Optimal Demand Response and Power Flow

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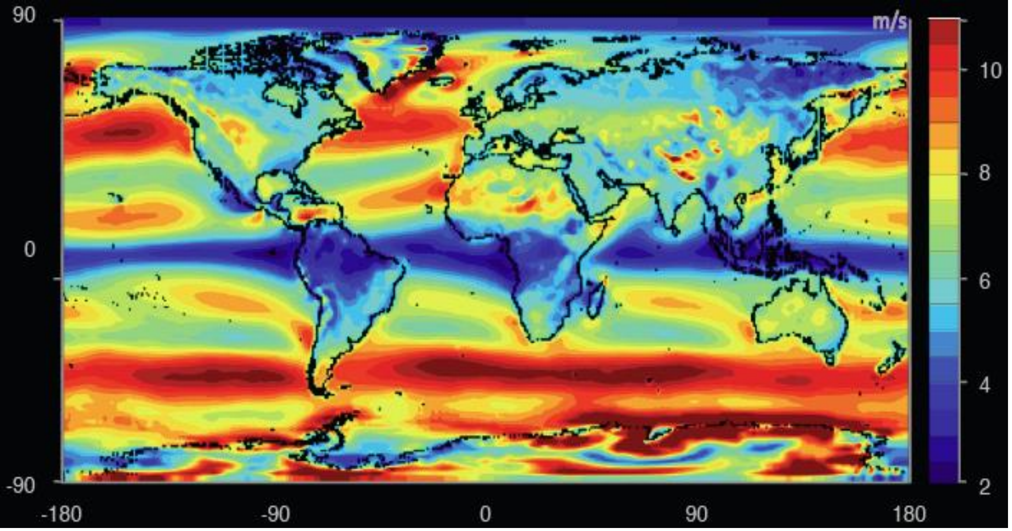
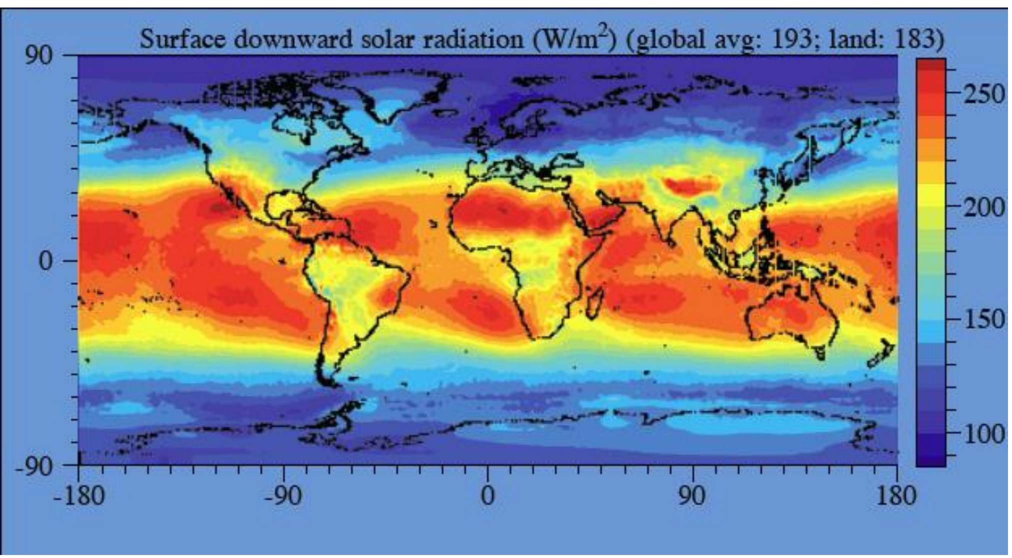


Wind power over land (exc. Antartica)
70 – 170 TW



Solar power over land
340 TW

Worldwide

energy demand:
16 TW

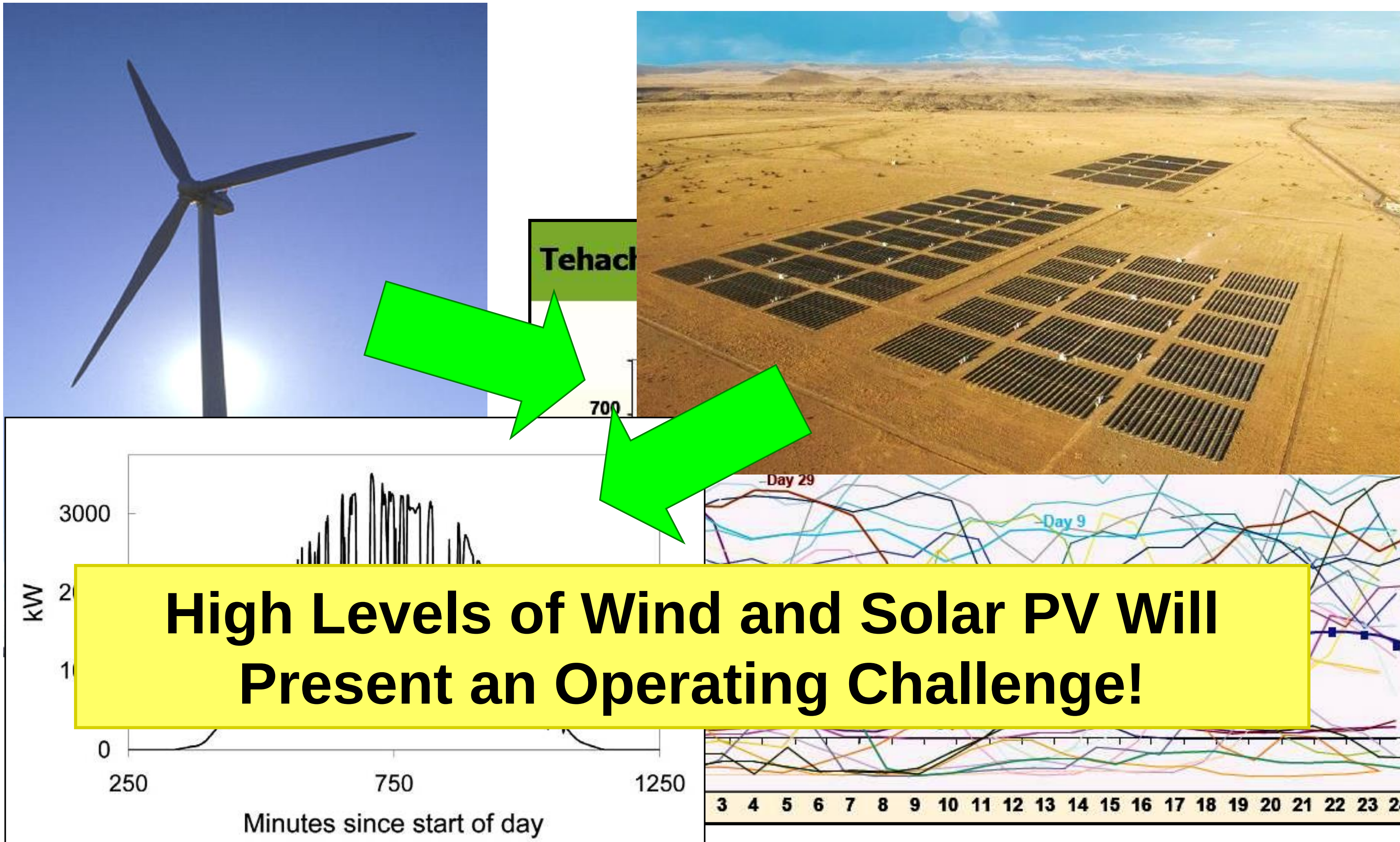
electricity demand:
2.2 TW

wind capacity (2009):
159 GW

grid-tied PV capacity (2009):
21 GW

Source: Renewable Energy Global Status Report, 2010
Source: M. Jacobson, 2011

Uncertainty



High Levels of Wind and Solar PV Will Present an Operating Challenge!

Source: Rosa Yang, EPRI

Implications

Current control paradigm works well today

- Low uncertainty, few active assets to control
- Centralized, open-loop, human-in-loop, worst-case preventive
- Schedule supplies to match loads

Future needs

- Fast computation to cope with rapid, random, large fluctuations in supply, demand, voltage, freq
- Simple algorithms to scale to large networks of active DER
- Real-time data for adaptive control, e.g. real-time DR

Outline

Optimal demand response

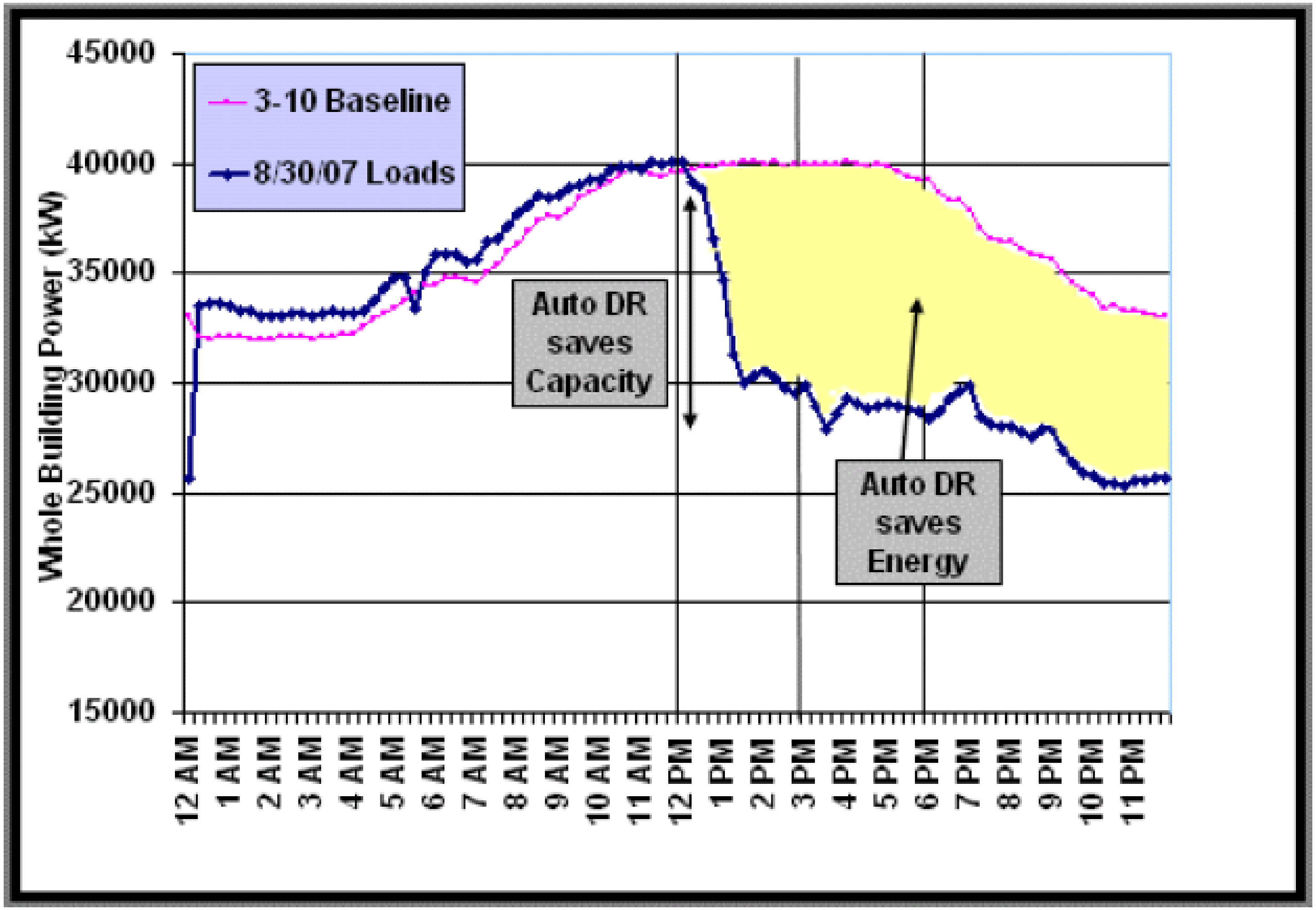
- With L. Chen, L. Jiang, N. Li

Optimal power flow

- With S. Bose, M. Chandy, C. Clarke, M. Farivar, D. Gayme, J. Lavaei

Automated Demand Response Saves Capacity and Energy

Electric load profile for PG&E participants on 8/30/2007



Source: Steven Chu, GridWeek 2009

Optimal demand response

Model

Results

- Uncorrelated demand: distributed alg
- Correlated demand: distributed alg
- Impact of uncertainty

Some refs:

- Kirschen 2003, S. Borenstein 2005, Smith et al 2007
- Caramanis & Foster 2010, 2011
- Varaiya et al 2011
- Ilic et al 2011

Optimal demand response

Model

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- L Chen, N. Li, L. Jiang and S. H. Low, Optimal demand response. In Control & Optimization Theory of Electric Smart Grids, Springer 2011
- L. Jiang and S. H. Low, CDC 2011, Allerton 2011

Features to capture

Wholesale markets

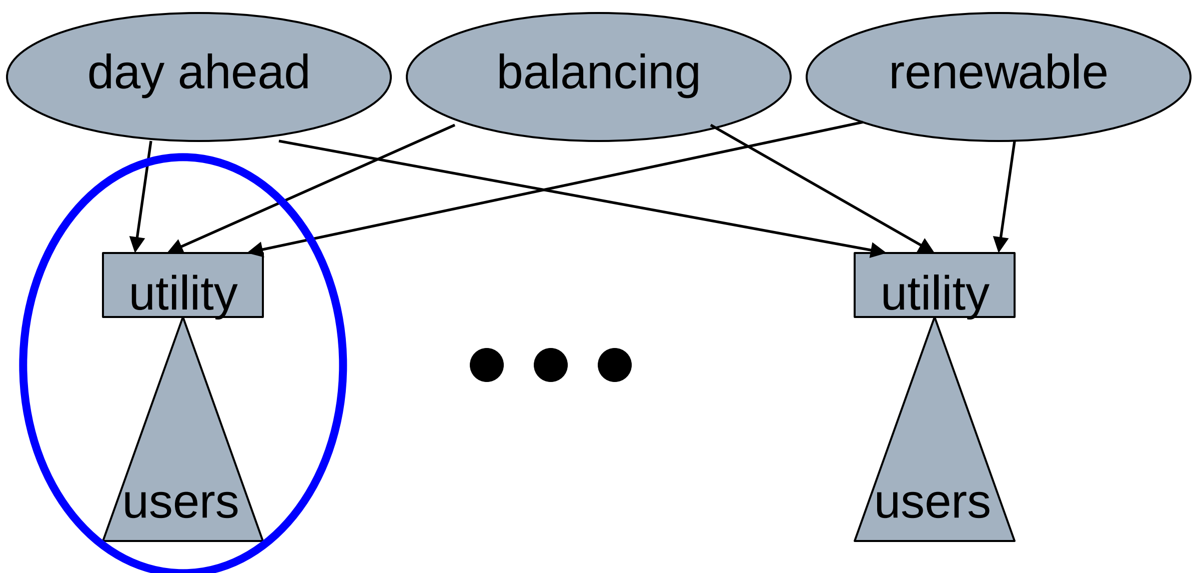
- Day ahead, real-time balancing

Renewable generation

- Non-dispatchable

Demand response

- Real-time control (through pricing)



Model: user

Each user has 1 appliance (wlog)

- Attains utility $u_i(x_i(t))$ when consumes $x_i(t)$

$$\underline{x}_i(t) \le x_i(t) \le \overline{x}_i(t) \qquad \sum_t x_i(t) \ge \overline{X}_i$$

Demand at t: $$D(t) := \sum_i x_i(t)$$

Model: LSE (load serving entity)

capacity

Power procurement

- Day-ahead power: $P_d(t), \; c_d\left(P_d(t)\right)$
  - Control, decided a day ahead

Model: LSE (load serving entity)

capacity

Power procurement

- Day-ahead power: $P_d(t), \;\; c_d\big(P_d(t)\big)$
  - Control, decided a day ahead
- Renewable power: $P_r(t), \;\; c_r\big(P_r(t)\big) = 0$
  - Random variable, realized in real-time

Model: LSE (load serving entity)

capacity

Power procurement

- Day-ahead power: $P_d(t),\ c_d\left(P_d(t)\right),\ c_o\left(\Delta x(t)\right)$
  - Control, decided a day ahead
- Renewable power: $P_r(t),\ c_r\left(P_r(t)\right) = 0$
  - Random variable, realized in real-time

energy

Model: LSE (load serving entity)

capacity

Power procurement

- Day-ahead power: $P_d(t)$, $c_d\left(P_d(t)\right)$, $c_o\left(\Delta x(t)\right)$
  - Control, decided a day ahead
- Renewable power: $P_r(t)$, $c_r\left(P_r(t)\right) = 0$
  - Random variable, realized in real-time
- Real-time balancing power: $P_b(t)$, $c_b\left(P_b(t)\right)$
  - $P_b(t) = D(t) - P_r(t) - P_d(t)$

energy

- Use as much renewable as possible
- Optimally provision day-ahead power
- Buy sufficient real-time power to balance demand

Simplifying assumption

- No network constraints

Objective

Day-ahead decision

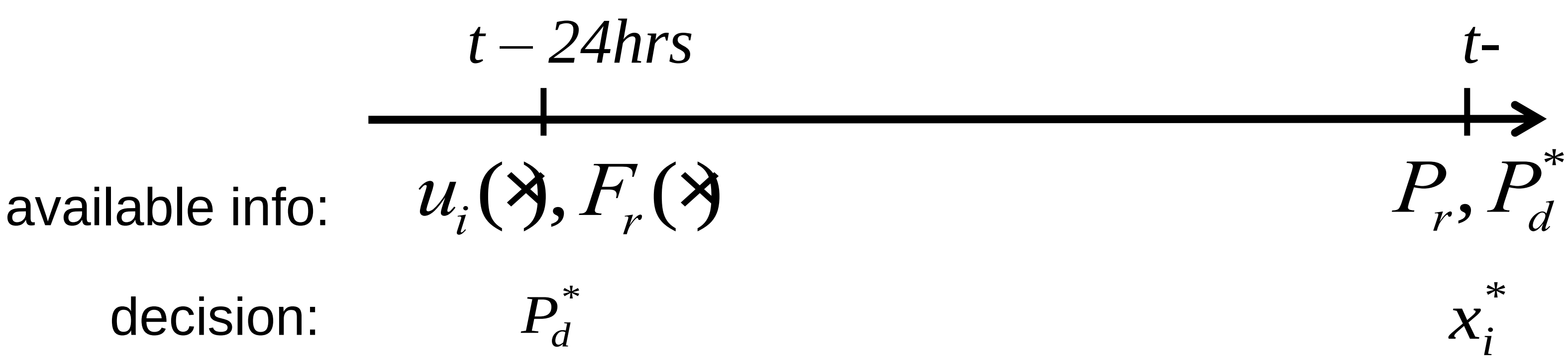
- How much power P_d should LSE buy from day-ahead market?

Real-time decision (at *t*-)

- How much x_i should users consume, given realization of wind power P_r and P_d ?

How to compute these decisions distributively?

How does closed-loop system behave ?



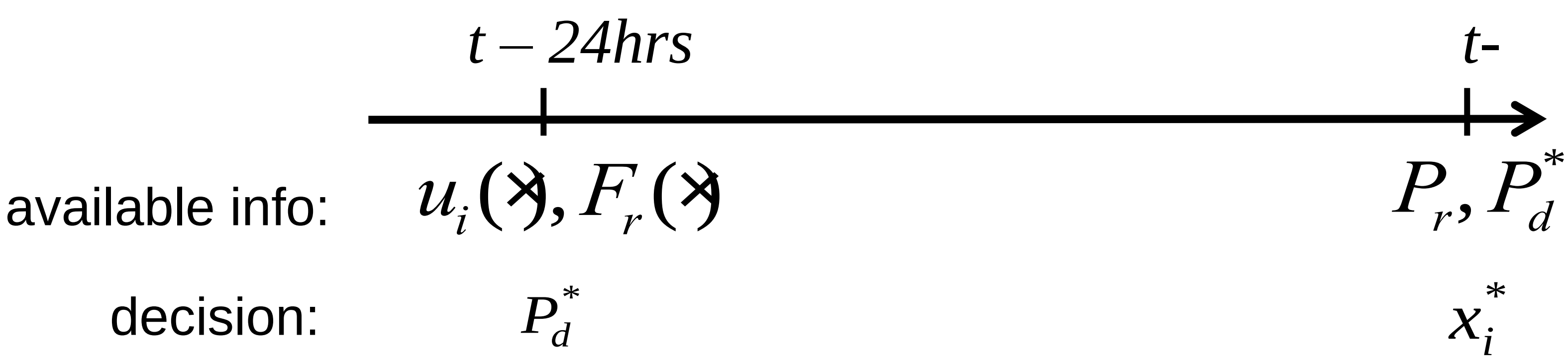
Objective

Real-time (at *t*-)

- Given P_d and realizations of P_r, choose optimal $x_i^* = x_i^*(P_d; P_r)$ to max social welfare

Day-ahead

- Choose optimal P_d^* that maximizes expected optimal social welfare



Optimal demand response

Model

Results

- Uncorrelated demand: distributed alg
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- Impact of uncertainty

Uncorrelated demand: T=1

Each user has 1 appliance (wlog)

- Attains utility $u_i(x_i(t))$ when consumes $x_i(t)$

$$\underline{x}_i(t) \le x_i(t) \le \overline{x}_i(t)$$

~~$\sum_t x_i(t) \ge \overline{X}_i$~~

Demand at t: $D(t) := \sum_i x_i(t)$

drop t for this case

Welfare function

Supply cost

$$c\left(P_d, x\right) = c_d\left(P_d\right) + c_o\left(D(x)\right)_0^{P_d} + c_b\left(D(x) - P_d\right)_+$$

$$D(x) := \sum_i x_i - P_r$$

excess demand

Welfare function

Supply cost

$$c\left(P_d, x\right) = c_d\left(P_d\right) + c_o\left(\Delta(x)\right)_0^{P_d} + c_b\left(\Delta(x) - P_d\right)_+$$

$$\Delta(x) := \sum_i x_i - P_r$$

← excess demand

Welfare function (random)

$$W\left(P_d, x\right) = \sum_i u_i(x_i) - c\left(P_d, x\right)$$

user utility ↑ supply cost ↑

Optimal operation

Welfare function (random)

$$W(P_d, x) = \sum_i u_i(x_i) - c(P_d, x)$$

Optimal real-time demand response

$$\max_x W(P_d, x)$$

given realization of P_r

Optimal operation

Welfare function (random)

$$W(P_d, x) = \sum_i u_i(x_i) - c(P_d, x)$$

Optimal real-time demand response

$$x^*(P_d) \quad := \quad \arg \max_x W(P_d, x)$$

given realization of P_r

Optimal operation

Welfare function (random)

$$W(P_d, x) = \sum_i u_i(x_i) - c(P_d, x)$$

Optimal real-time demand response

$$x^*(P_d) \;:=\; \arg \max_x W(P_d, x)$$

given realization of P_r

Optimal day-ahead procurement

$$P_d^* \;:=\; \arg \max_{P_d} \mathrm{E} W(P_d, x^*(P_d))$$

Overall problem: $\max_{P_d} \mathrm{E} \max_x W(P_d, x)$

Real-time DR vs scheduling

- Real-time DR: $$\max_{P_d} \mathrm{E} \max_{x} W\left(P_d, x\right)$$
- Scheduling: $$\max_{P_d} \max_{x} \mathrm{E}\ W\left(P_d, x\right)$$

Theorem

Under appropriate assumptions:

$$W^*_{real-time\ DR} = W^*_{scheduling} + \frac{Ng^2}{1+Ng} s^2$$

benefit increases with

- uncertainty s^2
- marginal real-time cost g

Algorithm 1 (real-time DR)

$$\max_{P_d} \; \mathrm{E} \; \underbrace{\max_{x} W\left(P_d, x\right)}_{\text{real-time DR}}$$

Active user i computes x_i^*

- Optimal consumption

LSE computes

- Real-time "price" m_b^*

Algorithm 1 (real-time DR)

Active user i :

$$x_i^{k+1} = \left(x_i^k + \gamma\left(u_i'\left(x_i^k\right) - \mu_b^k\right)\right)_{\underline{x}_i}^{\overline{x}_i}$$

inc if marginal utility > real-time price

LSE :

$$\mu_b^{k+1} = \left(\mu_b^k + \gamma\left(\Delta\left(x^k\right) - y_o^k - y_b^k\right)\right)_+$$

inc if total demand > total supply

- Decentralized
- Iterative computation at t-

Algorithm 1 (real-time DR)

Theorem: Algorithm 1

Socially optimal

- Converges to welfare-maximizing DR $x^* = x^*(P_d)$
- Real-time price aligns marginal cost of supply with individual marginal utility

$$\mu_b^* = c'\left(P_d, D\left(x^*\right)\right) = u_i'\left(x_i^*\right)$$

Incentive compatible

- x_i^* max *i*'s surplus given price μ_b^*

Algorithm 2 (day-ahead procurement)

Optimal day-ahead procurement

$$\max_{P_d} \; \mathrm{E}W\left(P_d, x^*\left(P_d\right)\right)$$

LSE: $$P_d^{m+1} = \left(P_d^m + g^m\left(m_o^m - c_d'\left(P_d^m\right)\right)\right)_+$$

calculated from Monte Carlo simulation of Alg 1 (stochastic approximation)

Algorithm 2 (day-ahead procurement)

Theorem

Algorithm 2 converges a.s. to optimal P_d^* for appropriate stepsize g^k

Optimal demand response

Model

Results

- Uncorrelated demand: distributed alg
- Correlated demand: distributed alg
- Impact of uncertainty

Impact of renewable on welfare

Renewable power:

$$P_r(t;a,b) := a \times m(t) + b \times V(t)$$

mean — $m(t)$; zero-mean RV — $V(t)$

Optimal welfare of (1+T)-period DP

$$W^*(a,b)$$

Impact of renewable on welfare

$$P_r(t;a,b) := a \times m(t) + b \times V(t)$$

<u>Theorem</u>

- $W^*(a,b)$ increases in a, decreases in b
- $W^*(s,s)$ increases in s (plant size)

With ramp rate costs

Day-ahead ramp cost $s_d(t) := f_d\big(P_d(t), P_d(t+1)\big)$

Real-time ramp cost $s_b(t) := f_b\big(P_b(t), P_b(t+1)\big)$

Social welfare

$$W^*(a,b) := E\left\{ \sum_{t=1}^{T} W_t(x(t), P_d(t); P_r(t)) - \sum_{t=1}^{T-1} \big(s_d(t) + s_b(t)\big) \right\}$$

<u>Theorem</u>

- $W^*(a,b)$ increases in a, decreases in b
- $W^*(s,s)$ increases in s (plant size)

Outline

Optimal demand response

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Optimal power flow

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Optimal power flow (OPF)

- OPF is solved routinely to determine
  - How much power to generate where
  - Market operation & pricing
  - Parameter setting, e.g. taps, VARs

- Non-convex and hard to solve
  - Huge literature since 1962
  - In practice, operators often use heuristics to find a feasible operating point
  - Or solve DC power flow (LP)

Optimal power flow (OPF)

Problem formulation

- Carpentier 1962

Computational techniques:

- Dommel & Tinney 1968
- Surveys: Huneault et al 1991, Momoh et al 2001, Pandya et al 2008

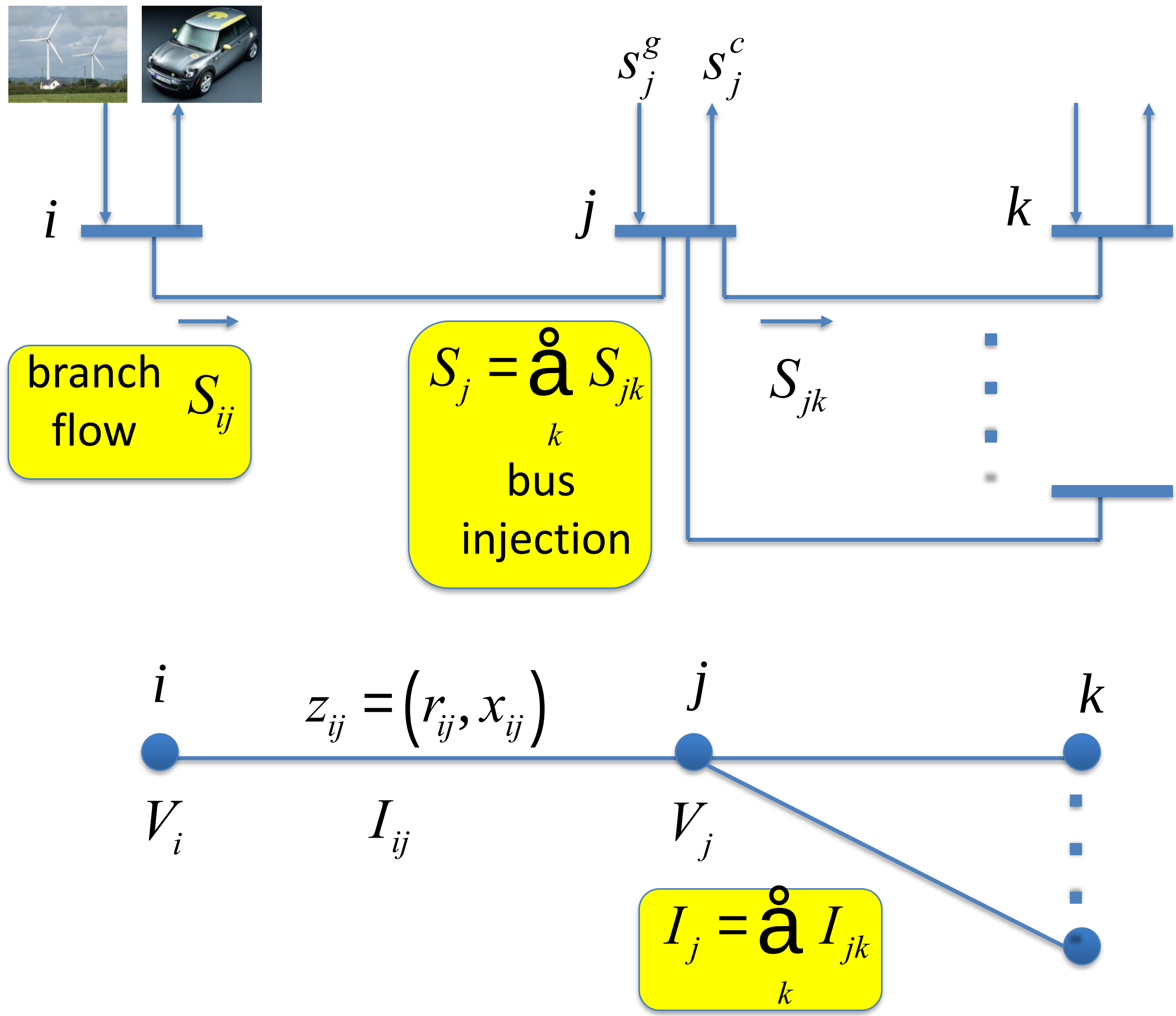
Bus injection model (SDP formulation):

- Bai et al 2008, 2009, Lavaei et al 2010
- Bose et al 2011, Sojoudi et al 2011, Zhang et al 2011
- Lesieutre et al 2011

Branch flow model

- Baran & Wu 1989, Chiang & Baran 1990, Farivar et al 2011

Models



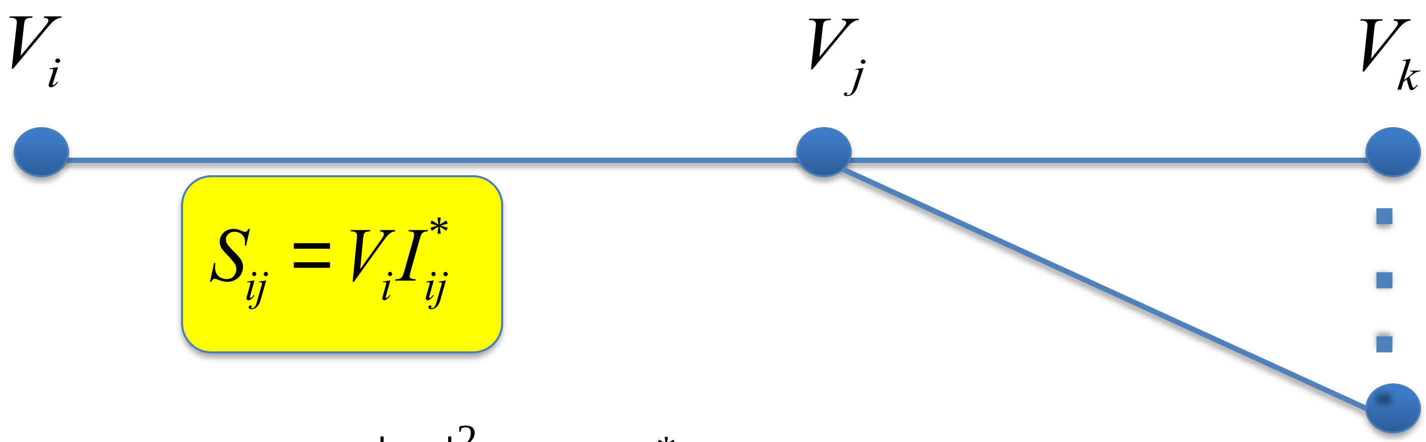
s_j^g
s_j^c
i
j
k
branch flow S_{ij}
$S_j = \sum_k S_{jk}$
bus injection
S_{jk}
$z_{ij} = (r_{ij}, x_{ij})$
V_i
I_{ij}
V_j
$I_j = \sum_k I_{jk}$

Models: Kirchhoff's law

$$S_i = \sum_j S_{ij} = V_i I_i^*$$

linear relation:

$$I = YV$$



$$S_{ij} = V_i I_{ij}^*$$

$$S_{ij} = \frac{|V_i|^2}{z_{ij}^*} - \frac{V_i V_j^*}{z_{ij}^*}$$

Outline: OPF

SDP relaxation

- Bus injection model

Conic relaxation

- Branch flow model

Application

Bus injection model

Nodes *i* and *j* are linked with an admittance $y_{ij} = g_{ij} - \mathbf{i} b_{ij}$

$$Y_{ij} = \begin{cases} y_{ii} + \sum_{j \sim i} y_{ij}, & \text{if } i = j \\ -y_{ij}, & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$

- Kirchhoff's Law: $I = YV$

Classical OPF

$$\min \sum_{k \in G} f_k\left(P_k^g\right)$$ ← Generation cost

$$\text{over} \quad \left(S_k^g,\ V_k\right)$$

subject to

$$\underline{S}_k^g \le S_k^g \le \overline{S}_k^g$$ Generation power constraints

$$\underline{V}_k \le |V_k| \le \overline{V}_k$$ Voltage magnitude constraints

$$I = YV$$ Kirchhoff law

$$V_k I_k^* = S_k^g - S_k^c$$ Power balance

Classical OPF

In terms of V:

$$P_k = \mathrm{tr}\, \Phi_k V V^*$$

$$Q_k = \mathrm{tr}\, \Psi_k V V^*$$

$$\Phi_k := \left(\frac{Y_k^* + Y_k}{2} \right)$$

$$\Psi_k := \left(\frac{Y_k^* - Y_k}{2\mathbf{i}} \right)$$

$$\min \quad \sum_{k \in G} \mathrm{tr}\, M_k V V^*$$

$$\text{over} \quad V$$

$$\text{s.t.} \quad \underline{P}_k^g - P_k^d \le \mathrm{tr}\, \Phi_k V V^* \le \overline{P}_k^g - P_k^d$$

$$\underline{Q}_k^g - Q_k^d \le \mathrm{tr}\, \Psi_k V V^* \le \overline{Q}_k^g - Q_k^d$$

$$\underline{V}_k^2 \le \mathrm{tr}\, J_k V V^* \le \overline{V}_k^2$$

Key observation [Bai et al 2008]:
OPF = rank constrained SDP

Classical OPF

$$\min \quad \sum_{k \in G} \operatorname{tr} M_k W$$

$$\text{over} \quad W \ \text{positive semidefinite matrix}$$

$$\text{s.t.} \quad \underline{P}_k \le \operatorname{tr} \Phi_k W \le \overline{P}_k$$

$$\underline{Q}_k \le \operatorname{tr} \Psi_k W \le \overline{Q}_k$$

$$\underline{V}_k^2 \le \operatorname{tr} J_k W \le \overline{V}_k^2$$

$W \ge 0$, ~~rank $W = 1$~~

convex relaxation: SDP

Semi-definite relaxation

Non-convex QCQP

↓

Rank-constrained SDP

↓

Relax the rank constraint and solve the SDP

↓

Does the optimal solution satisfy the rank-constraint?

yes → We are done!

no → Solution may not be meaningful

SDP relaxation of OPF

$$\min \quad \sum_{k \in G} \operatorname{tr} M_k W$$

over $\quad W$ positive semidefinite matrix

$$\text{s.t.} \quad \underline{P}_k \le \operatorname{tr} \Phi_k W \le \overline{P}_k \qquad \underline{\lambda}_k, \overline{\lambda}_k$$

$$\underline{Q}_k \le \operatorname{tr} \Psi_k W \le \overline{Q}_k \qquad \underline{\mu}_k, \overline{\mu}_k$$

$$\underline{V}_k^2 \le \operatorname{tr} J_k W \le \overline{V}_k^2 \qquad \underline{\gamma}_k, \overline{\gamma}_k$$

$$W \succeq 0$$

Lagrange multipliers

$$A(\lambda_k, \mu_k, \gamma_k) := \sum_{k \in G} M_k + \sum_k \left(\lambda_k \Phi_k + \mu_k \Psi_k + \gamma_k J_k \right)$$

Sufficient condition

Theorem

If A^{opt} has rank n-1 then

- W^{opt} has rank 1, SDP relaxation is exact
- Duality gap is zero
- A globally optimal V^{opt} can be recovered

All IEEE test systems (essentially) satisfy the condition!

J. Lavaei and S. H. Low: Zero duality gap in optimal power flow problem. Allerton 2010, TPS 2011

OPF over radial networks

Suppose

- tree (radial) network
- no lower bounds on power injections

Theorem

A^{opt} always has rank n-1

- W^{opt} always has rank 1 (exact relaxation)
- OPF always has zero duality gap
- Globally optimal V^{opt} solvable efficiently

S. Bose, D. Gayme, S. H. Low and M. Chandy, OPF over tree networks. Allerton 2011

OPF over radial networks

Suppose

- tree (radial) network
- no lower bounds on power injections

<u>Theorem</u>

A^{opt} always has rank n-1

- W^{opt} always has rank 1 (exact relaxation)
- OPF always has zero duality gap
- Globally optimal V^{opt} solvable efficiently

Also: B. Zhang and D. Tse, Allerton 2011
S. Sojoudi and J. Lavaei, submitted 2011

QCQP over tree

QCQP (C, C_k)

$$\begin{array}{lll} \text{min} & x^* C x & \\ \text{over} & x \in \mathbf{C}^n & \\ \text{s.t.} & x^* C_k x \le b_k & k \in K \end{array}$$

graph of QCQP

$$G(C, C_k) \text{ has edge } (i, j) \Leftrightarrow$$

$$C_{ij} \neq 0 \text{ or } [C_k]_{ij} \neq 0 \text{ for some } k$$

QCQP over tree

$$G(C, C_k) \text{ is a tree}$$

QCQP over tree

QCQP (C, C_k)

$$\min \quad x^* C x$$
$$\text{over} \quad x \in \mathbf{C}^n$$
$$\text{s.t.} \quad x^* C_k x \le b_k \qquad k \in K$$

Semidefinite relaxation

$$\min \quad \operatorname{tr} CW$$
$$\text{over} \quad W \succeq 0$$
$$\text{s. t.} \quad \operatorname{tr} C_k W \le b_k \qquad k \in K$$

QCQP over tree

QCQP (C, C_k)

$$\begin{array}{ll} \min & x^* C x \\ \text{over} & x \in \mathbf{C}^n \\ \text{s.t.} & x^* C_k x \le b_k \qquad k \in K \end{array}$$

Key assumption

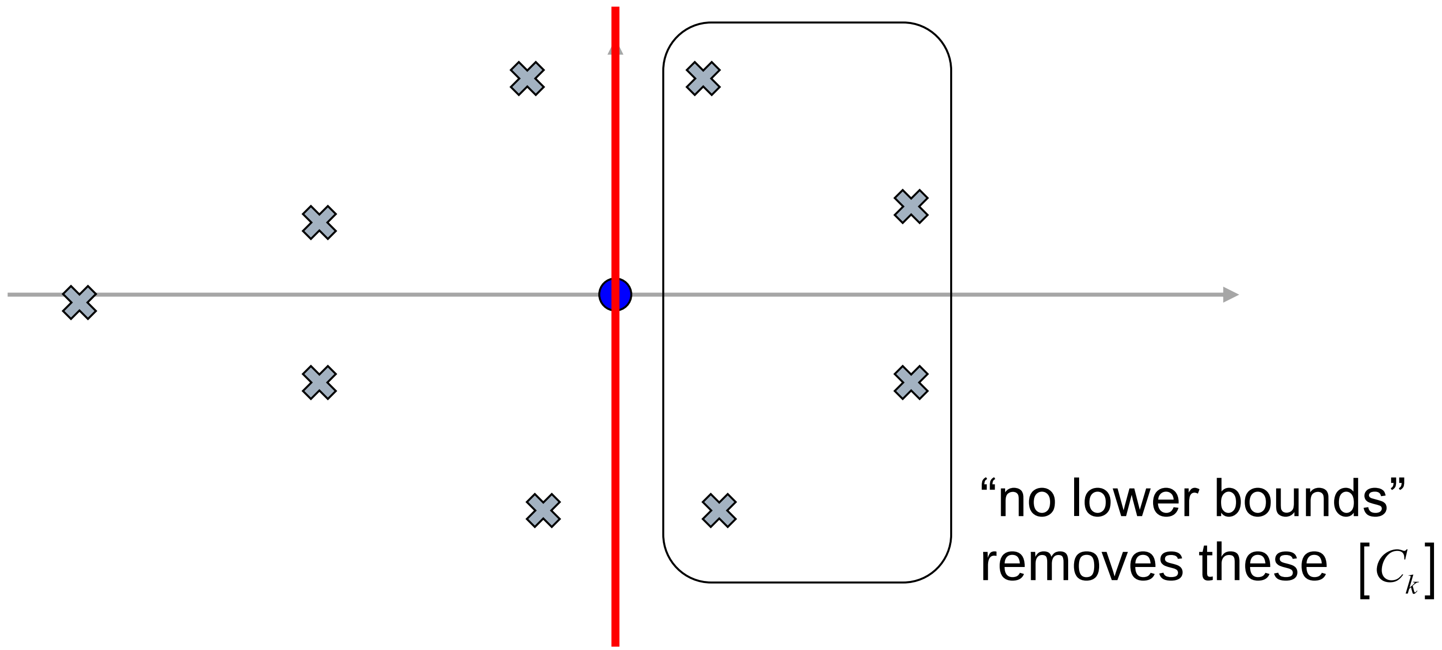
$$(i,j) \in G(C, C_k): \quad 0 \notin \text{int conv hull}\left(C_{ij}, [C_k]_{ij}, \ \forall k\right)$$

Theorem

Semidefinite relaxation is exact for QCQP over tree

S. Bose, D. Gayme, S. H. Low and M. Chandy, submitted March 2012

OPF over radial networks

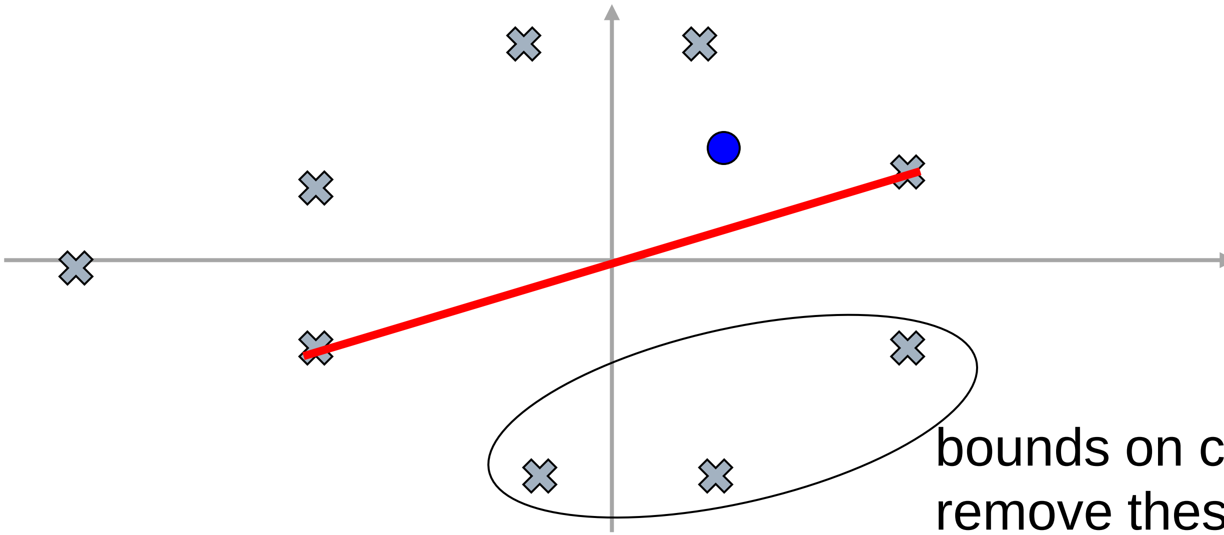


Theorem

A^{opt} always has rank *n-1*

- W^{opt} always has rank 1 (exact relaxation)
- OPF always has zero duality gap V^{opt}
- Globally optimal ... solvable efficiently

OPF over radial networks



bounds on constraints remove these $[C_k]_{ij}$

Theorem

A^{opt} always has rank n-1

- W^{opt} always has rank 1 (exact relaxation)
- OPF always has zero V^{opt} duality gap
- Globally optimal ... solvable efficiently

Outline: OPF

SDP relaxation

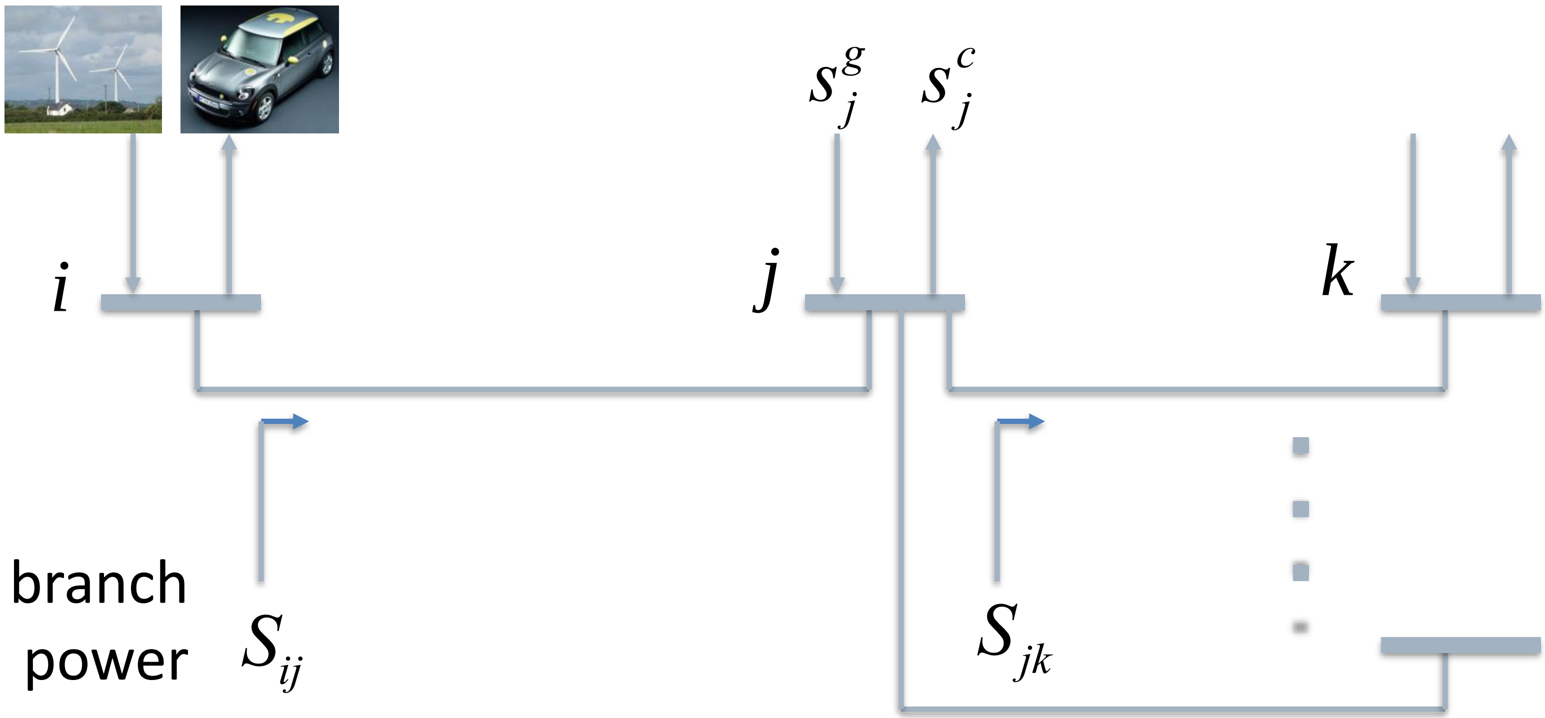
- Bus injection model

Conic relaxation

- Branch flow model

Application

Branch flow model

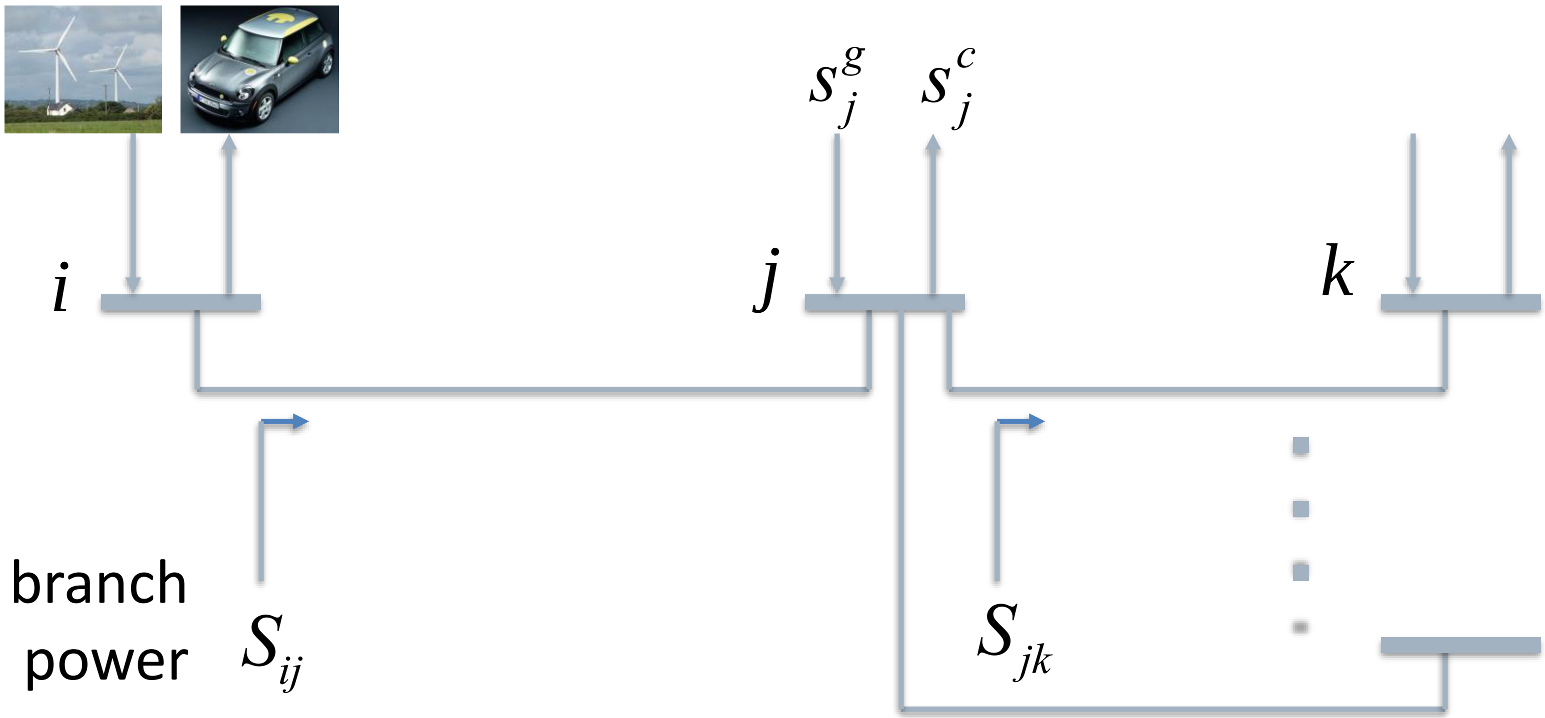


Kirchhoff's Law: $$S_{ij} = \sum_{k:j\sim k} S_{jk} + z_{ij}\left|I_{ij}\right|^2 + s_j^c - s_j^g$$

line loss

load - gen

Branch flow model



Kirchhoff's Law: $S_{ij} = \sum_{k:j\sim k} S_{jk} + z_{ij}\left|I_{ij}\right|^2 + s_j^c - s_j^g$

Ohm's Law: $V_j = V_i - z_{ij} I_{ij}$

$S_{ij} = V_i I_{ij}^*$

OPF using branch flow model

$$\min \quad \sum_{i\sim j} r_{ij} l_{ij} + \sum_{i} a_i v_i$$

$$l_{ij} := |I_{ij}|^2$$

$$v_i := |V_i|^2$$

real power loss

CVR (conservation voltage reduction)

OPF using branch flow model

$$\min \quad \sum_{i\sim j} r_{ij} l_{ij} + \sum_{i} a_i v_i$$

$$\text{over} \quad (S, I, V, s^g, s^c)$$

$$\text{s. t.} \quad \underline{s}_i^g \le s_i^g \le \overline{s}_i^g \qquad \underline{s}_i \le s_i^c$$

$$\underline{v}_i \le v_i \le \overline{v}_i$$

Kirchhoff's Law: $S_{ij} = \sum_{k: j\sim k} S_{jk} + z_{ij} \left| I_{ij} \right|^2 + s_j^c - s_j^g$

Ohm's Law: $V_j = V_i - z_{ij} I_{ij}$

$$S_{ij} = V_i I_{ij}^*$$

OPF using branch flow model

$$\min \quad \sum_{i\sim j} r_{ij} l_{ij} + \sum_{i} a_i v_i$$

$$\text{over} \quad (S, I, V, s^g, s^c)$$

$$\text{s. t.} \quad \underline{s}_i^g \le s_i^g \le \overline{s}_i^g \qquad \underline{s}_i \le s_i^c$$

$$\underline{v}_i \le v_i \le \overline{v}_i$$

demands

Kirchhoff's Law: $S_{ij} = \sum_{k:j\sim k} S_{jk} + z_{ij} \left| I_{ij} \right|^2 + s_j^c - s_j^g$

Ohm's Law: $V_j = V_i - z_{ij} I_{ij}$

$$S_{ij} = V_i I_{ij}^*$$

OPF using branch flow model

$$\min \quad \sum_{i\sim j} r_{ij} l_{ij} + \sum_{i} \alpha_i v_i$$

$$\text{over} \quad (S, I, V, s^g, s^c)$$

$$\text{s. t.} \quad \underline{s}_i^g \le s_i^g \le \overline{s}_i^g \qquad \underline{s}_i \le s_i^c$$

$$\underline{v}_i \le v_i \le \overline{v}_i$$

generation

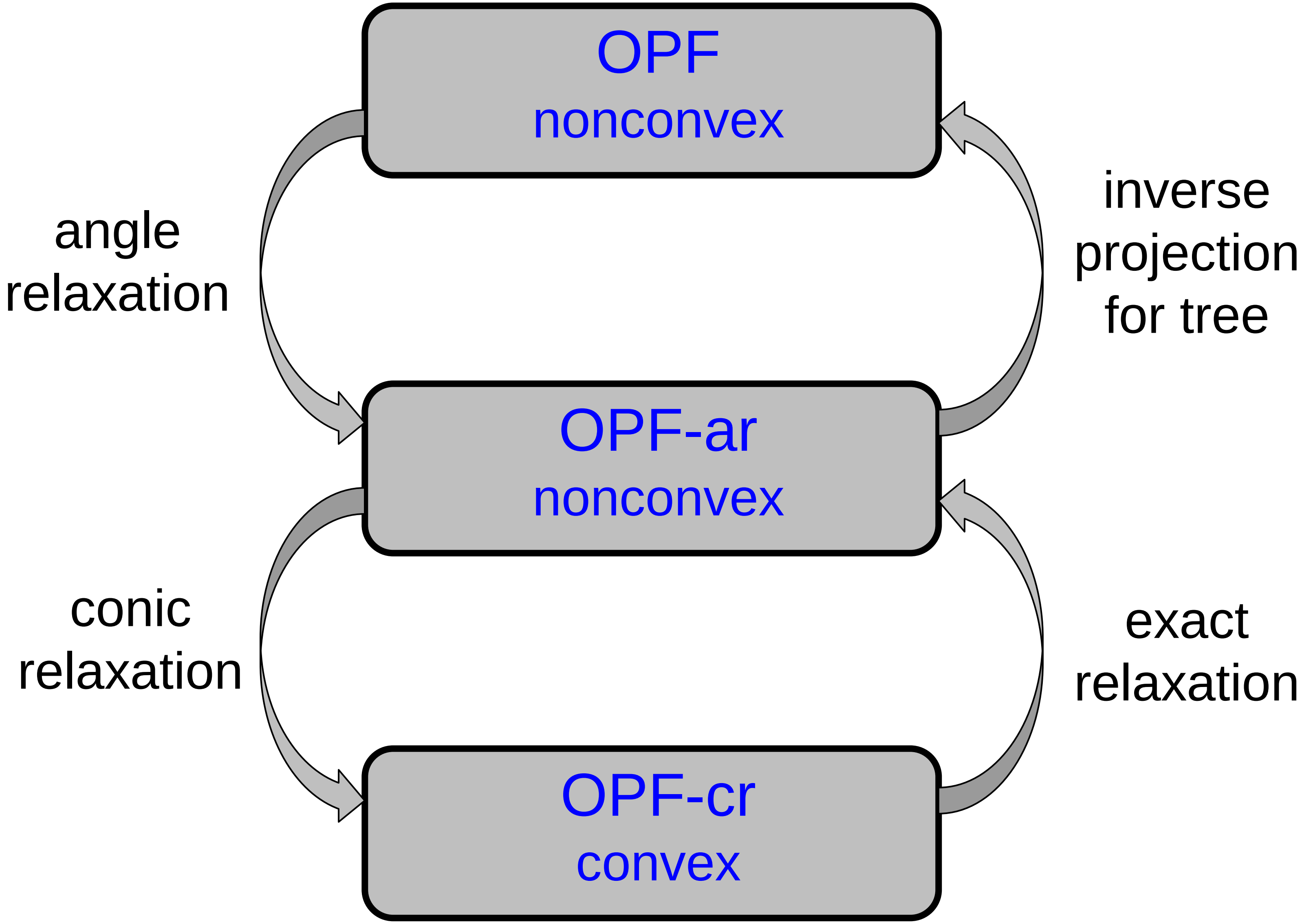
VAR control

Kirchhoff's Law: $S_{ij} = \sum_{k:j\sim k} S_{jk} + z_{ij} \left| I_{ij} \right|^2 + s_j^c - s_j^g$

Ohm's Law: $V_j = V_i - z_{ij} I_{ij}$

$$S_{ij} = V_i I_{ij}^*$$

Solution strategy



Angle relaxation

Kirchhoff's Law: $S_{ij} = \sum_{k:j\sim k} S_{jk} + z_{ij}|I_{ij}|^2 + s_j^c - s_j^g$

Angles of I_{ij}, V_i eliminated !

Points relaxed to circles

$$|V_i|^2 = |V_j|^2 + 2\left(r_{ij}P_{ij} + x_{ij}Q_{ij}\right) - \left(r_{ij}^2 + x_{ij}^2\right)|I_{ij}|^2$$

$$|I_{ij}|^2 = \left(\frac{P_{ij}^2 + Q_{ij}^2}{|V_i|^2}\right)$$

Baran and Wu 1989
for **radial** networks

Angle relaxation

$$P_{ij} = \sum_{k:j\sim k} P_{jk} + r_{ij}\left|I_{ij}\right|^2 + p_j^c - p_j^g$$

$$Q_{ij} = \sum_{k:j\sim k} Q_{jk} + x_{ij}\left|I_{ij}\right|^2 + q_j^c - q_j^g$$

$$\left|V_i\right|^2 = \left|V_j\right|^2 + 2\left(r_{ij}P_{ij} + x_{ij}Q_{ij}\right) - \left(r_{ij}^2 + x_{ij}^2\right)\left|I_{ij}\right|^2$$

$$\left|I_{ij}\right|^2 = \left(\frac{P_{ij}^2 + Q_{ij}^2}{\left|V_i\right|^2}\right)$$

Baran and Wu 1989
for **radial** networks

OPF-ar

$$l_{ij} := |I_{ij}|^2$$
$$v_i := |V_i|^2$$

- Linear objective
- Linear constraints
- Quadratic equality

$$\min \quad \sum_{i\sim j} r_{ij} l_{ij} + \sum_i a_i v_i$$

$$\text{over} \quad (S, l, v, s^g, s^c)$$

$$\text{s. t.} \quad P_{ij} = \sum_{k:j\sim k} P_{jk} + r_{ij} l_{ij} + p_j^c - p_j^g$$

$$Q_{ij} = \sum_{k:j\sim k} Q_{jk} + x_{ij} l_{ij} + q_j^c - q_j^g$$

$$v_i = v_j + 2\left(r_{ij} P_{ij} + x_{ij} Q_{ij}\right) - \left(r_{ij}^2 + x_{ij}^2\right) l_{ij}$$

$$l_{ij} = \left(\frac{P_{ij}^2 + Q_{ij}^2}{v_i}\right) \qquad \underline{s}_i \le s_i^g \le \overline{s}_i$$

$$\underline{v}_i \le v_i \le \overline{v}_i, \qquad \underline{s}_i \le \overline{s}_i^c$$

OPF-cr

$$\min \sum_{i\sim j} r_{ij} l_{ij} + \sum_{i} a_i v_i$$

$$\text{over} \quad (S, l, v, s^g, s^c)$$

$$\text{s. t.} \quad P_{ij} = \sum_{k:j\sim k} P_{jk} + r_{ij} l_{ij} + p_j^c - p_j^g$$

$$Q_{ij} = \sum_{k:j\sim k} Q_{jk} + x_{ij} l_{ij} + q_j^c - q_j^g$$

$$v_i = v_j + 2\left(r_{ij} P_{ij} + x_{ij} Q_{ij}\right) - \left(r_{ij}^2 + x_{ij}^2\right) l_{ij}$$

Quadratic inequality

$$l_{ij} \geq \left(\frac{P_{ij}^2 + Q_{ij}^2}{v_i}\right) \qquad \underline{s}_i \leq s_i^g \leq \overline{s}_i$$

$$\underline{v}_i \leq v_i \leq \overline{v}_i, \qquad \underline{s}_i \leq s_i^c$$

OPF over radial networks

Theorem

Both relaxation steps are exact

- OPF-cr is convex and exact
- Phase angles can be uniquely determined

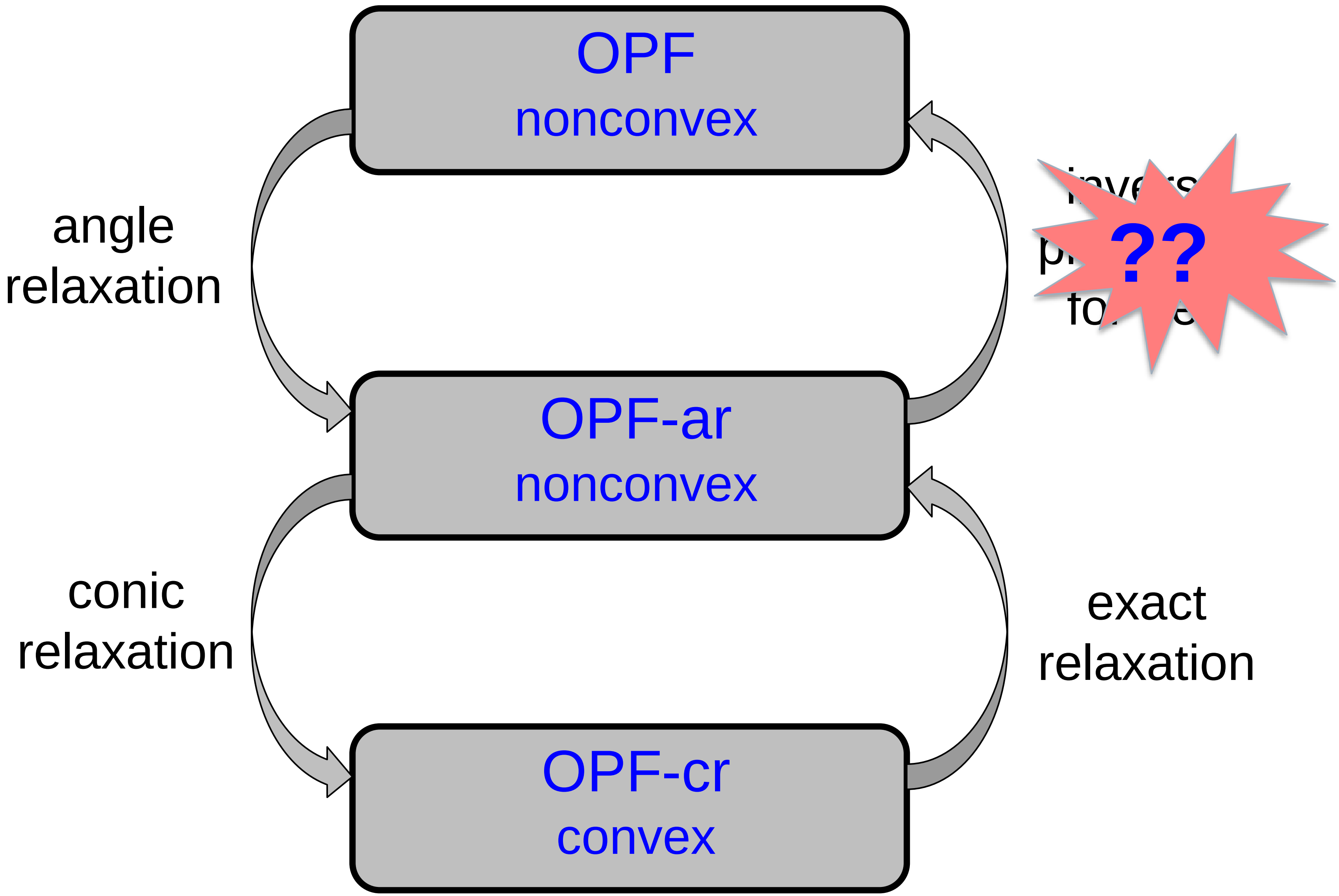
OPF-ar has zero duality gap

M. Farivar, C. Clarke, S. H. Low and M. Chandy, Inverter VAR control for distribution systems with renewables. SmartGridComm 2011

What about mesh networks ??

M. Farivar and S. H. Low, submitted March 2012

Solution strategy



OPF using branch flow model

$$\min \quad \sum_{i\sim j} r_{ij} l_{ij} + \sum_{i} a_i v_i$$

$$\text{over} \quad (S, I, V, s^g, s^c)$$

$$\text{s. t.} \quad \underline{s}_i^g \le s_i^g \le \overline{s}_i^g \qquad \underline{s}_i \le s_i^c$$

$$\underline{v}_i \le v_i \le \overline{v}_i$$

Kirchoff's Law: $S_{ij} = \sum_{k:j\sim k} S_{jk} + z_{ij} \left| I_{ij} \right|^2 + s_j^c - s_j^g$

Ohm's Law: $V_j = V_i - z_{ij} I_{ij}$

$$S_{ij} = V_i I_{ij}^*$$

Convexification of mesh networks

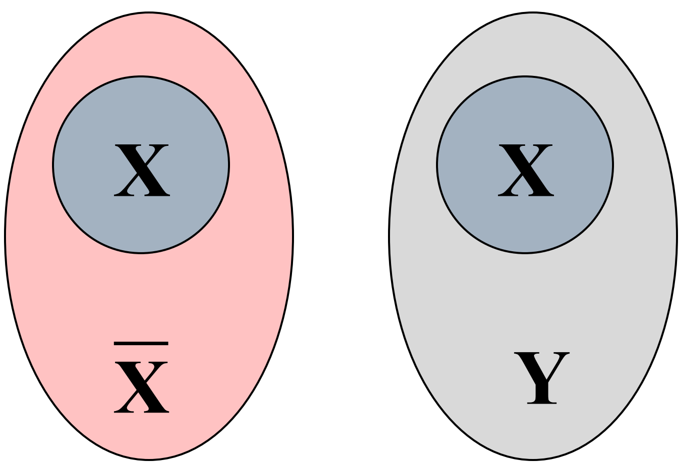
OPF $\qquad \min_{x} \; f\left(\hat{h}(x)\right) \quad \text{s.t.} \quad x \in \mathbf{X}$

OPF-ar $\qquad \min_{x} \; f\left(\hat{h}(x)\right) \quad \text{s.t.} \quad x \in \mathbf{Y}$

OPF-ps $\qquad \min_{x,f} \; f\left(\hat{h}(x)\right) \quad \text{s.t.} \quad x \in \overline{\mathbf{X}}$

<u>**Theorem**</u>

- $\overline{\mathbf{X}} = \mathbf{Y}$
- Need phase shifters only outside spanning tree



Key message

Radial networks computationally simple

- Exploit tree graph & convex relaxation
- Real-time scalable control promising

Mesh networks can be convexified

- Design for simplicity
- Need few phase shifters (sparse topology)

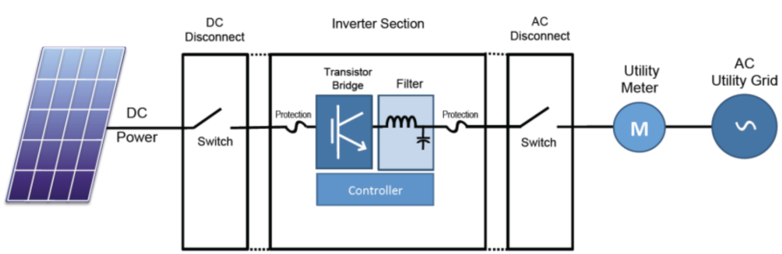
Application: Volt/VAR control

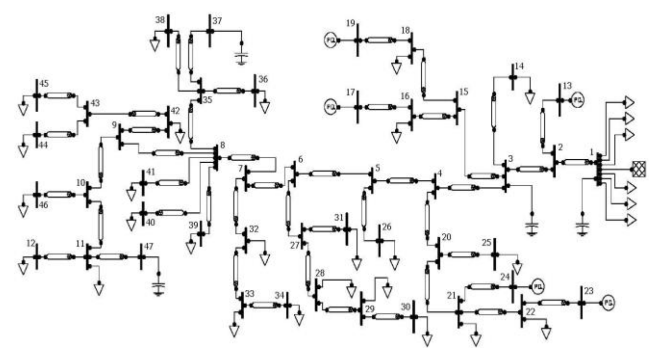
Motivation

- Static capacitor control cannot cope with rapid random fluctuations of PVs on distr circuits

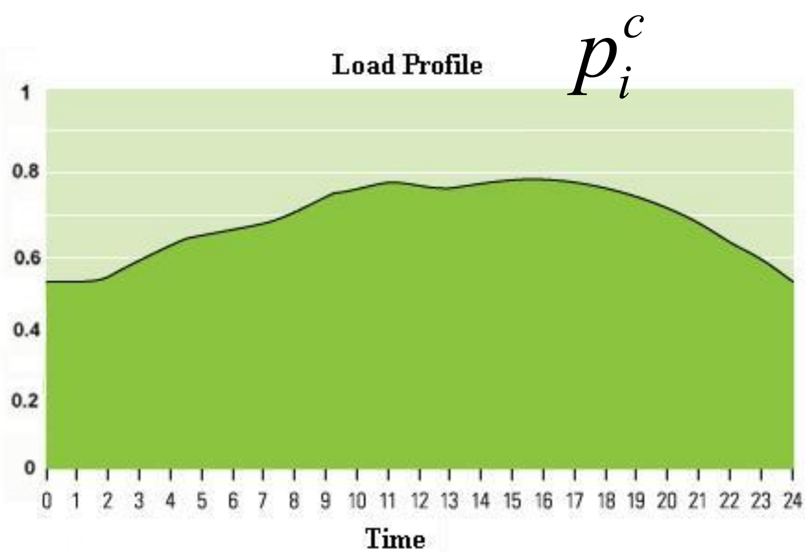
Inverter control

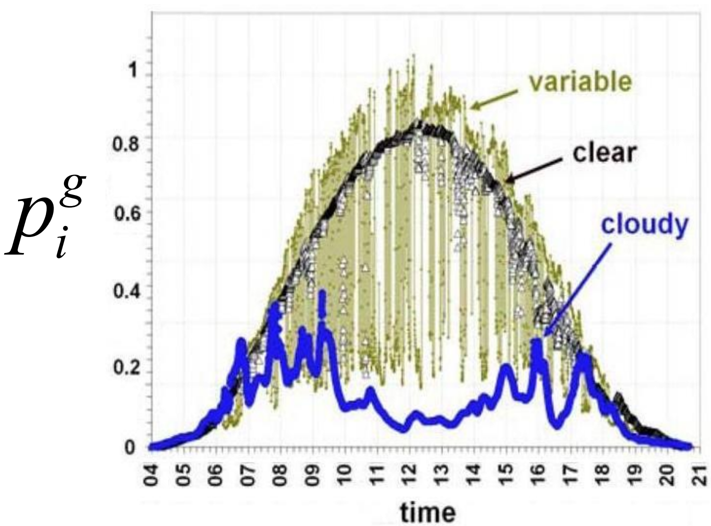
- Much faster & more frequent
- IEEE 1547 does not optimize VAR currently (unity PF)

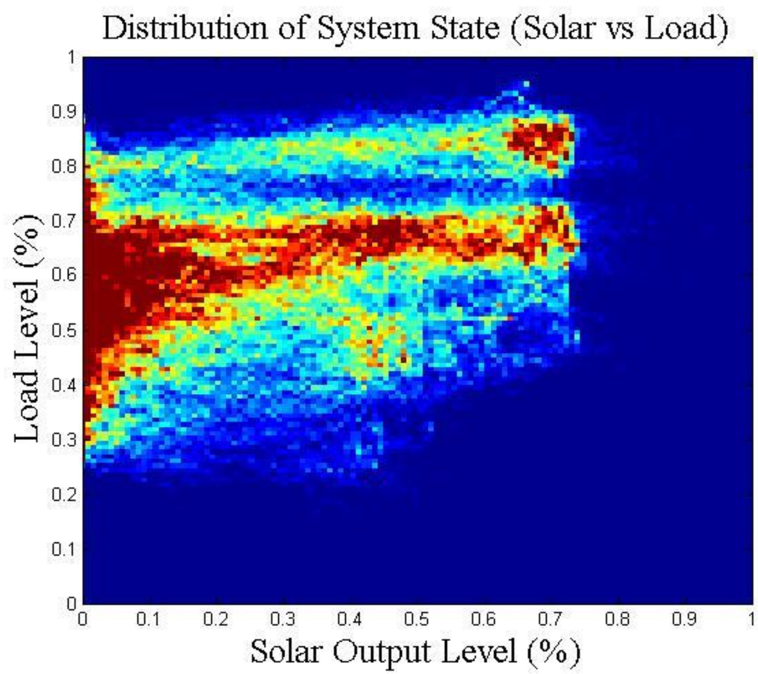




Load and Solar Variation



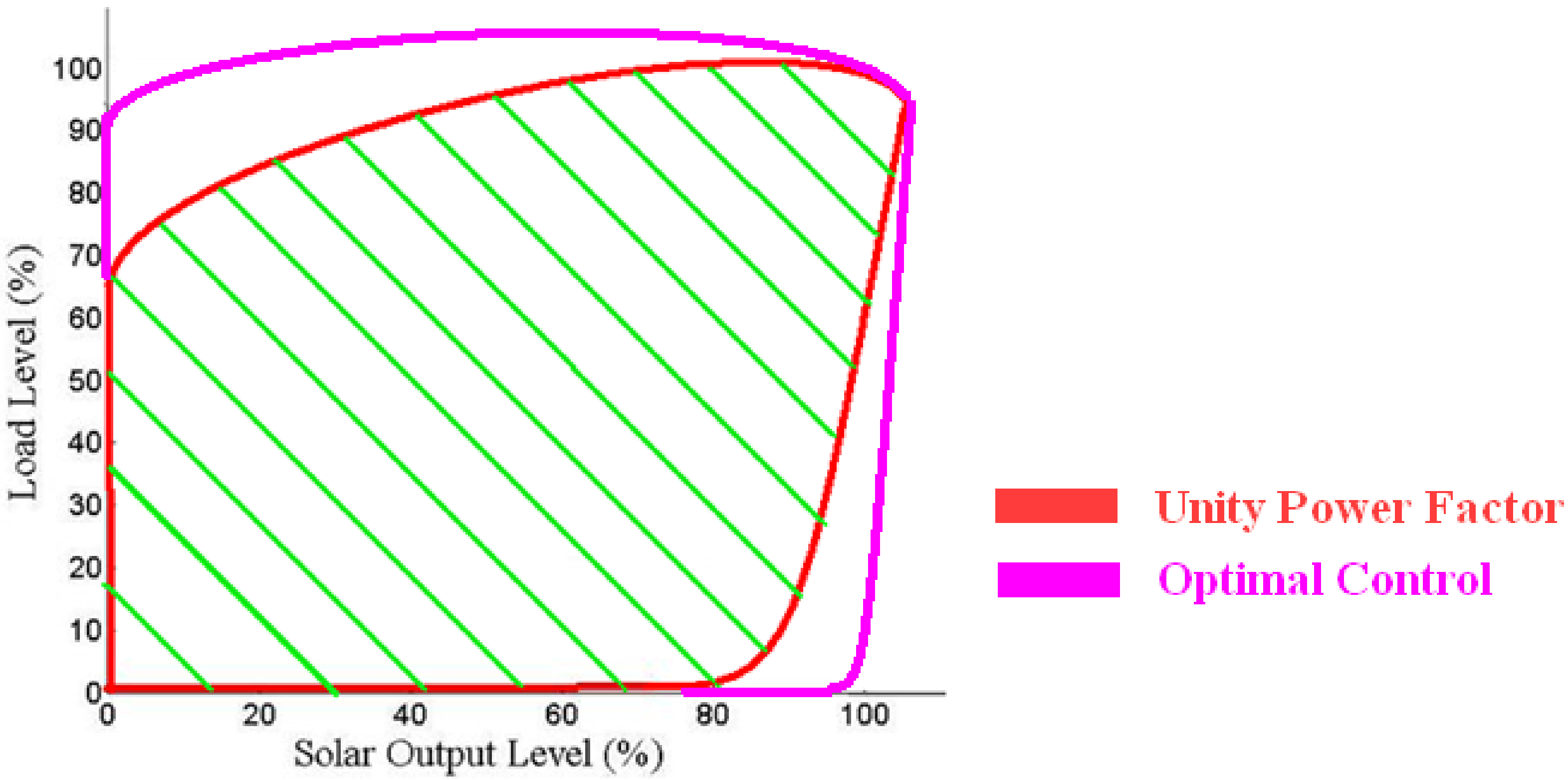




Empirical distribution
of (load, solar) for Calabash

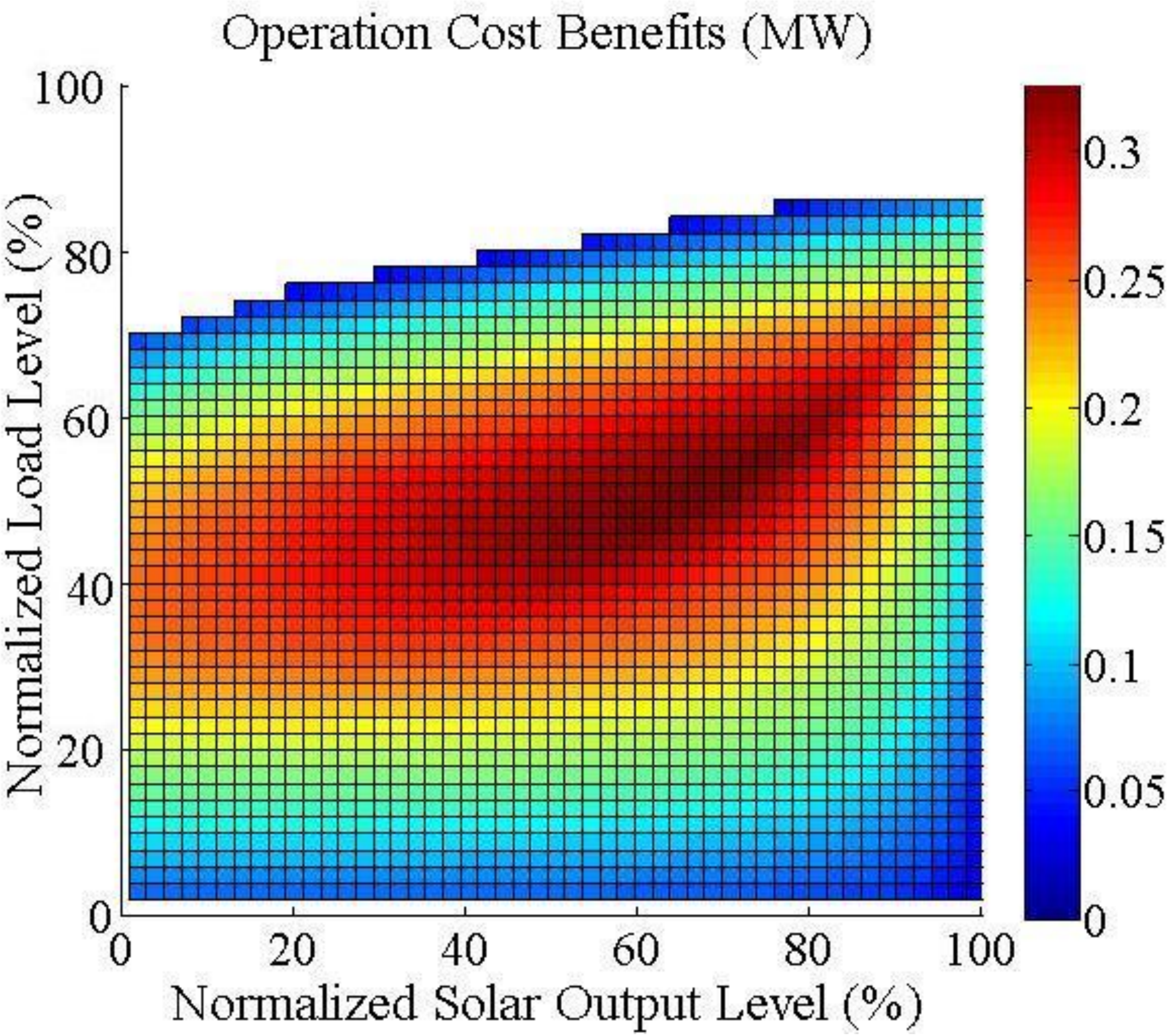
Improved reliability

$\left(p_i^g, p_i^c\right)$ for which problem is feasible



Implication: reduced likelihood of violating voltage limits or VAR flow constraints

Energy savings



Summary

RESULTS FOR SOME VOLTAGE TOLERANCE THRESHOLDS

Voltage Drop Tolerance(pu)	Annual Hours Saved Spending Outside Feasibility Region	Average Power Saving (%)
0.03	842.9	3.93%
0.04	160.7	3.67%
0.05	14.5	3.62%

- More reliable operation
- Energy savings