Power System Analysis

Chapter 12  Optimal power flow
Outline

1. Bus injection model
2. Branch flow model
3. OPF applications
4. Optimization algorithms
Outline

1. Bus injection model
   • Single-phase devices
   • Single-phase OPF
   • OPF as QCQP
   • Three-phase devices
   • Three-phase OPF
   • Three-phase OPF as QCQP

2. Branch flow model

3. OPF applications

4. Optimization algorithms
Single-phase devices

Voltage source $j$
- Ideal voltage source: terminal voltage $V_j = \text{internal voltage}$
- $V_j$ is variable if the source is controllable, or given otherwise

Current source $j$
- Ideal current source: terminal voltage $I_j = \text{internal voltage}$
- $I_j$ is variable if the source is controllable, or given otherwise

Power source $j$
- Ideal power source: terminal power $s_j = \text{internal power}$
- $s_j$ is variable if the source is controllable, or given otherwise

Impedance $j$
- Impedance $z_j$: constrains its terminal voltage & current $V_j = -z_j I_j$
Single-phase OPF

Assumptions

Assume WLOG

- Single-phase devices: voltage sources and power sources only
- Each bus has a single device with \( (V_j, s_j) \)

Formulate the simplest OPF to study general computational properties
Single-phase OPF

Simplest formulation

Optimization variable: \((V, s) := (V_j, s_j, j \in \mathbb{N})\)

- Represents voltage sources \(V_j\) and power sources \(s_j\) only

Cost function \(C_0(V, s)\)

- Fuel cost: \(C_0(V, s) := \sum_{j \text{ gens}} c_j \text{Re}(s_j)\)
- Total real power loss: \(C_0(V, s) := \sum_j \text{Re}(s_j)\)
Single-phase OPF

Simplest formulation

Power flow equations in BIM

- Equality constraints on \((V, s)\)

\[ s_j = \sum_{k:j \sim k} S_{jk}(V) := \sum_{k:j \sim k} \left( y^{s}_{jk} \right)^{H} \left( |V_j|^2 - V_j V_k^H \right) + \left( y^{m}_{jj} \right)^{H} |V_j|^2, \quad j \in \overline{N} \]

- Derivation:

\[ I_{jk}(V) := y^{s}_{jk}(V_j - V_k) + y^{m}_{jk} V_j \]

\[ S_{jk}(V) := V_j I_{jk}^H(V) := \left( y^{s}_{jk} \right)^{H} \left( |V_j|^2 - V_j V_k^H \right) + \left( y^{m}_{jk} \right)^{H} |V_j|^2 \]

- Can also use polar form and Cartesian form

- Nonlinear and global equality constraints, resulting in nonconvexity of OPF
Single-phase OPF

Simplest formulation

Operational constraints

- Injection limits (e.g. gen. or load capacity limits): \( s_j^\text{min} \leq s_j \leq s_j^\text{max} \)
- Voltage limits: \( v_j^\text{min} \leq |V_j|^2 \leq v_j^\text{max} \)
- Line limits: \( |I_{jk}(V)|^2 \leq I_{jk}^\text{max} \), \( |I_{kj}(V)|^2 \leq I_{kj}^\text{max} \)

\[
\begin{align*}
&\left| y_{jk}^r (V_j - V_k) + y_{jk}^m V_j \right|^2 \leq I_{jk}^\text{max}, \quad (j, k) \in E \\
&\left| y_{kj}^r (V_k - V_j) + y_{kj}^m V_k \right|^2 \leq I_{kj}^\text{max}, \quad (j, k) \in E
\end{align*}
\]

Line limits can also be on line powers \( (S_{jk}(V), S_{kj}(V)) \) or apparent powers \( (|S_{jk}(V)|, |S_{kj}(V)|) \)
Single-phase OPF

Simplest formulation

OPF in BIM

\[
\begin{align*}
\min_{(V,s)} & \quad C_0(V,s) \\
\text{subject to} & \quad f(V,s) = 0 \quad \text{power flow equations} \\
& \quad g(V,s) \leq 0 \quad \text{operational constraints}
\end{align*}
\]

- Does not need assumption \( y_{jk}^s = y_{kj}^s \)
- Can accommodate single-phase transformers with complex turns ratios
Single-phase OPF

1. Other devices
   - Can include other devices such as current sources, impedances, capacity taps
   - Allow multiple devices connected to same bus

2. Can formulate OPF in terms of $V$ only
   - Use power flow equations to express injections $s_j(V)$ as functions of $V$
   - Eliminate $s_j$ and power flow equations (equality constraints)

Next: explain each in turn
Single-phase OPF
Including other devices

Examples

- Current source (controllable): variable $I_j$ with local constraints $|I_j|^2 \leq I_j^{\text{max}}$, $s_j = V_j I_j^H$
- Impedance $z_j$: imposes additional constraint $s_j = |V_j|^2 / z_j^H$
- Capacitor tap (controllable): variable $y_j$ with local constraints $y_j^{\text{min}} \leq y_j \leq y_j^{\text{max}}$, $s_j = y_j^H |V_j|^2$
- Multiple devices: injection variables $s_{jk}$ with local constraints $s_{jk}^{\text{min}} \leq s_{jk} \leq s_{jk}^{\text{max}}$, $s_j = \sum_k s_{jk}$

Including other devices at bus $j$ imposes additional local constraints

- Additional optimization var $u_j$ may be introduced
- Equality constraints relating $(V_j, s_j)$ and $u_j$ (if present): $f_j(V_j, s_j, u_j) = 0$
- Inequality (operational) constraints (e.g., capacity limits): $g_j(u_j) \leq 0$
Single-phase OPF

In terms of $V$ only

Equality constraints (BIM in complex form)

$$s_j(V) = \sum_{k:j\sim k} S_{jk}(V) := \sum_{k:j\sim k} \left( y_{jk}^s \right)^H \left( |V_j|^2 - V_j V_k^H \right) + \left( y_{jj}^m \right)^H |V_j|^2, \quad j \in \mathcal{N}$$

- Expresses $s_j$ in terms of voltages $V$

Cost $C_0(V) := C_0(V, s(V))$ expressed as function of $V$

- Fuel cost:

$$C_0(V) := \sum_{j:\text{gens}} c_j \text{Re}(s_j(V)) = \sum_{j:\text{gens}} c_j \text{Re} \left( \sum_{k:j\sim k} \left( y_{jk}^s \right)^H \left( |V_j|^2 - V_j V_k^H \right) + \left( y_{jj}^m \right)^H |V_j|^2 \right)$$

- Total real power loss:

$$C_0(V) := \sum_{j} \text{Re}(s_j(V))$$
Single-phase OPF
Operational constraints

Injection limits (e.g. generation or load capacity limits) \( s_j^{\text{min}} \leq s_j(V) \leq s_j^{\text{max}} \):

\[
\sum_{k:j\sim k} \left( y_{jk}^s \right)^H \left( |V_j|^2 - V_j V_k^H \right) + \left( y_{jj}^m \right)^H |V_j|^2 \leq \bar{s}_j, \quad j \in \bar{N}
\]

- Polar form:

\[
p_j \leq \left( \sum_{k=0}^{N} g_{jk} \right) |V_j|^2 - \sum_{k \neq j} |V_j||V_k| \left( g_{jk} \cos \theta_{jk} - b_{jk} \sin \theta_{jk} \right) \leq \bar{p}_j
\]

\[
p_j \leq \left( \sum_{k=0}^{N} b_{jk} \right) |V_j|^2 - \sum_{k \neq j} |V_j||V_k| \left( b_{jk} \cos \theta_{jk} + g_{jk} \sin \theta_{jk} \right) \leq \bar{q}_j
\]
Single-phase OPF

Operational constraints

Voltage limits (same as before):
\[ v_{j}^{\text{min}} \leq |V_j|^2 \leq v_{j}^{\text{max}}, \quad j \in \mathcal{N} \]

Line limits (same as before):
\[
\begin{aligned}
&\left| y_{jk}^s (V_j - V_k) + y_{jk}^m V_j \right|^2 \leq I_{jk}^{\text{max}}, \quad (j, k) \in E \\
&\left| y_{kj}^s (V_k - V_j) + y_{kj}^m V_k \right|^2 \leq I_{kj}^{\text{max}}, \quad (j, k) \in E
\end{aligned}
\]

- Line limits can also be on line powers \( (S_{jk}(V), S_{kj}(V)) \) or apparent powers \( \left( |S_{jk}(V)|, |S_{kj}(V)| \right) \)
Single-phase OPF
In terms of $V$ only

Feasible set

$$\mathbb{V} := \{ V \in \mathbb{C}^{N+1} \mid V \text{ satisfies operational constraints} \}$$

OPF in BIM

$$\min_{V \in \mathbb{V}} C_0(V)$$

- Does not need assumption $y_{jk}^s = y_{kj}^s$
- Can accommodate single-phase transformers with complex turns ratios
Single-phase OPF

In terms of $V$ only

Feasible set

$$\mathbb{V} := \{ V \in \mathbb{C}^{N+1} \mid V \text{ satisfies operational constraints} \}$$

OPF in BIM

$$\min_{V \in \mathbb{V}} C_0(V)$$

We will mostly study this simple OPF

Can express it as a QCQP
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**OPF as QCQP**

**QCQP**

Quadratically constrained quadratic program:

\[
\min_{x \in \mathbb{C}^n} x^H C_0 x \\
\text{s.t.} \quad x^H C_l x \leq b_l, \quad l = 1, \ldots, L
\]

- \( C_l : n \times n \) Hermitian matrix
- \( b_l \in \mathbb{R} \)
- Homogeneous QCQP : all monomials are of degree 2
OPF as QCQP

Inhomogeneous QCQP

\[
\min_{x \in \mathbb{C}^n} x^H C_0 x + (c_0^H x + x^H c_0)
\]
\[
\text{s.t.} \quad x^H C_l x + (c_l^H x + x^H c_l) \leq b_l, \quad l = 1, \ldots, L
\]

Homogenization:

- Idea: \( |x|^2 + (c^H x + x^H c) \leq b \iff |x + ct|^2 - |c|^2 |t|^2 \leq b, \quad |t|^2 = 1 \)
- If \( (x, t = e^{i\theta}) \) satisfies 2nd inequality, then \( xt = xe^{i\theta} \) satisfies 1st inequality
**OPF as QCQP**

**QCQP**

Equivalent homogeneous QCQP

$$\min_{x \in \mathbb{C}^n, t \in \mathbb{C}} \begin{bmatrix} x^H & t^H \end{bmatrix} \begin{bmatrix} C_0 & c_0 \\ c_0^H & 0 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}$$

s.t. $$\begin{bmatrix} x^H & t^H \end{bmatrix} \begin{bmatrix} C_l & c_l \\ c_l^H & 0 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \leq b_l, \quad l = 1, \ldots, L$$

$$\begin{bmatrix} x^H & t^H \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} = 1$$

Homogenization:

- Idea: $$|x|^2 + (c^H x + x^H c) \leq b \iff |x + ct|^2 - |c|^2 |t|^2 \leq b, \quad |t|^2 = 1$$
- If $$(x, t = e^{i\theta})$$ satisfies 2nd inequality, then $xt = xe^{i\theta}$ satisfies 1st inequality
OPF as QCQP

To write OPF as QCQP:
- Assume cost function $C_0(V) = V^H C_0 V$ can be written as a quadratic form
- Need to rewrite operational constraints in terms of quadratic forms
**OPF as QCQP**

**Injection limits** \( s_j^{\text{min}} \leq s_j(V) \leq s_j^{\text{max}} \)

\[
s_j(V) = V_j I_j^H = \left( e_j^H V \right) \left( e_j^H I \right)^H = e_j^H V V^H Y^H e_j
\]

\[
s_j(V) = \text{tr} \left( e_j^H V V^H Y^H e_j \right) = \text{tr} \left( \left( Y^H e_j e_j^H \right) V V^H \right) = V^H Y_j V
\]
OPF as QCQP

Injection limits  \( s_{j}^{\text{min}} \leq s_{j}(V) \leq s_{j}^{\text{max}} \)

\[
s_{j}(V) = V_{j}I_{j}^{\text{H}} = (e_{j}^{\text{H}}V)(e_{j}^{\text{H}}I)^{\text{H}} = e_{j}^{\text{H}} VV^{\text{H}}Y_{j}^{\text{H}}e_{j}
\]

\[
s_{j}(V) = \text{tr} \left( e_{j}^{\text{H}}VV^{\text{H}}Y_{j}^{\text{H}}e_{j} \right) = \text{tr} \left( \left( Y_{j}^{\text{H}}e_{j}^{\text{H}} \right) VV^{\text{H}} \right) =: VV^{\text{H}}Y_{j}^{\text{H}}V
\]

- \( Y_{j} \) is not Hermitian so \( V^{\text{H}}Y_{j}^{\text{H}}V \) is generally complex
- Define  \( \Phi_{j} := \frac{1}{2} \left( Y_{j}^{\text{H}} + Y_{j} \right), \quad \Psi_{j} := \frac{1}{2i} \left( Y_{j}^{\text{H}} - Y_{j} \right) \)
- Then  \( \text{Re}(s_{j}) = V^{\text{H}}\Phi_{j}V, \quad \text{Im}(s_{j}) = V^{\text{H}}\Psi_{j}V \)

Hence  \( s_{j}^{\text{min}} \leq s_{j}(V) \leq s_{j}^{\text{max}} \) is equivalent to:

\[
p_{j}^{\text{min}} \leq V^{\text{H}}\Phi_{j}V \leq p_{j}^{\text{max}}, \quad q_{j}^{\text{min}} \leq V^{\text{H}}\Psi_{j}V \leq q_{j}^{\text{max}}
\]
OPF as QCQP

Voltage limits

Voltage magnitude is: \[ |V_j|^2 = V^H J_j V \] where \( J_j := e_j e_j^T \)

Hence voltage limits are: \( v_j^{\text{min}} \leq V^H J_j V \leq v_j^{\text{max}} \)
OPF as QCQP

Line limits

Write $I_{jk}$ in terms of voltage vector $V$:

$$I_{jk} = y^s_{jk}(V_j - V_k) + y^m_{jk}V_j = \left(y^s_{jk}(e_j - e_k)^T + y^m_{jk}e_j^T\right)V$$

Hence current limit is:

$$|I_{jk}|^2 = V^H\hat{Y}_{jk}V \leq I_{jk}^{\text{max}} \quad \text{where}$$

$$\hat{Y}_{jk} := \left(y^s_{jk}(e_j - e_k)^T + y^m_{jk}e_j^T\right)^H\left(y^s_{jk}(e_j - e_k)^T + y^m_{jk}e_j^T\right)$$
**OPF as QCQP**

**Simplest formulation**

\[
\min_{V \in \mathbb{C}^{N+1}} V^H C_0 V \\
\text{s.t.} \quad p_j^{\text{min}} \leq V^H \Phi_j V \leq p_j^{\text{max}}, \quad j \in \bar{N} \\
q_j^{\text{min}} \leq V^H \Psi_j V \leq q_j^{\text{max}}, \quad j \in \bar{N} \\
v_j^{\text{min}} \leq V^H J_j V \leq v_j^{\text{max}}, \quad j \in \bar{N} \\
V^H \hat{Y}_{jk} V \leq \bar{l}_{jk}^{\text{max}}, \quad (j, k) \in E \\
V^H \hat{Y}_{kj} V \leq \bar{l}_{kj}^{\text{max}}, \quad (j, k) \in E
\]
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1. Bus injection model
   - Single-phase devices
   - Single-phase OPF
   - OPF as QCQP
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   - Three-phase OPF as QCQP

2. Branch flow model

3. Optimization algorithms
Recall: overall 3-phase BIM
Device + network

1. **Device model** for each 3-phase device
   - Internal model on \( \left( V_j^{Y/\Delta}, I_j^{Y/\Delta}, s_j^{Y/\Delta} \right) \) + conversion rules
   - External model on \( \left( V_j, I_j, s_j \right) \)
   - Either can be used
   - Power source models are nonlinear; other devices are linear

**Our perspective:**
- Internal vars \( \left( V_j^{Y/\Delta}, I_j^{Y/\Delta}, s_j^{Y/\Delta} \right) \) are controllable, depending on types of device
- External vars \( \left( V_j, I_j, s_j \right) \) are not directly controllable

∴ use internal model + conversion rules
Recall: overall 3-phase BIM
Device + network

2. Network model relates terminal vars \((V, I, s)\)
   - Nodal current balance (linear): \(I = YV\)
   - Nodal power balance (nonlinear):
     \[
     s_j = \sum_{k:j \sim k} \text{diag} \left( V_j(V_j - V_k)^{H}y_{jk}^{sH} + V_jV_j^{H}y_{jk}^{mH} \right)
     \]
   - Either can be used

For OPF, our formulation uses \((V, s)\):
   - Relate \((V, s)\) through power flow equations
   - Power sources lead to nonlinear analysis, even if we use \(I = YV\) as network equation
   - Need to relate internal optimization vars to \( (V_j, s_j) \) using conversion rules
Three-phase devices

Voltage source $V_{j}^{Y/\Delta}$

- *Internal* optimization variable $u_{j} := V_{j}^{Y/\Delta}$ ($y_{j}^{Y}$ assumed given)
- Local constraints that relate internal vars to $(V_{j}, s_{j})$

$$Y : \quad V_{j} = V_{j}^{Y} + y_{j}^{Y} 1$$

$$\Delta : \quad \Gamma V_{j} = V_{j}^{\Delta}$$

**Note:**

- Choosing $V_{j}^{\Delta}$ does not uniquely determine $V_{j}$
- Optimization over $V_{j}$ implicitly chooses an optimal $y_{j}^{\Delta} := \frac{1}{3}^{\text{T}}V_{j}$
- If $y_{j}^{\Delta}$ is given, then $\Gamma V_{j} = V_{j}^{\Delta}$ should be replaced by $V_{j} = \Gamma^{\dagger} V_{j}^{\Delta} + y_{j}^{\Delta} 1$
Three-phase devices

Current source $I_j^{Y/\Delta}$

- *Internal* optimization variable $u_j := I_j^{Y/\Delta}$
- Local constraints that relate internal vars and $(V_j, s_j)$

\[
Y : \quad s_j = - \text{diag} \left( V_j I_j^{Y_H} \right)
\]

\[
\Delta : \quad s_j = - \text{diag} \left( V_j I_j^{\Delta H} \Gamma \right)
\]

**Note:**

- Optimization over $I_j^\Delta$ implicitly chooses an optimal $\beta_j^\Delta := \frac{1}{3} 1^T I_j^\Delta$

- If $\beta_j^\Delta$ is given, it imposes an additional constraint $I_j^\Delta = - \frac{1}{3} \Gamma I_j + \beta_j^\Delta 1$ (and express $I_j$ in terms of $(V_j, s_j)$)
Three-phase devices

Power source \( \left( s_j^\Delta, I_j^\Delta \right) \)

- Internal optimization variable \( u_j := (s_j^\Delta, I_j^\Delta) \) (assume \( y^\Delta_j := V^n_j = 0 \))
- Local constraints that relate internal vars and \( (V_j, s_j) \)
  \[
  Y : \quad s_j = - s_j^Y \\
  \Delta : \quad s_j = - \text{diag} \left( V_j I_j^{\Delta H} \Gamma \right), \quad s_j^\Delta = \text{diag} \left( \Gamma V_j I_j^{\Delta H} \right)
  \]

Impedance \( z_j^{\Delta/\Delta} \)

- Given parameter: \( z_j^{\Delta/\Delta} \) (assume \( y^\Delta_j := V^n_j = 0 \))
- Local constraints on terminal vars \( (V_j, s_j) \)
  \[
  Y : \quad s_j = - \text{diag} \left( V_j V_j^H y_j^{YH} \right) \\
  \Delta : \quad s_j = - \text{diag} \left( V_j V_j^H Y_j^{\Delta H} \right) \quad Y_j^\Delta := \Gamma^T y^\Delta \Gamma
  \]
Three-phase OPF

Variables:

- Terminal variables \( (V_j, s_j) \)
- Internal variables \( u_j \) depending on devices (discussed above)

Cost function: \( C_0 (V, s, u) \)

Equality constraints:

1. Power flow equations on \( (V, s) \) (global constraint): \( f(V, s) = 0 \)

\[
    s_j = \sum_{k:j \sim k} \text{diag} \left( V_j (V_j - V_k)^H \left( y_{jk}^s \right)^H + V_j V_j^H \left( y_{jk}^m \right)^H \right), \quad j \in \mathcal{N}
\]

2. Conversion rules relating internal optimization var \( u_j \) to \( (V_j, s_j) \) (local constraint, discussed above)

\[
    f_j^{\gamma \Delta} \left( V_j, s_j, u_j \right) = 0, \quad j \in \mathcal{N}
\]
Three-phase OPF

Inequality constraints:

1. Operational constraints on external vars: \( g(V, s) \leq 0 \)

   injection limits: \( s_j^{\phi_{\min}} \leq s_j^\phi \leq s_j^{\phi_{\max}}, \quad \phi \in \{a, b, c\}, \quad j \in \bar{N} \)

   voltage limits: \( v_j^{\phi_{\min}} \leq |V_j^{\phi}|^2 \leq v_j^{\phi_{\max}}, \quad \phi \in \{a, b, c\}, \quad j \in \bar{N} \)

   line limits: \( |I_{jk}^{\phi}(V)|^2 \leq I_{jk}^{\phi_{\max}}, \quad |I_{kj}^{\phi}(V)|^2 \leq I_{kj}^{\phi_{\max}}, \quad \phi \in \{a, b, c\}, \quad (j, k) \in E \)

Same constraints as single-phase OPF, but on single-phase equivalent circuit
Three-phase OPF

Inequality constraints:

2. Operational constraints on internal vars: \( g_j^{y/\Delta}(u_j) \leq 0 \)

   for \( \phi n \in \{an, bn, cn\}, \ \phi \phi \in \{ab, bc, ca\} \)

   voltage source: \( v_{j}^{\phi n \min} \leq |v_{j}^{\phi n}|^2 \leq v_{j}^{\phi n \max}, \quad v_{j}^{\phi \phi \min} \leq |v_{j}^{\phi \phi}|^2 \leq v_{j}^{\phi \phi \max} \)

   current source: \( |i_{j}^{\phi n}|^2 \leq i_{j}^{\max}, \quad |i_{j}^{\phi \phi}|^2 \leq i_{j}^{\max} \)

   power source: \( s_{j}^{\phi n \min} \leq s_{j}^{\phi n} \leq s_{j}^{\phi n \max}, \quad |i_{j}^{\phi \phi}|^2 \leq i_{j}^{\phi \phi \max} \)

Local constraints at each bus \( j \)
Three-phase OPF

Constraints summary

1. Constraints on terminal variables: \( f(V, s) = 0, \quad g(V, s) \leq 0 \)
   - Power flow equation and operational constraints (terminal power injection limits, voltage limits, line limits)
   - Global constraints
   - Extension of single-phase constraints to 3-phase setting, using single-phase equivalent

2. Conversion rules relating \( u_j \) and \( (V_j, s_j) \): \( f_j^{Y/\Delta}(u_j, V_j, s_j) = 0 \)
   - Local equality constraint for each device \( j \)

3. Operational constraints on internal variables: \( g_j^{Y/\Delta}(u_j) \leq 0 \)
   - Depending on type of device (voltage and capacity limits)
   - Local constraints for each device \( j \)
Three-phase OPF
Simplest formulation

OPF in BIM

\[
\begin{align*}
\min_{(V,s,u)} & \quad C_0(V,s,u) \\
\quad f(V,s) &= 0, \quad g(V,s) \leq 0 \quad \text{Global constraints on terminal vars} \\
\quad f_j^{Y/\Delta}(V_j,s_j,u_j) &= 0, \quad g_j^{Y/\Delta}(u_j) \leq 0, \quad j \in \mathcal{N} \quad \text{Local constraints at each bus } j
\end{align*}
\]
Three-phase OPF
As QCQP

1. Can formulate OPF in terms of \((V, u)\) only
   - Use power flow equations to express \(s_j(V) = V^H \left( Y_j^H \right) V\) and eliminate \(s_j\) and \(f(V, s) = 0\)
   - Same idea as before applied to single-phase equivalent

2. Can formulate OPF as QCQP
   - Express operational constraints \(g(V, s(V)) \leq 0\) in terms of quadratic forms in \(V\) (same idea applied to single-phase equivalent)
   - Express conversion rules \(f_j^{Y/\Delta} \left( V_j, s_j(V), u_j \right) = 0\) in terms of quadratic forms in \((V, u_j)\)

For details: see Lecture Notes
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Overview

BFM and BIM differ only in power flow equations

device models → nodal current/power balance → network models

BFM & BIM use same device models

line/transformer models

single-phase or 3-phase
Assumptions
Both single-phase & 3-phase OPF

Radial network
- BFM most useful for modeling distribution systems

\[ z^s_{jk} = z^s_{kj} \] or equivalently \[ y^s_{jk} = y^s_{kj} \]
- Does not include 3-phase transformers in \( \Delta Y \) or \( Y\Delta \) configuration (or single-phase transformers with complex gains)

\[ y^m_{jk} = y^m_{kj} = 0 \]
- Reasonable assumption for distribution line where \( |y^m_{jk}|, |y^m_{kj}| \ll |y^s_{jk}| \)

Includes only voltage sources and power sources
- Optimization variables are voltages (squared magnitudes) \( v_j \) and power injections \( s_j \) respectively
- A current source or an impedance will introduce additional var and constraint.
Single-phase OPF

Power flow equations
- All lines point away from bus 0 (root)

\[ \sum_{k:j \rightarrow k} S_{jk} = S_{ij} - z_{ij} \ell_{ij} + s_j, \quad j \in N \]

\[ v_j - v_k = 2 \text{Re} \left( z_{jk}^H S_{jk} \right) - |z_{jk}|^2 \ell_{jk}, \quad j \rightarrow k \in E \]

\[ \ell_{jk} = |S_{jk}|^2, \quad j \rightarrow k \in E \]

Operational constraints
- \( s_j^{\text{min}} \leq s_j \leq s_j^{\text{max}} \)
- \( v_j^{\text{min}} \leq v_j \leq v_j^{\text{max}} \)
- \( \ell_{jk} \leq I_{jk}^{\text{max}} \)
Single-phase OPF

Feasible set

$$\mathcal{T}_0 := \{x := (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies PF equations & operational constraints} \}$$

OPF in BFM

$$\min_{x \in \mathcal{T}_0} C(x)$$
Single-phase OPF
Equivalence

Recall for BIM:

- Feasible set: \( \mathbb{V} := \{ V \in \mathbb{C}^{N+1} \mid V \text{ satisfies operational constraints} \} \)
- OPF: \( \min_{V \in \mathbb{V}} C_0(V) \)

OPF in BFM is equivalent to OPF in BIM:

- Feasible sets \( \mathbb{T}_0 \) and \( \mathbb{V} \) are equivalent (Ch 6)
- … provided cost functions \( C(x) \) and \( C_0(V) \) are the same
Three-phase OPF

Variables \((x, \nu)\): 

1. Directly generalizes vars in single-phase OPF \((\mathbb{S}_+^n : \text{complex psd matrices})\)

\[
\begin{align*}
    s_j &\in \mathbb{C}^3, & v_j &\in \mathbb{S}_+^3, & j &\in \overline{N} \\
    \ell_{jk} &\in \mathbb{S}_+^3, & S_{jk} &\in \mathbb{C}^{3 \times 3}, & j &\rightarrow k \in E
\end{align*}
\]

To write conversion rule for power sources, introduce phasors as additional vars

\[
\left( V_j, j \in \overline{N} \right), \quad \left( \tilde{I}_{jk}, j \rightarrow k \in E \right)
\]

Let \(x := (s, v, \ell, V, \tilde{I}, S)\)
Three-phase OPF

Variables \((x,u)\):

2. Internal variables \(u := \left( u_j, j \in N \right)\) of 3-phase devices

  \[
  \text{voltage source : } u_j := V_j^{Y/\Delta} \in \mathbb{C}^3
  \]

  \[
  \text{power source : } u_j := \left( u_{j1}, u_{j2} \right) = \left( s_j^{Y/\Delta}, I_j^{Y/\Delta} \right) \in \mathbb{C}^6
  \]
Three-phase OPF

Equality constraints

1. Power flow equations (from Ch 10):

\[
\sum_{k \leftarrow j} \text{diag}(S_{jk}) = \text{diag} \left( S_{ij} - z_{ij} \ell_{ij} \right) + s_j, \quad j \in \overline{N}
\]

\[
v_j - v_k = \left( z_{jk} S_{jk}^H + S_{jk} z_{jk}^H \right) - z_{jk} \ell_{jk} z_{jk}^H, \quad j \rightarrow k \in E
\]

\[
\begin{bmatrix}
v_j s_{jk} \\
S_{jk}^H \ell_{jk}
\end{bmatrix} \geq 0, \quad j \rightarrow k \in E
\]

\[
\text{rank} \begin{bmatrix}
v_j s_{jk} \\
S_{jk}^H \ell_{jk}
\end{bmatrix} = 1, \quad j \rightarrow k \in E
\]

\[
v_j = V_j V_j^H, \quad \ell_{jk} = \tilde{I}_{jk} \tilde{I}_{jk}^H, \quad S_{jk} = V_j \tilde{I}_{jk}^H, \quad j \rightarrow k \in E
\]  

additional equations
Three-phase OPF

Equality constraints

1. Power flow equations (from Ch 10):

\[
\sum_{k:j\rightarrow k} \text{diag}(S_{jk}) = \text{diag}\left(S_{ij} - z_{ij} \ell_{ij}\right) + s_j, \quad j \in \overline{N}
\]

\[
v_j - v_k = \left(z_{jk} S_{jk}^H + S_{jk} z_{jk}^H\right) - z_{jk} \ell_{jk} z_{jk}^H, \quad j \rightarrow k \in E
\]

\[
\begin{bmatrix}
  v_j S_{jk} \\
  S_{jk}^H \ell_{jk}
\end{bmatrix} \geq 0, \quad j \rightarrow k \in E
\]

\[
\text{rank}\begin{bmatrix}
  v_j S_{jk} \\
  S_{jk}^H \ell_{jk}
\end{bmatrix} = 1, \quad j \rightarrow k \in E
\]

\[
v_j = V_j V_j^H, \quad \ell_{jk} = \tilde{I}_{jk} \tilde{I}_{jk}^H, \quad S_{jk} = V_j \tilde{I}_{jk}^H, \quad j \rightarrow k \in E
\]

Steven Low  OPF  Branch flow model
Three-phase OPF

Equality constraints

2. Conversion rules for voltage & power sources (assume $V_j^m = 0$)

- **Voltage source**:
  
  \[ Y : \quad v_j = V_j^Y V_j^{YH} = u_j u_j^H \]

  \[ \Delta : \quad \Gamma v_j \Gamma^T = V_j^\Delta V_j^{\Delta H} = u_j u_j^H \]

- **Power source**:
  
  \[ Y : \quad s_j = - \text{diag} \left( V_j u_j^H \right), \quad s_j = - u_j \]

  \[ \Delta : \quad s_j = - \text{diag} \left( V_j u_j^H \Gamma \right), \quad u_j = \text{diag} \left( \Gamma V_j u_j^H \right) \]
Three-phase OPF

Inequality constraints

1. Operational constraints on $\mathbf{x}$:

   - injection limits:
     \[ s_j^{\text{min}} \leq s_j \leq s_j^{\text{max}}, \quad j \in \overline{N} \]

   - voltage limits:
     \[ v_j^{\text{min}} \leq \text{diag}(v_j) \leq v_j^{\text{max}}, \quad j \in \overline{N} \]

   - line limits:
     \[ \text{diag}(\mathbf{c}_{jk}) \leq I_{jk}^{\text{max}}, \quad (j, k) \in E \]
Three-phase OPF

Inequality constraints

2. Operational constraints on internal vars $u_j$:

- **Voltage source:**
  
  \[
  v_j^{\phi n \min} \leq |V_j^{\phi n}|^2 \leq v_j^{\phi n \max}, \quad v_j^{\phi \phi \min} \leq |V_j^{\phi \phi}|^2 \leq v_j^{\phi \phi \max}
  \]

- **Power source:**
  
  \[
  s_j^{Y \min} \leq s_j^Y \leq s_j^{Y \max}, \quad |I_j^{\phi n}|^2 \leq I_j^{\phi n \max}
  \]

  \[
  s_j^{\Delta \min} \leq s_j^\Delta \leq s_j^{\Delta \max}, \quad |I_j^{\phi \phi}|^2 \leq I_j^{\phi \phi \max}
  \]
Three-phase OPF

Feasible set

\[ \mathcal{T}_{3p} := \{ (x, u) := (s, v, \ell, V, I, S, u) \mid (x, u) \text{ satisfies all constraints} \} \]

OPF in BFM

\[ \min_{(x,u) \in \mathcal{T}_{3p}} C(x, u) \]

Three-phase OPF in BFM is equivalent to three-phase OPF in BIM:

- Their feasible sets are equivalent (Ch 10)
- ... provided their cost functions are equivalent
Outline

1. Bus injection model
2. Branch flow model
3. OPF applications
   • Voltage control (distribution grid)
4. Optimization algorithms
Voltage control
Distribution system

Voltage instability: magnitudes fluctuate outside their limits
  • PVs may push magnitudes above upper limits
  • EVs may push magnitudes below lower limits

Traditional solution
  • Infrastructure upgrade: more/larger transformers, wires, etc

Non-wire solution
  • Distributed energy resources (DER) optimization
  • e.g. batteries, smart inverters, demand response
  • Can formulate as an OPF
Voltage control
Optimal battery operation

\[
\min_{u,V,b} \sum_t \sum_j \left( |V_j(t)|^2 - v_j^{\text{ref}}(t) \right)^2
\]

s.t. \( u_j(t) + \sigma_j(t) = \sum_{k:j \sim k} S_{jk}(V(t)) \), \( v_j \leq |V_j(t)|^2 \leq \bar{v}_j \)

\[
|S_{jk}(V(t))| \leq S_{jk}, \quad |S_{kj}(V(t))| \leq \bar{S}_{kj}
\]
Voltage control
Optimal battery operation

\[
\min_{u,V,b} \sum_t \sum_j \left( |V_j(t)|^2 - v_j^{\text{ref}(t)} \right)^2
\]
deviation from nominal voltages

\[
s.t. \quad u_j(t) + \sigma_j(t) = \sum_{k:j \sim k} S_{jk}(V(t)), \quad v_j \leq |V_j(t)|^2 \leq \bar{v}_j
\]
charging/discharging (100% efficiency)

\[
|S_{jk}(V(t))| \leq \bar{S}_{jk}, \quad |S_{kj}(V(t))| \leq \bar{S}_{kj}
\]

\[
b_j(t + 1) = b_j(t) - \text{Re} \left( u_j(t) \right)
\]
Voltage control
Optimal battery operation

\[
\begin{array}{c}
\min_{u, V, b} \quad \sum_t \sum_j \left( |V_j(t)|^2 - v_j^{\text{ref}}(t) \right)^2 \\
\text{s.t.} \quad u_j(t) + \sigma_j(t) = \sum_{k:j \sim k} S_{jk}(V(t)), \quad \nu_j \leq |V_j(t)|^2 \leq \bar{v}_j \\
\quad |S_{jk}(V(t))| \leq \bar{S}_{jk}, \quad |S_{kj}(V(t))| \leq \bar{S}_{kj} \\
\quad b_j(t + 1) = b_j(t) - \text{Re} \left( u_j(t) \right), \quad \text{charging/discharging (100% efficiency)} \\
\quad u_j \leq \text{Re} \left( u_j(t) \right) \leq \bar{u}_j, \quad 0 \leq b_j(t) \leq B_j, \quad \text{power limit} \quad \text{energy limit}
\end{array}
\]
Voltage control

Optimal battery placement

\[
\begin{align*}
\min_{u,V,b,B} \quad & \sum_t \sum_j \left( \left| V_j(t) \right|^2 - v_j^{\text{ref}}(t) \right)^2 + \sum_j c_j B_j \\
\text{s.t.} \quad & u_j(t) + \sigma_j(t) = \sum_{k:j \sim k} S_{jk}(V(t)), \quad v_j \leq \left| V_j(t) \right|^2 \leq \bar{v}_j \\
& |S_{jk}(V(t))| \leq \bar{S}_{jk}, \quad |S_{kj}(V(t))| \leq \bar{S}_{kj} \\
& b_j(t + 1) = b_j(t) - \text{Re} \left( u_j(t) \right) \\
& u_j \leq \text{Re} \left( u_j(t) \right) \leq \bar{u}_j, \quad 0 \leq b_j(t) \leq B_j \\
& B_j^{\text{opt}} > 0: \text{place battery at bus } j
\end{align*}
\]
Outline

1. Bus injection model
2. Branch flow model
3. Optimization algorithms
   - Newton-Raphson algorithm
   - Interior-point algorithm
Complex formulation

Even though OPF is often formulated in $\mathbb{C}$, it is converted to $\mathbb{R}$ before being solved iteratively.

**Example: QCQP**

$$\begin{align*}
\min_{x \in \mathbb{C}^n} & \quad x^H C_0 x \\
\text{s.t.} & \quad x^H C_l x \leq b_l, \quad l = 1, \ldots, L
\end{align*}$$

- $C_l : n \times n$ Hermitian matrix
- $b_l \in \mathbb{R}$

**Equivalent to:**

$$\begin{align*}
\min & \quad \begin{bmatrix} x_r^T & x_i^T \end{bmatrix} \begin{bmatrix} C_{0r} & -C_{0i} \\ C_{0i} & C_{0r} \end{bmatrix} \begin{bmatrix} x_r \\ x_i \end{bmatrix} \\
\text{s.t.} & \quad \begin{bmatrix} x_r^T & x_i^T \end{bmatrix} \begin{bmatrix} C_{lr} & -C_{li} \\ C_{li} & C_{lr} \end{bmatrix} \begin{bmatrix} x_r \\ x_i \end{bmatrix} \leq b_l, \quad l = 1, \ldots, L
\end{align*}$$

- $2n \times 2n$ symmetric matrices
Algorithms for OPF

Popular algorithms

Newton-Raphson algorithm
- 2nd order algorithm
- Interior-point algorithm

Interior-point algorithm
- Based on barrier functions
- Uses of Newton-Raphson algorithm for subproblems
Newton-Raphson algorithm

NR is algorithm for solving

\[ F(x) = 0, \quad F : \mathbb{R}^n \rightarrow \mathbb{R}^n \]

Iteratively:

\[ x(t + 1) = y(t) + \Delta x(t) \]
\[ J(y(t)) \Delta x(t) = -F(x(t)) \]

where \( J(x) := \frac{\partial F}{\partial x}(x) \) is Jacobian of \( F \)

Application to optimization problems:

- \( F(x) = 0 \) is KKT condition
- If NR converges, it computes a KKT point \( x^{\text{opt}} \)
- \( x^{\text{opt}} \) is a global optimal if the problem is convex (feasible otherwise)
Newton-Raphson algorithm

Describe NR progressively for solving
  • Linear equality constrained problems
  • Nonlinear equality constrained problems
  • Inequality constrained problems
Newton-Raphson algorithm

Linear equality constraint

Consider

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad Ax = b$$

where

- $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable
- $A \in \mathbb{R}^{m \times n}$
Newton-Raphson algorithm

Linear equality constraint

Consider

\[
\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad Ax = b
\]

Lagrangian:

\[
L(x, \lambda) := f(x) + \lambda^T(Ax - b)
\]

Jacobian of \( L(x, \lambda) \):

\[
F(x, \lambda) := \begin{bmatrix}
\nabla_x L(x, \lambda) \\
\nabla_\lambda L(x, \lambda)
\end{bmatrix} = \begin{bmatrix}
\nabla f(x) + A^T \lambda \\
Ax - b
\end{bmatrix}
\]

KKT condition to be solved by NR algorithm:

\[
F(x, \lambda) = 0
\]
Newton-Raphson algorithm

Linear equality constraint

Consider

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad Ax = b$$

Jacobian of $F(x, \lambda)$:

$$J(x, \lambda) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x) & A^T \\ A & 0 \end{bmatrix}$$

- KKT matrix
- Independent of $\lambda$

NR iteration:

$$\begin{bmatrix} x(t + 1) \\ \lambda(t + 1) \end{bmatrix} = \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} + \begin{bmatrix} \Delta x(t) \\ \Delta \lambda(t) \end{bmatrix} \quad \text{where} \quad \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x(t)) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x(t) \\ \Delta \lambda(t) \end{bmatrix} = - \begin{bmatrix} \nabla f(x(t)) + A^T \lambda(t) \\ Ax(t) - b \end{bmatrix}$$
Newton-Raphson algorithm
Nonlinear equality constraint

Consider

$$\min_{x \in \mathbb{R}^n} \quad f(x) \quad \text{s.t.} \quad g(x) = 0$$

where

- $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are twice continuously differentiable

Follow the same procedure as for linear equality constrained problems
Newton-Raphson algorithm
Nonlinear equality constraint

Consider

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad g(x) = 0$$

Lagrangian:

$$L(x, \lambda) := f(x) + \lambda^T g(x)$$

Jacobian of $L(x, \lambda)$:

$$F(x, \lambda) := \begin{bmatrix} \nabla_x L(x, \lambda) \\ \nabla_\lambda L(x, \lambda) \end{bmatrix} = \begin{bmatrix} \nabla f(x) + \frac{\partial g}{\partial x}(x)^T \lambda \\ g(x) \end{bmatrix}$$

KKT condition to be solved by NR algorithm:

$$F(x, \lambda) = 0$$
Newton-Raphson algorithm

Nonlinear equality constraint

Consider

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad g(x) = 0$$

Jacobian of $F(x, \lambda)$:

$$J(x, \lambda) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x) + \sum_k \frac{\partial^2 g_k}{\partial x^2} \lambda_k & \frac{\partial g}{\partial x}(x)^T \\ \frac{\partial g}{\partial x}(x) & 0 \end{bmatrix}$$

NR iteration:

$$\begin{bmatrix} x(t+1) \\ \lambda(t+1) \end{bmatrix} = \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} + \begin{bmatrix} \Delta x(t) \\ \Delta \lambda(t) \end{bmatrix} \quad \text{where} \quad J(x, \lambda) \begin{bmatrix} \Delta x(t) \\ \Delta \lambda(t) \end{bmatrix} = - \begin{bmatrix} \nabla f(x(t)) + \frac{\partial g}{\partial x}(x(t))^T \lambda(t) \\ g(x(t)) \end{bmatrix}$$
Newton-Raphson algorithm

Inequality constraint

Consider

\[
\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad g(x) \leq 0
\]

where

- \( f: \mathbb{R}^n \to \mathbb{R} \) and \( g: \mathbb{R}^n \to \mathbb{R}^m \) are twice continuously differentiable

Two common solution approaches

1. Introduce slack var \( z \geq 0 \) to reduce the inequality into a simple inequality constraint:

\[
\min_{(x,z) \in \mathbb{R}^{n+m}} f(x) \quad \text{s.t.} \quad g(x) + z = 0, \quad z \geq 0
\]

2. Replace constraint by a penalty term and reduce to unconstrained problem:

\[
\min_{x \in \mathbb{R}^n} f(x) + \frac{1}{t} \phi(x)
\]

This is the approach of interior-point algorithms!
**Interior-point algorithm**

**Basic idea**

Consider

$$\min_{x \in \mathbb{R}^n} f_0(x) \quad \text{s.t.} \quad f(x) \leq 0, \quad g(x) = 0$$

where

- $f_0 : \mathbb{R}^n \to \mathbb{R}, f : \mathbb{R}^n \to \mathbb{R}^m, g : \mathbb{R}^n \to \mathbb{R}^p$ are twice continuously differentiable

**Basic idea:**

- Approximate problem by equality constrained problem by replacing $f(x) \leq 0$ by a **barrier function**
- Solve the approximate problem by Newton-Raphson methods
Interior-point algorithm

Log barrier function

Log barrier function $\phi : \mathbb{R}^n \to \mathbb{R}$ is

$$\phi(x) := -\sum_{i=1}^{m} \log(-f_i(x))$$

over $\text{dom} \phi := \{x \in \mathbb{R}^n : f_i(x) < 0, i = 1, \ldots, m\}$

Properties:

- $\phi(x) \to \infty$ as $f_i(x) \to 0$ for any $i$
- $\nabla \phi(x) = \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla f_i(x)$
- $\frac{\partial^2 \phi}{\partial x^2}(x) = \sum_i \frac{1}{f_i^2(x)} \nabla f_i(x) \nabla f_i^T(x) + \sum_i \frac{1}{-f_i(x)} \frac{\partial^2 f_i}{\partial x^2}(x)$
Interior-point algorithm

Approximate problem

Consider

\[
\min_{x \in \mathbb{R}^n} f_0(x) \quad \text{s.t.} \quad f(x) \leq 0, \quad g(x) = 0
\]

Approximate problem

\[
\min_{x \in \mathbb{R}^n} f_0(x) + \frac{1}{t} \phi(x) \quad \text{s.t.} \quad g(x) = 0
\]

or

\[
\text{Problem}(t): \quad \min_{x \in \mathbb{R}^n} tf_0(x) + \phi(x) \quad \text{s.t.} \quad g(x) = 0
\]

- Larger \( t > 0 \) \( \implies \) more accurate approximation
Barrier method
A popular interior-point method

Basic idea

• Solve Problem($t$) for an increasing sequence of $t > 0$ until solution is accurate enough
• For each $t$, solve Problem($t$) using Newton-Raphson algorithm

Questions

• How to choose the sequence of $t$?
• When to terminate?

Answer these question for convex problems
Barrier method

Assumptions

1. Original problem is convex, i.e., $f_0, f_1, \ldots, f_m$ are convex and $g(x) = Ax - b$

2. For each $t > 0$, Newton-Raphson algorithm converges to the unique optimal solution $x(t)$ of the approximate problem

- Central point: optimal solution $x(t)$
- Central path: set $\{x(t) : t > 0\}$ of central points
Barrier method

Central point $x(t)$

1. Original problem is convex, i.e., $f_0, f_1, \ldots, f_m$ are convex and $g(x) = Ax - b$

2. For each $t > 0$, Newton-Raphson algorithm converges to the unique optimal solution $x(t)$ of the approximate problem

**Theorem**

For each $t > 0$

1. $x(t)$ is feasible for original problem

2. Objective value is at most $m/t$ away from optimal value, i.e., $f_0(x(t)) - f_0^{\text{opt}} \leq m/t$

   In particular $f_0(x(t)) \to f_0^{\text{opt}}$ as $t \to \infty$
Barrier method

**Input:** *strictly* feasible $x$, initial $t := t_0$, scaling factor $\gamma > 1$, tolerance $\varepsilon$.

**Output:** an approximate solution $x$:

1. while $t \leq \frac{m}{\varepsilon}$ do

   (a) Solve Problem($t$) to compute $x(t)$ using the Newton-Raphson algorithm starting from $x$.
   
   (b) $x \leftarrow x(t)$.
   
   (c) $t \leftarrow \gamma t$.

2. Return: $x$.

In principle, one can solve Problem($t$) with $t := m/\varepsilon$ instead of solving a sequence of Problem($t$). In practice, barrier method works better.