Outline

1. Relaxation of QCQP
2. Application to OPF
3. Exactness condition: linear separability
4. Exactness condition: small angle difference
5. Condition for global optimality
Outline

1. Relaxation of QCQP
   - SDP relaxation
   - Partial matrices and completions
   - Feasible sets
   - Relaxations and solution recovery
   - Tightness of relaxations

2. Application to OPF

3. Exactness condition: linear separability

4. Exactness condition: small angle difference

5. Condition for global optimality
QCQP

Quadratically constrained quadratic program:

\[
\min_{x \in \mathbb{C}^n} x^H C_0 x \\
\text{s.t. } x^H C_l x \leq b_l, \quad l = 1, \ldots, L
\]

- \( C_l : n \times n \) Hermitian matrix
- \( b_l \in \mathbb{R} \)
- Homogeneous QCQP: all monomials are of degree 2
- OPF can be formulated as (nonconvex) QCQP

Steven Low     OPF     Semidefinite relaxation of QCQP
QCQP
Equivalent problem

Using $x^H C_l x = \text{tr} \left( C_l x x^H \right)$, this is equivalent to:

$$\begin{align*}
\min_{X \in \mathbb{S}_+^n, x \in \mathbb{C}^n} & \quad \text{tr} \left( C_0 X \right) \\
\text{s.t.} & \quad \text{tr} \left( C_l X \right) \leq b_l, \quad l = 1, \ldots, L \\
& \quad X = xx^H
\end{align*}$$

- Any psd rank-1 matrix $X \in \mathbb{S}^{n \times n}_+$ has a spectral decomposition $X = xx^H$ for some $x \in \mathbb{C}^n$
- $x$ is unique up to a rotation, i.e., $x$ satisfies $X = xx^H x e^{j\theta}$ for any $\theta \in \mathbb{R}$
- Therefore can eliminate $x$
QCQP
Equivalent problem

Eliminating $x \rightarrow$ minimization over psd matrices $X$:

$$\min_{X \in \mathbb{S}^n} \quad \text{tr} \left( C_0 X \right)$$

s.t. \quad \text{tr} \left( C_l X \right) \leq b_l, \quad l = 1, \ldots, L

$$X \succeq 0, \quad \text{rank}(X) = 1$$

- $\text{tr} \left( C_l X \right) \leq b_l$ is linear in $X$
- $X \succeq 0$ is convex in $X$
- $\text{rank}(X) = 1$ is nonconvex in $X$ \quad \text{Removing rank constraint yields SDP relaxation}
SDP relaxation

SDP relaxation of QCQP

\[
\min_{X \in \mathbb{S}^n} \quad \text{tr} \left( C_0 X \right)
\]

s.t. \quad \text{tr} \left( C_l X \right) \leq b_l, \quad l = 1, \ldots, L

\[ X \geq 0 \]

- This is a standard semidefinite program which is a convex problem
- Solution strategy:
  - Solve SDP for an optimal solution \( X^{\text{opt}} \)
    - If \( \text{rank} \left( X^{\text{opt}} \right) = 1 \), then \( x^{\text{opt}} \in \mathbb{C}^n \) from spectral decomposition from \( X^{\text{opt}} = x^{\text{opt}} \left( x^{\text{opt}} \right)^H \)
    - If \( \text{rank} \left( X^{\text{opt}} \right) > 1 \), then, in general, no feasible solution of QCQP can be directly obtained
SDP relaxation

SDP relaxation of QCQP

\[
\begin{align*}
\min_{X \in \mathbb{S}^n} & \quad \text{tr} \left( C_0 X \right) \\
\text{s.t.} & \quad \text{tr} \left( C_l X \right) \leq b_l, \quad l = 1, \ldots, L \\
& \quad X \succeq 0
\end{align*}
\]

- Even though SDP is convex, for large networks, it is still computationally impractical.
- How to exploit sparsity of large networks to reduce computational burden?

Ans: partial matrices and completions!
Partial matrices

A QCQP instance specified by \((C_0, C_l, b_l, l = 1, \ldots, L)\) induces graph \(F := (N, E)\)

- \(N : n \) nodes (where \(C_l \in \mathbb{C}^{n \times n}\))
- \(E \subseteq N \times N : m \) links \((j, k) \in E\) iff \(\exists l \in \{0, 1, \ldots, L\}\) s.t. \([C_l]_{jk} = [C_l]_{kj}^H \neq 0\)

A partial matrix \(X_F\) is a set of \(n + 2m\) complex numbers defined on \(F = (N, E)\)

\[
X_F := \left\{ [X_F]_{jj}, [X_F]_{jk}, [X_F]_{kj} : j \in N, (j, k) \in E \right\}
\]

- \(X_F\) can be interpreted as matrix with entries partially specified, or a partial matrix
- If \(F\) is complete graph, then \(X_F\) is full \(n \times n\) matrix

A completion \(X\) of \(X_F\) is a full \(n \times n\) matrix that agrees with \(X_F\) on graph \(F\)

\[
[X]_{jj} = [X_F]_{jj}, \quad [X]_{jk} = [X_F]_{jk}, \quad [X]_{kj} = [X_F]_{kj}
\]
Partial matrices

If $q$ is clique (fully connected subgraph) of $F$, then $X_F(q)$ is fully specified principal submatrix of $X_F$ on $q$:

\[
[X(q)]_{jj} := [X_F]_{jj}, \quad [X(q)]_{jk} := [X_F]_{jk}, \quad [X(q)]_{kj} := [X_F]_{kj},
\]
Hermitian, psd, rank-1, trace

Partial matrix

A partial matrix $X_F$ is

- **Hermitian** ($X_F = X_F^H$) if $[X_F]_{kj} = [X_F]_{jk}^H$

- **psd** ($X_F \succeq 0$) if $X_F$ is Hermitian and $X_F(q) \succeq 0$ for all cliques $q$ of $F$

- **rank-1** if $\text{rank}(X_F(q)) = 1$ for all cliques $q$ of $F$
Hermitian, psd, rank-1, trace

Partial matrix

A partial matrix $X_F$ is

- **Hermitian** $(X_F = X_F^H)$ if $[X_F]_{kj} = [X_F]_{jk}^H$
- **psd** $(X_F \succeq 0)$ if $X_F$ is Hermitian and $X_F(q) \succeq 0$ for all cliques $q$ of $F$
- **rank-1** if $\text{rank} \left( X_F(q) \right) = 1$ for all cliques $q$ of $F$

- **$2 \times 2$ psd** if $X_F(j, k)$ is psd for all $(j, k) \in E$
- **$2 \times 2$ rank-1** if $X_F(j, k)$ is rank-1 for all $(j, k) \in E$

where $X_F(j, k) := \begin{bmatrix} [X_F]_{jj} & [X_F]_{jk} \\ [X_F]_{kj} & [X_F]_{kk} \end{bmatrix}$
Hermitian, psd, rank-1, trace

Partial matrix

For partial matrix $X_F$

$$\text{tr} \left( C_l X_F \right) := \sum_{j \in N} [C_l]_{jj} [X_F]_{jj} + \sum_{(j,k) \in E} \left( [C_l]_{jk} [X_F]_{kj} + [C_l]_{kj} [X_F]_{jk} \right)$$

If both $C_l$ and $X_F$ are Hermitian, then $\text{tr} \left( C_l X_F \right)$ is real:

$$\text{tr} \left( C_l X_F \right) = \sum_{j \in N} [C_l]_{jj} [X_F]_{jj} + 2 \sum_{(j,k) \in E} \text{Re} \left( [C_l]_{jk} [X_F]_{kj} \right)$$
Chordal graph & extensions

\( F \) is a chordal graph if

- Either \( F \) has no cycles, or
- All minimal cycles (ones without chords) are of length 3

A chordal extension \( c(F) \) of \( F \) is a chordal graph that contains \( F \)

- \( X_{c(F)} \) is a chordal extension of \( X_F \)

Every graph has a (generally nonunique) chordal extension

- Complete supergraph of \( F \) is a \( c(F) \)

**Theorem** [Grone et al 1984]: every psd partial matrix has a psd completion iff underlying graph is chordal

- We will extend this to psd rank-1 submatrices
Partial matrix & chordal extensions

Example

\[
W_F = \begin{bmatrix}
    x_{11} & x_{12} & x_{13} \\
    x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\
    x_{31} & x_{32} & x_{33} & x_{34} \\
    x_{41} & x_{42} & x_{43} & x_{44} & x_{45} \\
    x_{52} & x_{53} & x_{54} & x_{55}
\end{bmatrix}
\]

\[
W_{c(F)} = \begin{bmatrix}
    x_{11} & x_{12} & x_{13} \\
    x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\
    x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\
    x_{42} & x_{43} & x_{44} & x_{45} \\
    x_{52} & x_{53} & x_{54} & x_{55}
\end{bmatrix}
\]

\[
W_{c(F)} = \begin{bmatrix}
    x_{11} & x_{12} & x_{13} \\
    x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\
    x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\
    x_{43} & x_{44} & x_{45} \\
    x_{52} & x_{53} & x_{54} & x_{55}
\end{bmatrix}
\]

2 cliques \(W_{c(F)}(q)\)

3 cliques \(W_{c(F)}(q)\)
Rank-1 characterization

Equivalent conditions

C1: \( X \succeq 0, \quad \text{rank}(X) = 1 \)

C2: \( X_{c(F)} \succeq 0, \quad \text{rank}(X_{c(F)}) = 1 \)

C3: \( X_{F(j,k)} \succeq 0, \quad \text{rank}(X_{F(j,k)}) = 1, \quad (j,k) \in E \)

\[ \sum_{(j,k) \in c} \angle [X_{F}]_{jk} = 0 \mod 2\pi \quad \text{cycle condition} \]

Theorem

Suppose \( X_{jj} > 0, \quad [X_{c(F)}]_{jj} > 0, \quad [X_{F}]_{jj} > 0 \). Then C1 \( \iff \) C2 \( \iff \) C3.
Outline

1. Relaxation of QCQP
   - SDP relaxation
   - Partial matrices and completions
   - Feasible sets
   - Relaxations and solution recovery
   - Tightness of relaxations

2. Application to OPF

3. Exactness condition: linear separability

4. Exactness condition: small angle difference

5. Condition for global optimality
Feasible sets

Feasible set of QCQP

\[ \forall := \{ x \in \mathbb{C}^n \mid x^H C_l x \leq b_l, \ l = 1, \ldots, L \} \]

psd rank-1 matrices \( X \)

\[ \mathcal{X} := \{ X \in \mathbb{S}^n \mid X \text{satisfies } \text{tr}(C_l X) \leq b_l, \ C_1 \} \]

psd rank-1 chordal extensions \( X_{c(F)} \)

\[ \mathcal{X}_{c(F)} := \{ X_{c(F)} \mid X_{c(F)} \text{satisfies } \text{tr}\left( C_l X_{c(F)} \right) \leq b_l, \ C_2 \} \]

psd rank-1 partial matrices \( X_F \)

\[ \mathcal{X}_F := \{ X_F \mid X_F \text{satisfies } \text{tr}\left( C_l X_F \right) \leq b_l, \ C_3 \} \]
Feasible sets
Equivalence

Corollary
Fix any connected $F$. Any partial matrix $X_{c(F)} \in \mathbb{S}_{c(F)}$ or $X_F \in \mathbb{S}_F$ has a unique psd rank-1 completion $X \in \mathbb{S}$

**Definition:** Two sets $A$ and $B$ are equivalent ($A \equiv B$) if there is a bijection between them

**Theorem**
$\forall \equiv \mathbb{S} \equiv \mathbb{S}_{c(F)} \equiv \mathbb{S}_F$

**Implication:** A feasible $x \in \mathbb{V}$ can be recovered from any partial matrix $X_{c(F)} \in \mathbb{S}_{c(F)}$ or $X_F \in \mathbb{S}_F$ through spectral decomposition (but there is a simpler way to compute $x \in \mathbb{V}$ than completion)
Equivalent problems

QCQP

$$\min_{x \in \mathbb{C}^n} x^H C_0 x \quad \text{subject to} \quad x \in \mathbb{V}$$

is equivalent to min over matrices and partial matrices:

$$\min_{X} x^H C_0 x \quad \text{subject to} \quad X \in \hat{\mathbb{X}}$$

where $\hat{\mathbb{X}} := \left\{ \mathbb{X}, \mathbb{X}_{c(F)}, \mathbb{X}_F \right\}$

Implications:
Instead of solving for $X \in \mathbb{X}$, solve for $X_{c(F)} \in \mathbb{X}_{c(F)}$ or $X_F \in \mathbb{X}_F$ which are much smaller for large sparse networks

Steven Low  OPF  Semidefinite relaxation of QCQP
Equivalent problems

QCQP

\[
\min_{x \in \mathbb{C}^n} x^H C_0 x \quad \text{subject to} \quad x \in \mathbb{V}
\]

is equivalent to min over matrices and partial matrices:

\[
\min_X X^H C_0 X \quad \text{subject to} \quad X \in \hat{\mathbb{X}}
\]

where \( \hat{\mathbb{X}} := \{ \mathbb{X}, \mathbb{X}_{c(F)}, \mathbb{X}_F \} \)

Computational challenges remain:
\( \mathbb{X}, \mathbb{X}_{c(F)}, \mathbb{X}_F \) are all nonconvex
Semidefinite relaxations

Convex supersets

\[ \mathbb{X}^+ := \{ X \in \mathbb{S}^n \mid X_F \text{ satisfies } \text{tr}(C_lX) \leq b_l, \ X \succeq 0 \} \]

\[ \mathbb{X}^+_{c(F)} := \{ X_{c(F)} \mid X_F \text{ satisfies } \text{tr}\left( C_lX_{c(F)} \right) \leq b_l, \ X_{c(F)} \succeq 0 \} \]

\[ \mathbb{X}^+_F := \{ X_F \mid X_F \text{ satisfies } \text{tr}\left( C_lX_F \right) \leq b_l, \ X_F(j,k) \geq 0, \ (j,k) \in E \} \]

Semidefinite relaxations:

\( \text{QCQP-sdp : } \min_X \ C\left( X_F \right) \ \text{ s.t. } \ X \in \mathbb{X}^+ \) \hspace{1cm} \text{most complex} 

\( \text{QCQP-ch : } \min_{X_{c(F)}} \ C\left( X_F \right) \ \text{ s.t. } \ X_{c(F)} \in \mathbb{X}^+_{c(F)} \)

\( \text{QCQP-socp : } \min_{X_F} \ C\left( X_F \right) \ \text{ s.t. } \ X_F \in \mathbb{X}^+_F \) \hspace{1cm} \text{simplest}
Semidefinite relaxations

Solution recovery

If a feasible / optimal solution $X$ of semidefinite relaxation lies in $\mathcal{X}$, $\mathcal{X}_c(F)$, or $\mathcal{X}_F$, then can recover feasible / optimal $x \in \mathcal{V}$ of QCQP

Recovery procedure: given $X_F \in \mathcal{X}_F$

1. Set $|x_1| := \sqrt{[X_F]_{11}}$ and $\angle x_1$ to arbitrary value

2. For $j = 1, \ldots, n$,

   $|x_j| := \sqrt{[X_F]_{jj}}$, \hspace{1em} $\angle x_j := \angle V_1 - \sum_{(i,k) \in \mathcal{P}_j} \angle [X_F]_{ik}$

   where $\mathcal{P}_j$ : path from bus 1 to bus $j$ in an arbitrary spanning tree rooted at bus 1
Outline

1. Relaxation of QCQP
   - SDP relaxation
   - Partial matrices and completions
   - Feasible sets
   - Relaxations and solution recovery
   - Tightness of relaxations

2. Application to OPF

3. Exactness condition: linear separability

4. Exactness condition: small angle difference

5. Condition for global optimality
Semidefinite relaxations

Convex supersets

\[ \mathcal{X}^+ := \{ X \in \mathbb{S}^n \mid X_F \text{ satisfies } \text{tr}(C_l X) \leq b_l, X \succeq 0 \} \]

\[ \mathcal{X}_{c(F)}^+ := \{ X_{c(F)} \mid X_F \text{ satisfies } \text{tr}\left( C_l X_{c(F)} \right) \leq b_l, X_{c(F)} \succeq 0 \} \]

\[ \mathcal{X}_F^+ := \{ X_F \mid X_F \text{ satisfies } \text{tr}\left( C_l X_F \right) \leq b_l, X_F(j,k) \succeq 0, (j,k) \in E \} \]

Semidefinite relaxations:

QCQP-sdp : \[ \min_X C\left( X_F \right) \quad \text{s.t.} \quad X \in \mathcal{X}^+ \quad \text{most complex} \]

QCQP-ch : \[ \min_{X_{c(F)}} C\left( X_F \right) \quad \text{s.t.} \quad X_{c(F)} \in \mathcal{X}_{c(F)}^+ \]

QCQP-socp : \[ \min_{X_F} C\left( X_F \right) \quad \text{s.t.} \quad X_F \in \mathcal{X}_F^+ \quad \text{simplest} \]
Tightness

Definition
1. $A$ is an effective subset of $B$ ($A \subseteq B$) if given any $a \in A$, $\exists b \in B$ with same cost $C_A(a) = C_B(b)$
2. $A$ is similar to $B$ ($A \simeq B$) if $A \subseteq B$ and $B \subseteq A$

Theorem [Tightness]
1. $\forall \subseteq \mathbb{X}^+ \simeq \mathbb{X}^+_{c(F)} \subseteq \mathbb{X}^+_F$
2. If $F$ is a tree, then $\forall \subseteq \mathbb{X}^+ \simeq \mathbb{X}^+_{c(F)} \simeq \mathbb{X}^+_F$

Corollary [Optimal values]
1. $C^{qcqp} \geq C^{sdp} = C^{ch} \geq C^{socp}$
2. If $F$ is a tree, then $C^{qcqp} \geq C^{sdp} = C^{ch} = C^{socp}$

Steven Low  OPF  Semidefinite relaxation of QCQP
Semidefinite relaxations

Implications

1. Radial networks: Solve QCQP-socp
   - Simplest computationally
   - Same tightness as QCQP-ch and QCQP-SDP

2. Meshed networks: Solve QCQP-ch or QCQP-socp
   - QCQP-ch strictly tighter than QCQP-socp, and same tightness as QCQP-sdp
   - QCQP-ch can be orders of magnitude simpler computationally than QCQP-sdp for large sparse networks
   - QCQP-ch is as complex as QCQP-sdp in the worst case
Outline

1. Relaxation of QCQP

2. Application to OPF
   • Single-phase networks
   • Definition: exact relaxation

3. Exactness condition: linear separability

4. Exactness condition: small angle difference

5. Condition for global optimality
OPF as QCQP

Recall

\[
\min_{V \in \mathbb{C}^{N+1}} \quad V^H C_0 V \\
\text{s.t.} \\
\begin{align*}
p_j^{\min} & \leq \text{tr} \left( \Phi_j V V^H \right) \leq p_j^{\max}, & j \in \bar{N} \\
q_j^{\min} & \leq \text{tr} \left( \Psi_j V V^H \right) \leq q_j^{\max}, & j \in \bar{N} \\
v_j^{\min} & \leq \text{tr} \left( J_j V V^H \right) \leq v_j^{\max}, & j \in \bar{N} \\
\text{tr} \left( \hat{Y}_{jk} V V^H \right) & \leq I_{jk}^{\max}, & (j, k) \in E \\
\text{tr} \left( \hat{Y}_{kj} V V^H \right) & \leq I_{kj}^{\max}, & (j, k) \in E
\end{align*}
\]

abbreviated as:
\[
\text{tr} \left( C_l V V^H \right) \leq b_l, \quad l = 1, \ldots, L
\]
Given $V \in \mathbb{C}^{N+1}$, define partial matrix $W_G$ by

$$[W_G]_{jj} := |V_j|^2, \quad j \in \overline{N}$$

$$[W_G]_{jk} := V_j V_k^H =: [W_G]_{kj}^H, \quad (j, k) \in E$$

Constraints in terms of $W_G$

$$p_j^\text{min} \leq \text{tr} \left( \Phi_j W_G \right) \leq p_j^\text{max}$$

$$q_j^\text{min} \leq \text{tr} \left( \Psi_j W_G \right) \leq q_j^\text{max}$$

$$v_j^\text{min} \leq \text{tr} \left( J_j W_G \right) \leq v_j^\text{max}$$

$$\text{tr} \left( \hat{Y}_{jk} W_G \right) \leq I_{jk}^\text{max}$$

$$\text{tr} \left( \hat{Y}_{kj} W_G \right) \leq I_{kj}^\text{max}$$

Abbreviated as:

$$\text{tr} \left( C_l W_G \right) \leq b_l, \ l = 1, \ldots, L$$
OPF and relaxations

OPF as QCQP

\[
\min_V C_0(V) \quad \text{s.t.} \quad \text{tr} \left( C_l V V^H \right) \leq b_l, \quad l = 1, \ldots, L
\]

Semidefinite relaxations:

- **OPF-sdp**: \( \min_{W \in \mathbb{S}^{N+1}} C_0(W_G) \quad \text{s.t.} \quad \text{tr} \left( C_l W \right) \leq b_l, \quad l = 1, \ldots, L, \quad W \succeq 0 \)

- **OPF-ch**: \( \min_{W_{c(G)}} C_0(W_G) \quad \text{s.t.} \quad \text{tr} \left( C_l W_{c(G)} \right) \leq b_l, \quad l = 1, \ldots, L, \quad W_{c(G)} \succeq 0 \)

- **OPF-socp**: \( \min_{W_G} C_0(W_G) \quad \text{s.t.} \quad \text{tr} \left( C_l W_G \right) \leq b_l, \quad l = 1, \ldots, L, \quad W_G(j,k) \succeq 0, \quad (j,k) \in E \)
Exact relaxation

Definition

1. OPF-sdp is exact if every optimal solution $W^{\text{sdp}}$ of OPF-sdp is psd rank-1
2. OPF-ch is exact if every optimal solution $W^{\text{ch}}_{c(G)}$ of OPF-ch is psd rank-1
3. OPF-socp is exact if every optimal solution $W^{\text{socp}}_G$ of OPF-docp
   • is $2 \times 2$ psd rank-1, i.e., $W^{\text{socp}}_G(j, k)$ are psd rank-1 for all $(j, k) \in E$, and
   • satisfies cycle condition, i.e., $\sum_{(j,k) \in c} \angle [W^{\text{socp}}_G]_{jk} = 0 \mod 2\pi$
Outline

1. Relaxation of QCQP
2. Application to OPF
3. Exactness condition: linear separability
   - Sufficient condition for QCQP
   - Application to OPF
4. Exactness condition: small angle difference
5. Condition for global optimality
QCQP and SOCP relaxation

QCQP:

\[
\min_{x \in \mathbb{C}^n} \quad x^H C_0 x \\
\text{s.t.} \quad x^H C_l x \leq b_l, \quad l = 1, \ldots, L
\]

SOCP relaxation:

\[
\min_{X_G} \quad \text{tr} (C_0 X_G) \\
\text{s.t.} \quad \text{tr} (C_l X_G) \leq b_l, \quad l = 1, \ldots, L \\
X_G(j, k) \geq 0, \quad (j, k) \in E
\]

- \( C_l \) : \( n \times n \) Hermitian matrix, \( b_l \in \mathbb{R} \)
Sufficient condition

C13.1: $C_0$ is positive definite

C13.2: for every $(j, k) \in E$, $\exists \alpha_{jk}$ s.t. $\angle [C_l]_{jk} \in [\alpha_{ij}, \alpha_{ij} + \pi]$ for all $l = 0, \ldots, L$

**Theorem**

Suppose $G$ is a tree and C13.2 holds. Then

1. $C_{opt} = C_{socp}$

2. An optimal solution of QCQP can be recovered from every optimal solution of its SOCP relaxation

An optimal solution of SOCP relaxation may not be $2 \times 2$ rank-1 when optimal solutions of SOCP relaxation are nonunique
**Sufficient condition**

C13.1: \( C_0 \) is positive definite

C13.2: for every \((j, k) \in E\), \( \exists \alpha_{jk} \text{ s.t. } \angle[C_l]_{jk} \in [\alpha_{ij}, \alpha_{ij} + \pi] \) for all \( l = 0, \ldots, L \)

**Corollary**

Suppose \( G \) is a tree and both C13.1 and C13.2 hold. Then SOCP relaxation is exact, i.e., every optimal solution \( W_{G}^{\text{socp}} \) is \( 2 \times 2 \) psd rank-1

- Cycle condition is vacuous since \( G \) is a tree
Application to OPF
Recall OPF as QCQP

\[
\min_{V \in \mathbb{C}^{N+1}} \quad V^H C_0 V
\]

s.t.

\[
p_j^{\min} \leq \text{tr} \left( \Phi_j V V^H \right) \leq p_j^{\max}, \quad j \in \bar{N}
\]

\[
q_j^{\min} \leq \text{tr} \left( \Psi_j V V^H \right) \leq q_j^{\max}, \quad j \in \bar{N}
\]

\[
v_j^{\min} \leq \text{tr} \left( J_j V V^H \right) \leq v_j^{\max}, \quad j \in \bar{N}
\]

\[
\text{tr} \left( \hat{Y}_{jk} V V^H \right) \leq \bar{I}_{jk}^{\max}, \quad (j, k) \in E
\]

\[
\text{tr} \left( \hat{Y}_{kj} V V^H \right) \leq \bar{I}_{kj}^{\max}, \quad (j, k) \in E
\]

abbreviated as:

\[
\text{tr} \left( C_l V V^H \right) \leq b_l, \quad l = 1, \ldots, L
\]
Application to OPF

Exactness condition

Corollary
Suppose $G$ is a tree and both C13.1 and the diagram hold.
Then SOCP relaxation is exact
Outline

1. Relaxation of QCQP
2. Application to OPF
3. Exactness condition: linear separability
4. Exactness condition: small angle difference
   • Sufficient condition
   • 2-bus example
5. Condition for global optimality
Assumptions

Assume

1. Voltage magnitudes $|V_j|$ are fixed
2. Reactive powers are ignored
3. Shunt admittances are zero $y_{jk}^m = y_{kj}^m := 0$
**OPF formulation**

\[
\begin{align*}
\min_{p, P, \theta} & \quad C(p) \\
\text{s.t.} & \quad p_j^{\text{min}} \leq p_j \leq p_j^{\text{max}}, \quad j \in \mathcal{N} \\
& \quad \theta_{jk}^{\text{min}} \leq \theta_{jk} \leq \theta_{jk}^{\text{max}}, \quad (j, k) \in E \\
& \quad p_j = \sum_{k: k \sim j} P_{jk}, \quad j \in \mathcal{N} \\
& \quad P_{jk} = g_{jk} - g_{jk} \cos \theta_{jk} - b_{jk} \sin \theta_{jk}, \quad (j, k) \in E
\end{align*}
\]

constraints on line flows, line losses, or stability

nodal power balance

power flow equation (polar form)

where \( V_j = |V_j| e^{i\theta_j} \) with \(|V_j| := 1\) and \( \theta_{jk} := \theta_j - \theta_k \)

Eliminate \( P_{jk} \) and \( \theta_{jk} \)
OPF formulation

Define injection region

\[ \mathbb{P}_\theta := \left\{ p \in \mathbb{R}^n \mid p_j = \sum_{k:k \sim j} \left( g_{jk} - g_{jk} \cos \theta_{jk} - b_{jk} \sin \theta_{jk} \right), \quad \theta_{jk} \leq \theta_{jk} \leq \bar{\theta}_{jk} \right\} \]

\[ \mathbb{P}_p := \{ p \in \mathbb{R}^n \mid \underline{p}_j \leq p_j \leq \overline{p}_j, j \in N \} \]

**OPF:**

\[ \min_{p} C(p) \quad \text{s.t.} \quad p \in \mathbb{P}_\theta \cap \mathbb{P}_p \]

**SOCP relaxation:**

\[ \min_{p} C(p) \quad \text{s.t.} \quad p \in \text{conv} (\mathbb{P}_\theta) \cap \mathbb{P}_p \]

**Definition:** SOCP relaxation is **exact** if every optimal solution lies in \( \mathbb{P}_\theta \cap \mathbb{P}_p \)
Pareto front

Definitions

A point \( x \in A \subseteq \mathbb{R}^n \) is a **Pareto optimal point** in \( A \) if there does not exist another \( x' \in A \) such that

- \( x' \leq x \), and
- \( x'_j < x_j \) for at least one \( j \)

The **Pareto front** of \( A \):

\[
\mathcal{O}(A) := \{ \text{all Pareto optimal points} \}
\]
Sufficient condition

C13.3: $C(p)$ is strictly increasing in each $p_j$

C13.4: for every $(j, k) \in E$, $\tan^{-1} \frac{b_{jk}}{g_{jk}} < \theta_{jk}^{\min} \leq \theta_{jk}^{\max} < \tan^{-1} \frac{-b_{jk}}{g_{jk}}$

Theorem

Suppose $G$ is a tree and C13.3, C13.4 hold. Then

1. $\mathcal{P}_\theta \cap \mathcal{P}_p = \emptyset(\text{conv}(\mathcal{P}_\theta) \cap \mathcal{P}_p)$ feasible set is Pareto front of its relaxation

2. SOCP relaxation is exact
Geometric insight

2-bus network

For each line \((j, k) \in E\), line flows \(P := (P_{jk}, P_{kj})\) and angle differences \(\theta_{jk} := \theta_j - \theta_k\) satisfy

\[
P - g_{jk}1 = A \begin{bmatrix} \cos \theta_{jk} \\ \sin \theta_{jk} \end{bmatrix} \quad \text{where} \quad A := \begin{bmatrix} -g_{jk} & -b_{jk} \\ -g_{jk} & b_{jk} \end{bmatrix}
\]

1. \(P\) traces out an ellipse in \(\mathbb{R}^2\) as \(\theta_{jk}\) ranges over \([-\pi, \pi]\).

   Hence feasible set (subset of ellipse) is nonconvex.

2. C13.4 restricts \(\mathbb{P}_\theta\) to lower half of ellipse.
**Geometric insight**

**2-bus network**

For each line $(j, k) \in E$, line flows $P := \left( P_{jk}, P_{kj} \right)$ and angle differences $\theta_{jk} := \theta_j - \theta_k$ satisfy

$$P - g_{jk}^1 = A \begin{bmatrix} \cos \theta_{jk} \\ \sin \theta_{jk} \end{bmatrix} \quad \text{where} \quad A := \begin{bmatrix} -g_{jk} & -b_{jk} \\ -g_{jk} & b_{jk} \end{bmatrix}$$

1. $P$ traces out an ellipse in $\mathbb{R}^2$ as $\theta_{jk}$ ranges over $[-\pi, \pi]$. Hence feasible set (subset of ellipse) is noncovex.

2. C13.4 restricts $\mathbb{P}_\theta$ to lower half of ellipse

3. C13.3 implies Pareto front of relaxed feasible set coincides with feasible set, i.e., relaxation is exact

---

### Figure 13.6

The set $\text{conv}(P_{jk})$ is the intersection of the ellipse, including its interior, and a half-space.

---

### Figure 13.7

With lower bounds $p$ on power injections, the feasible set of OPF-socp (13.25) is the shaded region. (a) When the feasible set of OPF (13.24) is restricted to the lower half of the ellipse (small $|q_{jk}|$), the Pareto front remains on the ellipse itself, $P_{jk} \setminus P_{p} = O(\text{conv}(P_{jk}) \setminus P_{p})$, and hence the relaxation is exact. (b) When the feasible set of OPF includes upper half of the ellipse (large $|q_{jk}|$), the Pareto front may not lie on the ellipse if $p$ is large, making the relaxation not exact.

---

**Remark 13.5 (Tree topology)**. The tree topology allows the extension of the argument for a single line to a radial network with multiple lines, in two ways. First let $F_{jk}$ denotes the set of branch power flows on...
Outline

1. Relaxation of QCQP
2. Application to OPF
3. Exactness condition: linear separability
4. Exactness condition: small angle difference
5. Condition for global optimality
   - Sufficient condition
   - Application to OPF
No spurious local optima

\[
\begin{align*}
\text{minimize} & \quad f(x) \quad & f: \text{continuous, convex} \\
\text{subject to} & \quad x \in \mathcal{X} \quad & \mathcal{X}: \text{compact, nonconvex}
\end{align*}
\]

Convex relaxation:

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in \hat{\mathcal{X}}. \quad \hat{\mathcal{X}}: \text{compact, convex, } \mathcal{X} \subseteq \hat{\mathcal{X}} \subseteq K^n
\end{align*}
\]
No spurious local optima

\[
\begin{align*}
\text{minimize} & \quad f(x) & \quad f : \text{continuous, convex} \\
\text{subject to} & \quad x \in \mathcal{X} & \quad \mathcal{X} : \text{compact, nonconvex}
\end{align*}
\]

Convex relaxation: \[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in \hat{\mathcal{X}}. & \hat{\mathcal{X}} : \text{compact, convex}, \mathcal{X} \subseteq \hat{\mathcal{X}} \subseteq K^n
\end{align*}
\]

Relaxation (2) is **exact** if there exists optimal solution of (2) that is optimal for (1)

**Key result** [Zhou 2022]: Lyapunov-like conditions for
- Relaxation (2) is exact; and
- Any local optimum of (1) is globally optimal
No spurious local optima

**Definition:** A path from \( x \in \hat{X} \setminus X \) to \( X \) is a continuous function \( h_x: [0,1] \to \hat{X} \) such that \( h_x(0) = x \) and \( h_x(1) \in X \)

**Lemma** [Zhou 2022]

(2) is exact \( \iff \forall x \in \hat{X} \setminus X \) there is a path \( h_x \) from \( x \) to \( X \) such that
- \( f(h_x(t)) \) nonincreasing in \( t \)
- \( f(h_x(1)) < f(h_x(0)) \)
No spurious local optima

**Definition:** A *Lyapunov-like function* is a continuous function $V: \mathbb{R} \to \mathbb{R}_+$ such that

$$V(x) = \begin{cases} 
0 & x \in X \\
> 0 & x \in \tilde{x} \setminus X 
\end{cases}$$
No spurious local optima

Standard Lyapunov function

- Dynamical system:  \( \dot{y} = f(y(t)) \)
- Global asymptotic stability:  \( y(t) \to y^* \)
- Stability certificate: Lyapunov function  \( V(y) \) s.t.
  1.  \( V(y) > 0 \) if  \( y \neq y^* \),  \( =0 \) if  \( y = y^* \)
  2.  \( \dot{V}(y(t)) < 0 \) along trajectory  \( y(t) \)
No spurious local optima

Standard Lyapunov function

- Dynamical system: \( \dot{y} = f(y(t)) \)
- Global asymptotic stability: \( y(t) \to y^* \)
- Stability certificate: Lyapunov function \( V(y) \) s.t.
  1. \( V(y) > 0 \) if \( y \neq y^* \), \( =0 \) if \( y = y^* \)
  2. \( \dot{V}(y(t)) < 0 \) along trajectory \( y(t) \)

Our case (dynamical system replaced by optimization)

- Trajectory (path \( y(t) = h_x(t) \)) is not specified
- Goal is to enter \( X: x = y(0) \to y(1) \in X \)
- Lyapunov-like \( V(y) \) s.t.
  1. \( V(y) > 0 \) if \( y \neq y^* \), \( =0 \) if \( y = y^* \)
  2. \( \text{C1: } V(y(t)) \) non-increasing along trajectory \( y(t) \)

- Cost \( f(y(t)) \) must be non-increasing along \( y(t) \) and
  \( f(y(1)) < f(y(0)) \)
No spurious local optima

**Conditions:**  $\exists$ paths $\{h_x : x \in \mathbb{X} \setminus X\}$ and a Lyapunov-like function $V$ such that  
- C1: both $f(h_x(t))$ and $V(h_x(t))$ are non-increasing for $t \in [0, 1]$, and $f(h_x(0)) > f(h_x(1))$  
- C2: $\{h_x : x \in \mathbb{X} \setminus X\}$ is uniformly bounded and uniformly equicontinuous
No spurious local optima

**Conditions:** $\exists$ paths $\{h_x : x \in \hat{X} \setminus X\}$ and a Lyapunov-like function $V$ such that

- **C1:** both $f(h_x(t))$ and $V(h_x(t))$ are non-increasing for $t \in [0, 1]$, and $f(h_x(0)) > f(h_x(1))$
- **C2:** $\{h_x : x \in \hat{X} \setminus X\}$ is uniformly bounded and uniformly equi-continuous

**Theorem** [Zhou 2022]

- C1, C2 $\iff$ all local optima of (1) globally optimal & (2) exact

Are C1, C2 sufficient?
No spurious local optima

**Conditions:** \( \exists \) paths \( \{h_x : x \in \bar{X} \setminus X\} \) and a Lyapunov-like function \( V \) such that

- **C1:** both \( f(h_x(t)) \) and \( V(h_x(t)) \) are non-increasing for \( t \in [0, 1] \), and \( f(h_x(0)) > f(h_x(1)) \)
- **C2:** \( \{h_x : x \in \bar{X} \setminus X\} \) is uniformly bounded and uniformly equicontinuous

Local algorithm may converge to any local optimum:

**Examples**
- Global optimum (g.o.): \( b \)
- Pseudo local optimum (p.l.o.): \( c \)
- Genuine local optimum (g.l.o.): \( a, d \)
Main Results

Setup and Motivation

Relaxation and Conditions for Fengyu Zhou

Optimality

Local

No spurious local optima

Conditions: \( \exists \) paths \( \{h_x : x \in \mathcal{X} \setminus X\} \) and a Lyapunov-like function \( V \) such that

- C1: both \( f(h_x(t)) \) and \( V(h_x(t)) \) are non-increasing for \( t \in [0,1] \), and \( f(h_x(0)) > f(h_x(1)) \)
- C2: \( \{h_x : x \in \mathcal{X} \setminus X\} \) is uniformly bounded and uniformly equicontinuous
- C3: \( \exists k > 0 \) such that \( f(h_x(t)) - f(h_x(s)) \geq k\|h_x(t) - h_x(s)\| \)

Local algorithm may converge to any local optimum:

Examples

Global optimum (g.o.): \( b \)
Pseudo local optimum (p.l.o.): \( c \)
Genuine local optimum (g.l.o.): \( a, d \)

- C1, C2 eliminate genuine local optimal \( (a, d) \)
- C3 eliminates pseudo local optimum \( (c) \)
No spurious local optima

**Conditions:** ∃ paths \( \{h_x: x \in \hat{X} \setminus X\} \) and a Lyapunov-like function \( V \) such that

- **C1:** both \( f(h_x(t)) \) and \( V(h_x(t)) \) are non-increasing for \( t \in [0,1] \), and \( f(h_x(0)) > f(h_x(1)) \)
- **C2:** \( \{h_x: x \in \hat{X} \setminus X\} \) is uniformly bounded and uniformly equicontinuous
- **C3:** \( \exists k > 0 \) such that \( f(h_x(t)) - f(h_x(s)) \geq k\|h_x(t) - h_x(s)\| \)

**Theorem** [Zhou 2022]

- **C1, C2** \( \iff \) all local optima of (1) globally optimal & (2) exact
- **C1, C2, C3** \( \implies \) all local optima of (1) globally optimal & (2) exact

Applications: OPF, low rank SDP, …
Suitable for problems with convex cost but nonconvex feasible set
No spurious local optima

**Conditions:** \( \exists \) paths \( \{h_x: x \in \hat{X} \setminus X\} \) and a Lyapunov-like function \( V \) such that

- **C1:** both \( f(h_x(t)) \) and \( V(h_x(t)) \) are non-increasing for \( t \in [0, 1] \), and \( f(h_x(0)) > f(h_x(1)) \)
- **C2:** \( \{h_x: x \in \hat{X} \setminus X\} \) is uniformly bounded and uniformly equicontinuous

- **C3:** \( \exists k > 0 \) such that \( f(h_x(t)) - f(h_x(s)) \geq k \|h_x(t) - h_x(s)\| \)
Application to OPF

Non-convex problem:

\[
\min_{s,v,\ell,S} \quad f(s)
\]
\[
\text{s.t.} \quad \text{convex constr.}
\]
\[
v_{j\ell jk} = |S_{jk}|^2
\]

Relaxed problem:

\[
\min_{s,v,\ell,S} \quad f(s)
\]
\[
\text{s.t.} \quad \text{convex constr.}
\]
\[
v_{j\ell jk} \geq |S_{jk}|^2
\]

Baran-Wu 1989 DistFlow model
Application to OPF

Non-convex problem:

\[
\min_{s,v,\ell,S} f(s) \\
\text{s.t. convex constr.} \\
\nu_j \ell_{jk} = |S_{jk}|^2
\]

Relaxed problem:

\[
\min_{s,v,\ell,S} f(s) \\
\text{s.t. convex constr.} \\
\nu_j \ell_{jk} \geq |S_{jk}|^2
\]

Construction

\[
V := \sum_{jk} \nu_k \ell_{jk} - |S_{jk}|^2
\]

\[h_x: \text{linearly decrease } \ell_{jk} \text{ and linearly adjust } s, S \text{ accordingly.}\]

This construction satisfies C1, C2, C3

Theorem

If there are no lower bounds for \( s_j \), i.e., bus injections, then any local optimum of the original non-convex OPF is also a global optimum.

First result on the local optimality for non-convex OPF problem. [Zhou, Low CDC2020]
Application to OPF

Non-convex problem:

$$\min_{s,v,\ell,S} f(s)$$
$$\text{s.t.} \quad \text{convex constr.}$$
$$v_j \ell_{jk} = |S_{jk}|^2$$

Relaxed problem:

$$\min_{s,v,\ell,S} f(s)$$
$$\text{s.t.} \quad \text{convex constr.}$$
$$v_j \ell_{jk} \geq |S_{jk}|^2$$

Construction (a 2-bus example)

- $V := v_1 \ell_{12} - |S_{12}|^2$
- For $x \in \hat{X} \setminus X$, we have $|S_{12}|^2 - v_1 \ell_{12} < 0$.
- Let $\Delta$ be the positive root of
  $\frac{|z_{12}|^2}{4} a^2 + (v_1 - \text{Re}(z_{12} S_{12}^H)) a + |S_{12}|^2 - v_1 \ell_{12}$
- Consider the path:
  $$\tilde{s}_j(t) = s_j - \frac{t}{2} z_{12} \Delta - \frac{t}{2} z_{12} \Delta,$$
  $$\tilde{v}_j(t) = v_j,$$
  $$\tilde{\ell}_{12}(t) = \ell_{12} - t \Delta,$$
  $$\tilde{S}_{12}(t) = S_{12} - \frac{t}{2} z_{12} \Delta.$$

Construction satisfies C1, C2, C3
- SOCP relaxation is exact
- Local optima are globally optimal

F. Zhou
Summary

OPF is nonconvex & NP hard

OPF is “easy” in practice

- Semidefinite relaxations often exact
- Local algorithms often globally optimal

Analytical properties

- Exact relaxation
- No spurious local optima