\hat J_j = y_k(\hat U_j - \hat U_0), \qquad \hat J_k = y_k(\hat U_k - \hat U_0)$$

$$y_0 \hat U_0 = \hat J_0 + \hat J_j + \hat J_k$$

Admittance matrix in $\mathbb{C}^{9\times 9}$

$$\begin{bmatrix} \hat J_0 \\ \hat J_j \\ \hat J_k \end{bmatrix} = \begin{bmatrix} \sum_i y_i & -y_j & -y_k \\ -y_j & y_j & 0 \\ -y_k & 0 & y_k \end{bmatrix} \begin{bmatrix} \hat U_0 \\ \hat U_j \\ \hat U_k \end{bmatrix}$$

Since $\hat J_0 = 0$, can eliminate $\hat U_0$ to obtain Kron reduced admittance matrix

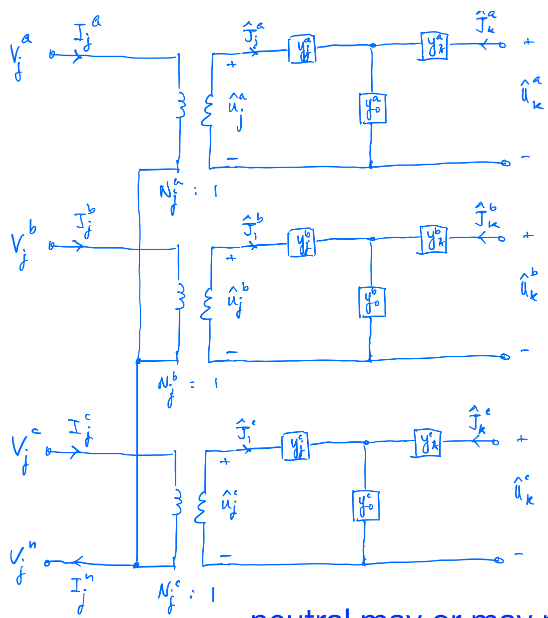
$$\hat J = Y_{\text{uvn}} \hat U$$

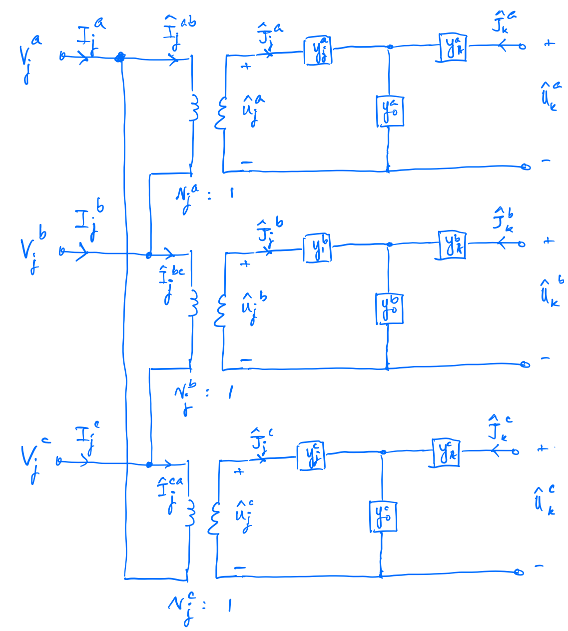
where

$$Y_{\text{uvn}} := \left(\mathbb{I}_2 \otimes \left(\sum_i y_i \right)^{-1} \right) \begin{bmatrix} y_j(y_0 + y_k) & -y_j y_k \\ -y_j y_k & y_k(y_0 + y_j) \end{bmatrix}$$

Three-phase transformers

External model: primary circuit





neutral may or may not be grounded!

Y config:

$$\hat{U}_j = M_j \left(V_j - V_j^n 1 \right), \qquad \hat{J}_j = M_j^{-1} I_j$$

Δ config:

$$\hat{U}_j = M_j \Gamma V_j, \qquad \hat{J}_j = M_j^{-1} I_j^{\Delta}, \qquad I_j = \Gamma^{\mathsf{T}} I_j^{\Delta}$$

Three-phase transformers

External model: conversion rule

Primary circuit

Y configuration: $\hat{U}_j = M_j \left(V_j - V_j^n 1 \right)$, $\hat{J}_j = M_j^{-1} I_j$

Δ configuration: $\hat{U}_j = M_j \Gamma V_j$, $\hat{J}_j = M_j^{-1} I_j^{\Delta}$, $I_j = \Gamma^{\mathsf{T}} I_j^{\Delta}$

Secondary circuit

Y configuration: $\hat{U}_k = M_k \left(V_k - V_k^n 1 \right)$, $\hat{J}_k = M_k^{-1} I_k$

Δ configuration: $\hat{U}_k = M_k \Gamma V_k$, $\hat{J}_k = M_k^{-1} I_k^{\Delta}$, $I_k = \Gamma^{\mathsf{T}} I_k^{\Delta}$

Three-phase transformers

External model: admittance matrix

Eliminate internal vars $(\hat{U}, \hat{J})$:

$$\boxed{I \;=\; D^{\mathsf{T}}(MY_{\mathsf{uvn}}M)D\left(V-\gamma\right)}$$

where

$\gamma := \left(V_j^n \mathbf{1}, V_k^n \mathbf{1}\right)$: neutral voltages in YY configuration

$D \in \mathbb{C}^{6\times 6}$: configuration dependent

YY config: $\quad D := \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{bmatrix}$

$\Delta\Delta$ config: $\quad D := \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix}$

ΔY config: $\quad D := \begin{bmatrix} \Gamma & 0 \\ 0 & \mathbb{I} \end{bmatrix}$

$Y\Delta$ config: $\quad D := \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \Gamma \end{bmatrix}$

For both single-phase & three-phase transformers:

- This model is equivalent to T equivalent circuit
- Different from simplified circuit (approximation)
- If shunt adm = 0, then all 3 models are equivalent