

Quadratically constrained quadratic programs on acyclic graphs with application to power flow

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Abstract

This paper proves that non-convex quadratically constrained quadratic programs have an exact semidefinite relaxation when their underlying graph is acyclic, provided the constraint set satisfies a certain technical condition. When the condition is not satisfied, we propose a heuristic to obtain a feasible point starting from a solution of the relaxed problem. These methods are then demonstrated to provide exact solutions to a richer class of optimal power flow problems than previously solved.

Index Terms

Optimal power flow, distribution circuits, radial networks, energy storage, SDP relaxation, minimum semidefinite rank.

I. INTRODUCTION

A quadratically constrained quadratic program (QCQP) is an optimization problem in which the objective function and the constraints are quadratic. In general, QCQPs are non-convex, and therefore lack computationally efficient solution methods. Many engineering problems including

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different types of optimal power flow (OPF) problems can be represented as QCQPs with complex variables. The contribution of this paper is to expand the class of non-convex QCQPs for which globally optimal solutions can be guaranteed.

There is a large literature on optimal or approximate algorithms for QCQPs. Many solution techniques involve semidefinite relaxation (SDR). One such method employs a convex semidefinite program that is a rank relaxation of the given QCQP problem. In some instances, the optimal solution of this relaxation is also the optimal solution of the original problem, in which case the relaxation is said to be exact. SDR provides a tractable way to approach QCQPs [1], [2] and the development of interior-point methods has led to its wide use as a polynomial-time computational tool [3]–[5]. For example, it has been applied to a variety of engineering problems such as, MIMO antenna beam-forming [6]–[9], sensor network localization [10], principal component analysis [11] and stability analysis [12]. SDR has also been shown to be an important tool for systems and control theory [13], [14]. Several authors have investigated exact relaxations, e.g., [15], [16], while others have applied SDR-based approximation techniques to NP-hard combinatorial problems and non-convex QCQPs, e.g., [17]–[19]. The accuracy of these approximations has also been extensively studied, e.g., [20]–[22].

In this paper, we prove a sufficient condition under which QCQPs with underlying acyclic graph structures admit an exact SDR. We apply our result to the optimal power flow (OPF) problem on radial distribution circuits. OPF is generally a non-convex, NP-hard problem that seeks to minimize some cost function, such as power loss, generation cost and/or user utilities, subject to network constraints. A full description of the OPF problem and various solution techniques can be found in surveys, e.g., [23]–[27] and the original 1962 formulation of Carpentier [28]. Recently, the authors in [29], [30] have observed that OPF can be cast as a QCQP and hence it admits an SDR. They show the relaxation to be exact on various IEEE test systems. Using Lagrangian duality, a sufficient condition is derived in [31] under which the SDR is exact and is shown to hold on IEEE benchmark systems and many randomly generated electricity networks. Results specialized to OPF over radial networks were reported in [32]. A similar OPF related result in the context of the geometry of the feasible set of power injections was independently obtained in [33] around the same time. This paper extends the previously known class of OPF problems with efficient algorithms that can guarantee optimal solutions.

The paper is organized as follows. In Section II, we provide conditions on QCQP problems that

guarantee an exact SDR. In Section III, we formulate the OPF problem and apply the result of Section II to show that this SDR of OPF over a radial network is exact under certain conditions that can be interpreted in terms of the power flow constraints. In Section IV we describe a heuristic to obtain solutions with bounds for problems that do not meet these conditions. We obtain numerical results for the OPF problem and show that when the SDR is not exact the deviation of the objective function from a theoretical bound is generally small. We conclude in Section V.

II. QCQP'S AND SEMIDEFINITE RELAXATION

Consider the following QCQP with complex variable $x \in \mathbb{C}^n$.

Primal problem P :

$$\begin{aligned} & \underset{x \in \mathbb{C}^n}{\text{minimize}} && x^* C x \\ & \text{subject to:} && x^* C_k x \leq b_k, \quad k \in \mathcal{K}. \end{aligned}$$

Here, x^* denotes the conjugate transpose of x , C is an $n \times n$ complex positive definite (Hermitian) matrix, i.e., $C \succ 0$. Given a finite index set \mathcal{K} , let C_k be an $n \times n$ complex Hermitian matrix and b_k be a scalar for each $k \in \mathcal{K}$. Problem P may not be convex because the matrices C_k are not necessarily positive semidefinite for all $k \in \mathcal{K}$. Thus, P is generally NP-hard. Hereafter, we assume that P has a feasible solution and denote the optimal value of any variable z as z_* .

The following semidefinite problem RP is a convex relaxation of P where W is an $n \times n$ complex positive semidefinite matrix.

Relaxed Problem RP :

$$\begin{aligned} & \underset{W \succeq 0}{\text{minimize}} && \text{tr}(CW) \\ & \text{subject to:} && \text{tr}(C_k W) \leq b_k, \quad k \in \mathcal{K}. \end{aligned} \tag{1}$$

RP is a relaxation of P because any feasible solution x of P , $W := xx^*$ is also a feasible solution of RP , i.e., the feasible solutions of P are a subset of those of RP . Hence, the optimal value of RP is less than or equal to the optimal value of P . If an optimal solution W_* of RP is of rank 1, then it has a unique decomposition $W_* = x_*[x_*]^*$ and x_* is the optimal solution of P . Moreover, if $\text{rank } W_* = 0$, then $W_* = 0$, and an optimal solution to P is $x_* = 0$. Thus,

an optimal solution W_* to RP can be used to compute an optimal solution x_* to P provided $\text{rank } W_* \leq 1$.

Our main result proves that for a particular class of QCQPs, an optimal solution W_* to RP must have $\text{rank } W_* \leq 1$. Therefore RP is an exact relaxation of P , and P can be solved in polynomial time by solving the convex problem RP .

To this end, we define the graph of a QCQP as follows. Let $[n] := \{1, 2, \dots, n\}$ and M_{ij} represent the element corresponding to the i^{th} row and the j^{th} column of a matrix $M \in \mathbb{C}^{n \times n}$. We denote the graph of the problem P as $\mathcal{G}(P)$, which is defined as an undirected graph on n vertices with indices in $[n]$ such that an edge exists between nodes i and j in $[n]$ if and only if either C_{ij} or some $[C_k]_{ij}$ is nonzero. Thus, for $i \neq j$ in $[n]$,

$$(i, j) \in \mathcal{G}(P) \iff C_{ij} \neq 0 \text{ or } [C_k]_{ij} \neq 0 \text{ for some } k \in \mathcal{K}. \quad (2)$$

Next, we restrict our attention to QCQPs that satisfy the following condition.

- Assumption 1:*
- 1) $\mathcal{G}(P)$ is acyclic, i.e., it is either a tree or a forest (collection of disjoint trees).
 - 2) For all $(i, j) \in \mathcal{G}(P)$, the origin of the complex plane does not lie in the interior of the convex hull of C_{ij} and $[C_k]_{ij}$ for all $k \in \mathcal{K}$.

We now present the main result of the paper and prove it through a sequence of lemmas.

Theorem 1: If Assumption 1 holds then the relaxation RP is exact, i.e. the optimal solution of P can be obtained from that of RP . Hence, P can be solved in polynomial time.

In the rest of this section, we prove the theorem for the case where the underlying graph is a tree. In what follows, we use the notation $\mathcal{G}(X) = \mathcal{T}$ to indicate that the underlying graph of a problem X is a tree. Our proof generalizes to problems over any acyclic graph, e.g., when $\mathcal{G}(P)$ is a forest.

Let the Lagrange multipliers for the inequalities in (1) be $\lambda_k \geq 0$ for each $k \in \mathcal{K}$. Then the Lagrangian dual of RP is

Dual problem DP :

$$\begin{aligned} & \underset{\lambda \geq 0}{\text{maximize}} && - \sum_{k \in \mathcal{K}} \lambda_k b_k \\ & \text{subject to} && C + \sum_{k \in \mathcal{K}} \lambda_k C_k \succeq 0. \end{aligned}$$

Let p_* , r_* and d_* denote the optimal values of P , RP and DP respectively. Then

$$p_* \geq r_* \geq d_*.$$

We prove the theorem by showing that when Assumption 1 holds,

$$p_* = r_* = d_*.$$

Define the $n \times n$ matrix $A(\lambda)$ as

$$A(\lambda) := C + \sum_{k \in \mathcal{K}} \lambda_k C_k. \quad (3)$$

Lemma 2: 1) $r_* = d_*$ and RP/DP has a finite primal-dual optimal point (W_*, λ_*) .

2) If $\text{rank } A(\lambda_*) \geq n - 1$, then $\text{rank } W_* \leq 1$ and $p_* = r_* = d_*$.

Proof:

- 1) RP is feasible because P is feasible. Since $C \succ 0$, choose $\lambda > 0$ arbitrarily small as compared to the smallest eigenvalue of C . For this choice of $\lambda > 0$, $A(\lambda) \succ 0$ and hence λ is a strictly feasible point of DP . The rest follows from Slater's condition.
- 2) The complementary slackness condition for optimality of (W_*, λ_*) implies

$$\text{tr}[W_* A(\lambda_*)] = \sum_i \rho_i w_i^* [A(\lambda_*)] w_i = 0$$

where $W_* = \sum_i \rho_i w_i w_i^*$ is the spectral decomposition of W_* . This implies that the eigenvectors w_i of W_* corresponding to nonzero eigenvalues ρ_i are all in the null space of $A(\lambda_*)$. Since $A(\lambda_*)$ is positive semidefinite with rank at least $n - 1$, its null space has rank at most 1, implying that $\text{rank } W_* \leq 1$. Hence $p_* = r_*$. ■

With an abuse of notation we define the graph $\mathcal{G}(H)$ for an $n \times n$ Hermitian matrix H as an undirected graph on n vertices with indices in $[n]$, such that for all i, j in $[n]$ where $i \neq j$, there is an edge between vertices i and j if and only if $H_{ij} \neq 0$, and no edges from a vertex to itself. A graph on n nodes is a *tree* if it has $n - 1$ edges and no cycles. Now we can express the graph of the problem P as

$$\mathcal{G}(P) = \mathcal{G}(C) \cup \left(\bigcup_{k \in \mathcal{K}} \mathcal{G}(C_k) \right).$$

Define

$$A_* := A(\lambda_*).$$

Lemma 3: If $\mathcal{G}(A_*) = \mathcal{T}$, then $\text{rank } W_* \leq 1$ and $p_* = r_* = d_*$.

Proof: The following statement is proved in e.g., [34], [35, Theorem 3.4] and [36, Corollary 3.9]). If an $n \times n$ Hermitian matrix A is positive semidefinite and the associated graph $\mathcal{G}(A)$ is a tree, then $\text{rank } A \geq n - 1$. Using this result, the proof follows from Lemma 2. ■

Lemma 3 addresses the case where we have $[A_*]_{ij} \neq 0$ for all edges (i, j) in tree \mathcal{T} and thus Theorem 1 follows. We now consider the case where $[A_*]_{ij} = 0$ for one or more of the edges (i, j) in \mathcal{T} . The key idea here is to perturb the original problem RP/DP so that Lemma 3 holds in the perturbed problem and finally recover a rank $W_* \leq 1$ for the original problem using a continuity argument. In particular, define the perturbed problems for $\epsilon > 0$:

Perturbed QCQP P^ϵ :

$$\begin{aligned} & \underset{x \in \mathbb{C}^n, \alpha \geq 0}{\text{minimize}} && x^* C x + \epsilon \sum_k \alpha_k \\ & \text{subject to:} && x^* C_k x + \alpha_k = b_k, \quad k \in \mathcal{K}. \end{aligned}$$

Perturbed relaxed problem RP^ϵ :

$$\begin{aligned} & \underset{W \succeq 0, \alpha \geq 0}{\text{minimize}} && \text{tr}(CW) + \epsilon \sum_k \alpha_k \\ & \text{subject to:} && \text{tr}(C_k W) + \alpha_k = b_k, \quad k \in \mathcal{K}. \end{aligned}$$

Perturbed dual problem DP^ϵ :

$$\begin{aligned} & \underset{\lambda}{\text{maximize}} && - \sum_{k \in \mathcal{K}} \lambda_k b_k \\ & \text{subject to} && A(\lambda) \succeq 0, \quad \lambda_k \geq \epsilon, \quad k \in \mathcal{K}. \end{aligned}$$

For any variable z in the original problem, let z^ϵ denote the corresponding variable in the perturbed problem. The following extends Lemmas 2 and 3 to the perturbed problems.

Lemma 4: P^ϵ is feasible and there exists $\epsilon_0 > 0$ such that for all ϵ in the open interval $(0, \epsilon_0)$:

- 1) $r_*^\epsilon = d_*^\epsilon$ and RP^ϵ/DP^ϵ has a finite optimum primal-dual point $\lambda_*^\epsilon, W_*^\epsilon, \alpha_*^\epsilon$.
- 2) $\mathcal{G}(A_*^\epsilon) = \mathcal{T}$ and hence $\text{rank } W_*^\epsilon \leq 1$ and $p_*^\epsilon = r_*^\epsilon$.

Proof:

1) Since $C \succ 0$ and $A(\lambda) = C + \sum_k \lambda_k C_k$, there exists a sufficiently small $\epsilon_0 > 0$ such that

$$A(\epsilon_0 \mathbf{1}) \succ 0,$$

where $\mathbf{1}$ is a vector of all ones of appropriate size. Thus $\lambda = \epsilon_0 \mathbf{1}$ is a strictly feasible point for problem DP^ϵ for all $\epsilon \in (0, \epsilon_0)$. The rest follows from Slater's condition.

2) We prove this by contradiction. For some edge (i, j) in \mathcal{T} let

$$[A_*^\epsilon]_{ij} = [C]_{ij} + \sum_{k \in \mathcal{K}} (\lambda_*^\epsilon)_k [C_k]_{ij} = 0.$$

Since $\lambda_*^\epsilon \geq \epsilon \mathbf{1} > 0$, this implies that zero is strictly in the interior of the convex hull of $[C]_{ij}$ and $[C_k]_{ij}$, $k \in \mathcal{K}$, thus violating the hypothesis in Assumption 1. Lemma 3 completes the proof. ■

Thus we have

$$p_*^\epsilon = r_*^\epsilon = d_*^\epsilon, \quad \text{and}$$

$$\text{rank } W_*^\epsilon \leq 1 \quad \text{for all } \epsilon \in (0, \epsilon_0).$$

Consider a decreasing sequence of $\epsilon \in (0, \epsilon_0)$ that converges to $\epsilon = 0$. Consider a feasible point W_f of RP . Note that W_f is also a feasible point of RP^ϵ . Since $C \succ 0$, we have

$$W_f \succeq W_*^\epsilon \succeq 0.$$

Hence, W_*^ϵ lies in a compact space for $\epsilon \in (0, \epsilon_0)$. Since the set of positive semidefinite matrices with rank at most 1 is closed [37], the limit W_* of the convergent subsequence of the optimal solutions W_*^ϵ to RP^ϵ can have at most rank 1. It can be checked that W_* satisfies the KKT condition for RP and is therefore optimal for RP . This completes the proof of Theorem 1.

III. OPTIMAL POWER FLOW: AN APPLICATION

In this section, we apply the results of Section II to the optimal power flow (OPF) problem. We start by summarizing some of the recent results on OPF relaxations in Section III-A. In Section III-B we formulate OPF in the complex domain. We then restrict attention to OPF over distribution circuits that are usually radial, i.e., they have a tree topology. In Section III-C, we show that the SDR of OPF over radial networks is exact and interpret the required conditions in terms of the power flow constraints. In Section III-D, we outline how these results generalize for a network with storage devices.

A. Prior work

As previously discussed, the observation that OPF can be written as a quadratically constrained quadratic program and therefore approximated by a semidefinite program was first reported in [29], [30]. Their simulations indicate that the relaxation is exact for many IEEE benchmark systems. Analyzing the SDP formulation, the authors in [31], [38] derive a sufficient condition under which the SDR is exact and propose obtaining the solution of the OPF by solving its convex Lagrangian dual. The limitations of the SDR approach were most recently explored in in [39]. A number of other examples of non-convexity in the OPF setting have also been examined, see e.g., [40]–[42].

Recently, the role of network structure in OPF has been of considerable interest. It has been independently reported in [32], [33], [43] that the SDR of OPF is exact for radial networks provided certain conditions on the power flow constraints are satisfied. Various second-order cone programming (SOCP) approaches have also been used for similar problems on radial networks, e.g., [44] investigates an SOCP relaxation on the branch flow model. A different approach is taken in [45], [46] using the model first introduced in [47], [48]. While [46] explores a linear approximation of the model, [45] proposes an SOCP relaxation and proves it to be exact under certain conditions. It is interesting that the conditions required for exact SOCP relaxation in [45] is similar to the conditions required for exact SDR in [32], [33], [43]. In this section, we use Theorem 1 to generalize some of the prior results on SDR of OPF in tree networks.

B. Problem Formulation

Consider a power systems network with n nodes (buses). The admittance-to-ground at bus i is y_{ii} and the admittance of the line between connected nodes i and j (denoted by $i \sim j$) is $y_{ij} = g_{ij} - \mathbf{i}b_{ij}$. We assume both $g_{ij} > 0$ and $b_{ij} > 0$, i.e., the lines are resistive and inductive. Define the corresponding $n \times n$ admittance matrix Y as

$$Y_{ij} = \begin{cases} y_{ii} + \sum_{j \sim i} y_{ij}, & \text{if } i = j, \\ -y_{ij}, & \text{if } i \neq j \text{ and } i \sim j, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

Remark 1: Y is symmetric but not necessarily Hermitian.

The remaining circuit parameters and their relations are defined as follows.

- V and I are n -dimensional complex voltage and current vectors, where V_k, I_k denote the voltage and the injection current at bus $k \in [n]$ respectively. The voltage magnitude at each bus is bounded as

$$0 < \underline{W}_k \leq |V_k|^2 \leq \overline{W}_k, \quad k \in [n].$$

- $S = P + \mathbf{i}Q$ is the n -dimensional complex power vector, where P and Q respectively denote the real and reactive powers and

$$S_k = P_k + \mathbf{i}Q_k = V_k I_k^*, \quad k \in [n]. \quad (5)$$

- P_k^D and Q_k^D are the real and reactive power demands at bus $k \in [n]$, which are assumed to be fixed and given.
- P_k^G and Q_k^G are the real and reactive power generation at bus k . They are decision variables that satisfy the constraints $\underline{P}_k^G \leq P_k^G \leq \overline{P}_k^G$ and $\underline{Q}_k^G \leq Q_k^G \leq \overline{Q}_k^G$.

Power balance at each bus $k \in [n]$ requires $P_k^G = P_k^D + P_k$ and $Q_k^G = Q_k^D + Q_k$, which leads us to define

$$\begin{aligned} \underline{P}_k &:= \underline{P}_k^G - P_k^D, & \overline{P}_k &:= \overline{P}_k^G - P_k^D \\ \underline{Q}_k &:= \underline{Q}_k^G - Q_k^D, & \overline{Q}_k &:= \overline{Q}_k^G - Q_k^D. \end{aligned}$$

The power injections at each bus $k \in [n]$ are then bounded as

$$\underline{P}_k \leq P_k \leq \overline{P}_k, \quad \underline{Q}_k \leq Q_k \leq \overline{Q}_k.$$

The branch power flows and their limits are defined as follows.

- $S_{ij} = P_{ij} + \mathbf{i}Q_{ij}$ is the sending-end complex power flow from node i to node j , where P_{ij} and Q_{ij} are the real and reactive power flows respectively. The real power flows are constrained as $|P_{ij}| \leq \overline{F}_{ij}$ where \overline{F}_{ij} is the line-flow limit between nodes i and j and $\overline{F}_{ij} = \overline{F}_{ji}$.
- $L_{ij} = P_{ij} + P_{ji}$ is the power loss over the line between nodes i and j , satisfying $L_{ij} \leq \overline{L}_{ij}$ where \overline{L}_{ij} is the thermal line limit and $\overline{L}_{ij} = \overline{L}_{ji}$. Also, observe that since $L_{ij} \geq 0$, we have $|P_{ij}| \leq \overline{F}_{ij}, |P_{ji}| \leq \overline{F}_{ji}$ if and only if $P_{ij} \leq \overline{F}_{ij}, P_{ji} \leq \overline{F}_{ji}$.

Let $J_k = e_k e_k^*$ where e_k is the k -th canonical basis vector in \mathbb{C}^n . Define $Y_k := e_k e_k^* Y$. Substituting these expressions into (5) yields

$$\begin{aligned} S_k &= e_k^* V I^* e_k = \text{tr}(V V^* (Y^* e_k e_k^*)) = V^* Y_k^* V \\ &= \left(V^* \underbrace{\left(\frac{Y_k^* + Y_k}{2} \right)}_{=: \Phi_k} V \right) + \mathbf{i} \left(V^* \underbrace{\left(\frac{Y_k^* - Y_k}{2\mathbf{i}} \right)}_{=: \Psi_k} V \right), \end{aligned} \quad (6)$$

where Φ_k and Ψ_k are Hermitian matrices. Thus, the two quantities $V^* \Phi_k V$ and $V^* \Psi_k V$ are real numbers; moreover

$$P_k = V^* \Phi_k V, \quad Q_k = V^* \Psi_k V.$$

The real power flow from i to j can be expressed as a quadratic form as follows.

$$P_{ij} = \text{Re}\{V_i (V_i - V_j)^* y_{ij}^*\} = V^* M^{ij} V, \quad (7)$$

where M^{ij} is an $n \times n$ Hermitian matrix. Further details of the OPF problem formulation are provided in the appendix.

The thermal loss of the line connecting buses i and j is

$$L_{ij} = L_{ji} = P_{ij} + P_{ji} = V^* T^{ij} V \quad (8)$$

where $T^{ij} = T^{ji} := M^{ij} + M^{ji} \succeq 0$.

For a Hermitian positive semidefinite $n \times n$ matrix C , we have

Optimal power flow problem OPF:

$$\underset{V \in \mathbb{C}^n}{\text{minimize}} \quad V^* C V$$

subject to:

$$\underline{P}_k \leq V^* \Phi_k V \leq \overline{P}_k, \quad k \in [n], \quad (9a)$$

$$\underline{Q}_k \leq V^* \Psi_k V \leq \overline{Q}_k, \quad k \in [n], \quad (9b)$$

$$\underline{W}_k \leq V^* J_k V \leq \overline{W}_k, \quad k \in [n], \quad (9c)$$

$$V^* M^{ij} V \leq \overline{F}_{ij}, \quad i \sim j, \quad (9d)$$

$$V^* T^{ij} V \leq \overline{L}_{ij}, \quad i \sim j, \quad (9e)$$

where (9a)–(9e) are constraints on the real and reactive powers, the voltage magnitudes, the line flows and thermal losses. Note that since $T^{ij} \succeq 0$, (8) implies that $P_{ij} + P_{ji} \geq 0$. This means that (9d) holds if and only if $|P_{ij}| \leq \bar{F}_{ij}$, i.e., (9d) actually bounds the line flows on both ends.

Remark 2 (Objective Functions): Different optimality criteria can be modeled by changing C as follows.

- Voltages: $C = \mathcal{I}_{n \times n}$ (identity matrix) where we aim to minimize $\|V\|^2 = \sum_k |V_k|^2$.
- Power loss: $C = (Y + Y^*)/2$ where we aim to minimize $\sum_i g_{ii}|V_i|^2 + \sum_{i < j} g_{ij}|V_i - V_j|^2$.
- Production costs: $C = \sum_k c_k \Phi_k$ where we aim to minimize $\sum_k c_k P_k^G$, $c_k \geq 0$.

We assume C is positive semidefinite.

C. Semidefinite relaxation of OPF over radial networks

Assume hereafter that *OPF* is feasible. To conform to the notations of Section II, we replace the the expression in (9a) by the equivalent constraints

$$\begin{aligned} V^*[\Phi_k]V &\leq \bar{P}_k, & k \in [n] \\ V^*[-\Phi_k]V &\leq -\underline{P}_k, & k \in [n]. \end{aligned}$$

We similarly rewrite (9b) and (9c). Note that the set of matrices C_k , $k \in \mathcal{K}$ in P is defined as

$$(C_k, k \in \mathcal{K}) := (\Phi_k, -\Phi_k, \Psi_k, -\Psi_k, J_k, -J_k, k \in [n]) \cup (M^{ij}, T^{ij}, i \sim j) \quad (10)$$

for the OPF problem. Then, the relaxed problem *RP*, the definition of the Lagrange multipliers λ , the dual problem *DP* and the matrix A defined in (3) naturally carry over. Then, the sufficient condition for exact rank relaxation of *OPF* (also derived in [31], [43]) is as follows.

Corollary 5: If A_* has rank $n - 1$, then the SDR of *OPF* is exact. Hence, *OPF* can be solved in polynomial time.

For the objective functions considered in the OPF, the matrix C may not be positive definite and thus Lemma 2 may not be applicable. However, the result still holds. The details of the modified proof are provided in the appendix.

We now restrict attention to *OPF* where the graph of the power network is a tree \mathcal{T} on n nodes. Thus the bus-admittance matrix Y satisfies

$$\mathcal{G}(Y) = \mathcal{T}. \quad (11)$$

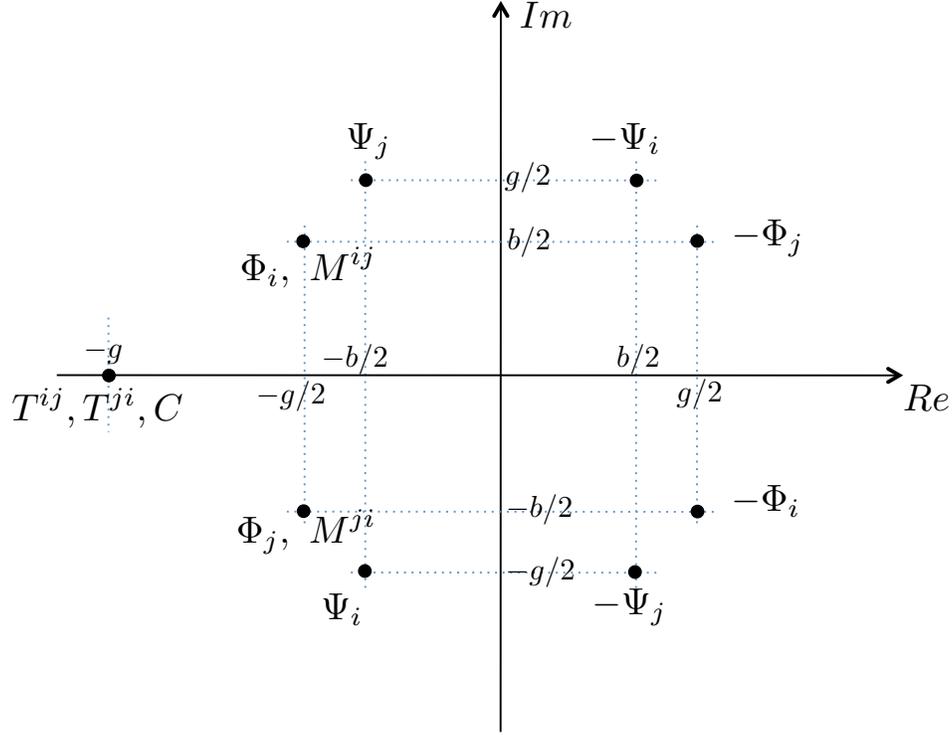


Fig. 1: C_{ij} and $([C_k]_{ij}, k \in \mathcal{K})$ on the complex plane for OPF for a fixed line (i, j) in tree \mathcal{T} .

Also, it follows that the graph of OPF is the same tree, i.e.,

$$\mathcal{G}(OPF) = \mathcal{T}.$$

Thus OPF on a radial network is a QCQP over a tree graph. Using Lemma 3, we get:

Corollary 6: If $\mathcal{G}(A_*) = \mathcal{T}$, then the SDR of OPF is exact and OPF can be solved in polynomial time.

Now, we explore constraint patterns under which the SDR of OPF over a radial network is exact, which implies that this OPF can be solved in polynomial time. Using Theorem 1, we outline how general constraint patterns can be generated and illustrate the scheme through examples.

Example 1: Consider an edge (i, j) in \mathcal{T} . The admittance of the line joining buses i and j is $g_{ij} - \mathbf{i}b_{ij}$. Assume $g_{ij} > b_{ij} > 0$. Now consider $([C_k]_{ij}, k \in \mathcal{K})$:

(a) $[\Phi_i]_{ij} = -g_{ij}/2 + \mathbf{i}b_{ij}/2,$

- (b) $[\Phi_j]_{ij} = -g_{ij}/2 - \mathbf{i}b_{ij}/2,$
- (c) $[\Psi_i]_{ij} = -b_{ij}/2 - \mathbf{i}g_{ij}/2,$
- (d) $[\Psi_j]_{ij} = -b_{ij}/2 + \mathbf{i}g_{ij}/2,$
- (e) $[M^{ij}]_{ij} = -g_{ij}/2 + \mathbf{i}b_{ij}/2,$
- (f) $[M^{ji}]_{ij} = -g_{ij}/2 - \mathbf{i}b_{ij}/2,$
- (g) $[T^{ij}]_{ij} = [T^{ji}]_{ij} = -g_{ij}.$

This follows from the matrix definitions given in the appendix. Also, for the objective functions considered, we have:

- (a) Voltages: $C_{ij} = 0,$
- (b) Power loss: $C_{ij} = -g_{ij},$
- (c) Production costs: $C_{ij} = -g_{ij}(c_i + c_j)/2 + \mathbf{i}b_{ij}(c_i - c_j)/2.$

In this example, let us consider power-loss minimization. Thus, $C_{ij} = -g_{ij}$. We draw the points $([C_k]_{ij}, k \in \mathcal{K})$ and C_{ij} in Figure 1 (subscript (i,j) is dropped in the diagram). Now, consider the (i,j) -th elements of the following set of matrices (represented as points in Figure 1):

$$(\Phi_i, \Phi_j, \Psi_i, \Psi_j, -\Psi_i, M^{ij}, M^{ji}, T^{ij} = T^{ji}, C).$$

The origin of the complex plane lies on the boundary of the convex hull of these points but not in its interior. With this set of points, associate a constraint pattern defined as follows. For any point in the diagram that is not a part of this set, the inequality associated with that matrix is removed from OPF . For example, the matrices $-\Psi_j$, $-\Phi_i$ and $-\Psi_j$ do not feature on the list of points. Hence,

$$\underline{P}_j = \underline{P}_i = \underline{Q}_j = -\infty. \quad (12)$$

This can be generalized to a constraint pattern over the tree \mathcal{T} by removing the lower bounds on real power at all nodes and the lower bound on reactive power at alternate nodes. Similarly, if the constraints on OPF are such that for every edge (i,j) in \mathcal{T} , the set of points in the complex plane represented by C_{ij} and $[C_k]_{ij}$ does not contain the origin in the interior of its convex hull, Theorem 1 implies that OPF with that constraint pattern has a polynomial time solution. Using this idea, we present a result that follows from the example discussed.

For any of the objective functions discussed:

Corollary 7: If $\underline{P}_k = \underline{Q}_k = -\infty$ for all nodes k in a tree \mathcal{T} , then the semidefinite relaxation of OPF is exact and OPF can be solved in polynomial time.

Remark 3: Removing the lower bounds in real and reactive power can be interpreted as load over-satisfaction, i.e., the real and reactive powers supplied to a node k can be greater than their respective real and reactive power demands P_k^D and Q_k^D . This result has been reported in [32], [33], [43].

Example 2: Let us consider voltage minimization, i.e., $C = \mathcal{I}_{n \times n}$. Also, for all adjacent nodes $i \sim j$ in \mathcal{T} , let $g_{ij} > b_{ij} > 0$. Now, consider the (i, j) -th elements of the following set of matrices

$$(-\Phi_i, \Phi_j, -\Phi_j, \Psi_i, -\Psi_j, C).$$

The origin of the complex plane is again on the boundary of the convex hull of these points. The constraint pattern associated with this set of points is

$$\bar{P}_i = \bar{Q}_j = \bar{L}_{ij} = \bar{L}_{ji} = \bar{F}_{ij} = \bar{F}_{ji} = +\infty, \quad \text{and} \quad \underline{Q}_i = -\infty.$$

This constraint pattern is consistent with Assumption 1 over the edge (i, j) . Similarly we can construct a constraint pattern for $b_{ij} \geq g_{ij} > 0$. These define a constraint pattern for the OPF problem, such that its SDR is exact.

D. Storage over a radial network

Here, we outline how the results of the previous sections can be extended to include storage devices distributed throughout the radial network. The charge/discharge dynamics of the storage elements lead to an optimization over a finite time horizon as in [49]. Consider discrete time steps $t = 1, 2, \dots, \tau$. Let $r_k(t)$ denote the average power used to charge/discharge the storage unit at bus $k \in [n]$ at time t . At each time t the ramp rates are bounded as

$$\underline{R}_k \leq r_k(t) \leq \bar{R}_k, \quad k \in [n]. \quad (13)$$

Let $B_k \geq 0$ be the storage capacity at bus k and b_k^0 is the storage level at node k at time $t = 0$. Thus, for each time t ,

$$0 \leq b_k^0 + \sum_{u=1}^t r_k(u) \leq B_k, \quad k \in [n], \quad (14)$$

where by abuse of notation the energy transacted over a time-step, the storage capacity B_k and initial condition b_k^0 in (14) are divided by the length of the time-step in order to convert them to units of power. This transformation allows us to write all equations in power units.

Optimal power flow with storage (SOPF):

$$\underset{V_k(t), r_k(t)}{\text{minimize}} \sum_{t=1}^T V(t)^* C V(t),$$

subject to: (13), (14) and

$$\underline{P}_k(t) \leq V(t)^* \Phi_k V(t) + r_k(t) \leq \overline{P}_k(t), \quad k \in [n] \quad (15a)$$

$$\underline{Q}_k(t) \leq V(t)^* \Psi_k V(t) \leq \overline{Q}_k(t), \quad k \in [n] \quad (15b)$$

$$\underline{W}_k(t) \leq V(t)^* J_k V(t) \leq \overline{W}_k(t), \quad k \in [n] \quad (15c)$$

$$V(t)^* M^{ij} V(t) \leq \overline{F}_{ij} \quad i \sim j, \quad (15d)$$

$$V(t)^* T^{ij} V(t) \leq \overline{L}_{ij}, \quad i \sim j, \quad (15e)$$

where $t = 1, 2, \dots, \tau$. As in [49], let $\lambda(t)$ denote the Lagrange multipliers for the constraints at time t . It can be verified that the dual problem can be written as τ linear matrix inequalities $A[\lambda(t)] \succeq 0$. Since the graph of each of these matrices $A[\lambda(t)]$ is a tree, the results of Section III-C are applicable.

IV. NUMERICAL EXAMPLES

A. Numerical techniques

For a QCQP over a tree, Assumption 1 guarantees W_* such that $\text{rank } W_* \leq 1$. When it does not, W_* may or may not be rank ≤ 1 . If the rank relaxation is exact, we recover the optimum solution x_* to P from $W_* = (x_*)(x_*)^*$. However, if $\text{rank } W_* > 1$, then W_* does not identify a feasible solution to P . In such cases, we describe a heuristic approach to obtain a feasible point \tilde{x} of P , starting from W_* . The feasible point \tilde{x} satisfies

$$\begin{aligned} \text{objective value of } RP \text{ at } W_* &\leq \text{optimum objective value of } P \\ &\leq \text{objective value of } P \text{ at } \tilde{x}. \end{aligned} \quad (16)$$

In many practical problems where $\text{rank } W_* > 1$, the principal eigenvalue of W_* is often orders of magnitude greater than the other eigenvalues. We use this principal eigenvector (appropriately scaled) as a starting point to search for a “nearby” feasible point of P . Specifically, let $w_* \in \mathbb{C}^n$

be the principal eigenvector of W_* . Define the starting point x_0 of the algorithm as

$$x_0 := w_* \sqrt{\text{tr}(CW_*)}.$$

This scaling ensures that the objective value at x_0 is the optimum objective value of RP at W_* .

If x_0 satisfies all constraints in P , then the algorithm ends with $\tilde{x} = x_0$. Otherwise, we construct a sequence of points (x_1, x_2, \dots) where x_{m+1} is constructed from x_m as follows.

- 1) For $k \in \mathcal{K}$, linearize the function $f_k(x) = x^* C_k x$ around the point x_m and call this function $f_k^{(m)}(x)$, i.e.,

$$f_k^{(m)}(x) = x_m^* C_k x_m + 2 \text{Re} [x_m^* C_k (x - x_m)].$$

- 2) For $k \in \mathcal{K}$, define

$$s_k^{(m)}(x) := \begin{cases} \underline{b}_k - f_k^{(m)}(x), & \text{if } f_k^{(m)}(x) \leq \underline{b}_k, \\ 0 & \text{if } \underline{b}_k \leq f_k^{(m)}(x) \leq \bar{b}_k, \\ f_k^{(m)}(x) - \bar{b}_k, & \text{if } \bar{b}_k \leq f_k^{(m)}(x). \end{cases}$$

We can interpret $s_k^{(m)}(x)$ as the amount by which the linearized function $f_k^{(m)}$ violates the inequality constraint $\underline{b}_k \leq f_k^{(m)}(x) \leq \bar{b}_k$.

- 3) Compute x_{m+1} using

$$\begin{aligned} x_{m+1} = \arg \min_{x \in \mathbb{C}^n} & \sum_{k \in \mathcal{K}} [s_k^{(m)}(x)]^2 \\ \text{subject to:} & \|x - x_m\|_1 \leq \theta, \end{aligned}$$

where $\|\cdot\|_1$ denotes the L^1 norm and θ is the maximum allowable step-size. This is a parameter for the algorithm and should be chosen such that the linearization $f_k^{(m)}(x)$ is a reasonably good approximation of the quadratic form $f_k(x)$ for all $k \in \mathcal{K}$ in the L^1 ball centered around x_m with radius θ .

- 4) If x_{m+1} satisfies all constraints in P , then the algorithm ends with $\tilde{x} = x_{m+1}$.

This heuristic approach either ends at a feasible point \tilde{x} of P or it fails to produce one within a fixed number of iterations. We show through numerical examples on OPF, that it yields a ‘‘nearby’’ feasible solution of OPF when the SDR is not exact.

B. OPF test examples

SDR of the OPF problem was tested on a sample distribution circuit from Southern California and other randomly generated radial circuits. The rank relaxation of OPF on these circuits was solved in MATLAB using YALMIP [50]. If the solution yielded W_* such that $\text{rank } W_* = 1$ then the optimal voltage profile (V_*) to the OPF problem was calculated from $W_* = (V_*)(V_*)^*$. However, if the relaxation was not exact, we used the heuristic approach to get a feasible point of OPF. Though the feasible point is not optimal for the original problem, we show its efficacy by calculating the objective value at the feasible point and showing it to be close to the objective value at the infeasible optimal point of the relaxed problem. To quantify the results, we report the statistics of the following quantity:

$$\eta := \frac{\text{Objective value at heuristically reached feasible point}}{\text{Objective value at optimal point of relaxed problem}} - 1.$$

Let $y = (a, b)$ denote that y is drawn from a uniform distribution over the interval $[a, b]$. Using this notation, we describe the test systems used for simulations.

- 1) *SoCal Distribution Circuit*: The sample industrial distribution system in Southern California was previously reported in [45]. It has a peak load of approximately 11.3 MW and has installed PV generation capacity of 6.4MW. We modified this circuit by removing the 30MW load at the substation bus (that represented other distribution circuits fed by the same substation) and simulated it with other parameters as in Table I. To scale the problem correctly, we cast the problem in per unit (p.u.) quantities using the base values given in Table I.
- 2) *Random Test Circuits*: The random test circuits have been generated using parameters typical of sparsely loaded rural circuits, as detailed in [51] and employed (with suitable modifications) in [52], [53]. We assume that 15-60% of the nodes have an installed PV capacity of 2 kW each. Other parameters are described in Table I.

We ran tests with both voltage and power-loss minimization as objective functions. The optimization results are summarized in Table II. For power-loss minimization, we always obtained a rank 1 optimal W_* and hence the relaxation was exact. For voltage minimization, the relaxation was inexact in many cases and the heuristic approach was used. Specifically, we used the voltage magnitudes and the phase angles at all buses except the substation bus (node 1) as our variables and allowed arbitrarily large step-size ($\theta = +\infty$). This approach always yielded a feasible point

Test system	SoCal distribution circuit	Random radial networks
Number of nodes (n)	47	50-150
Line impedances $(y_{ij})^{-1}$	[45, Table 1]	$(0.33 + i0.38)\Omega/km$, length = $(0.2km, 0.3km)$
Voltage limits $\sqrt{W_k}, \sqrt{W_k}$	1 ± 0.05 p.u. at all nodes.	1 ± 0.05 p.u. at all nodes.
Real power demand P_k^D	[45, Table 1]	$(0, 4.5kW)$
Reactive power demand Q_k^D	Computed with $p.f. = (0.80, 0.98)$ lagging	$(0.2P_k^D, 0.3P_k^D)$
Real power gen. limits $\bar{P}_k^G, \underline{P}_k^G$	PV nodes: $\bar{P}_k^G = (0.2, 1.0)$ times capacity, Substation node: $\bar{P}_1^G = 10MW$. At all nodes, $\underline{P}_k^G = 0$.	PV nodes: $\bar{P}_k^G = (0, 2kW)$, Substation node: \bar{P}_1^G scaled with n . At all nodes, $\underline{P}_k^G = 0$.
Reactive power gen. limits	$\bar{Q}_k^G = 0.3\bar{P}_k^G, \underline{Q}_k^G = -0.3\bar{P}_k^G$ at all nodes.	$\bar{Q}_k^G = 0.3\bar{P}_k^G, \underline{Q}_k^G = -0.3\bar{P}_k^G$ at all nodes.
Base quantities	$P_{base} = 1MW, V_{base} = 12.35kV(L - L)$.	$P_{base} = 1MW, V_{base} = 12.47kV(L - L)$.

TABLE I: Circuit Parameters for SDR of *OPF*

Test system	SoCal distribution circuit		Random radial networks	
Minimize	Power-loss	Voltage	Power-loss	Voltage
rank W_*	1	≥ 1	1	≥ 1
Mean η	N/A	1.8%	N/A	0.5%
Maximum η	N/A	4.1%	N/A	1.5%

TABLE II: Summary of simulation results

within 5 iterations. The statistics of the parameter η suggests that the heuristics locate a feasible point of *OPF* with an objective value close to the optimal objective value of the relaxed problem.

V. CONCLUSION

This paper broadens the class of QCQP problems for which efficient optimal solutions are known. We have also presented a heuristic to find an approximate solution with an optimality bound for problems outside this expanded class. Numerical results on OPF problems are used to illustrate the algorithm.

VI. APPENDIX: FURTHER DETAILS OF THE OPF PROBLEM

A. Proof of Lemma 5

Let $\bar{\gamma}_k, \underline{\gamma}_k$ be the Lagrange multipliers for the voltage constraints (9c) and $\tilde{\lambda}$ denote the multipliers for all other constraints. Choose $\bar{\gamma}_k = 1 + \delta$, $\underline{\gamma}_k = 1$ and $\tilde{\lambda} > 0$ sufficiently small as compared to δ , the smallest eigenvalue of $C + \delta\mathcal{I}$. Note that for this choice of the Lagrange multipliers, the matrix A is indeed positive definite and the rest follows from Slater's condition and infeasibility of $V = 0$ for OPF .

B. Matrices $(C_k, k \in \mathcal{K})$ for OPF

From (6), (7), (8), we have the following relations for $k \in [n]$, and (p, q) and (i, j) in \mathcal{T} :

$$[\Phi_k]_{ij} = \begin{cases} \frac{1}{2}Y_{ij} = \frac{1}{2}(-g_{ij} + \mathbf{i}b_{ij}) & \text{if } k = i \\ \frac{1}{2}Y_{ij}^* = \frac{1}{2}(-g_{ij} - \mathbf{i}b_{ij}) & \text{if } k = j \\ 0 & \text{if } k \neq i, k \neq j, \end{cases} \quad (17)$$

$$[\Psi_k]_{ij} = \begin{cases} \frac{-1}{2\mathbf{i}}Y_{ij} = \frac{1}{2}(-b_{ij} - \mathbf{i}g_{ij}) & \text{if } k = i \\ \frac{1}{2\mathbf{i}}Y_{ij}^* = \frac{1}{2}(-b_{ij} + \mathbf{i}g_{ij}) & \text{if } k = j \\ 0 & \text{if } k \neq i, k \neq j, \end{cases} \quad (18)$$

$$[M^{pq}]_{ij} = \begin{cases} g_{pq} & \text{if } i = j = p \\ \frac{1}{2}(-g_{pq} + \mathbf{i}b_{pq}) & \text{if } (i, j) = (p, q) \\ \frac{1}{2}(-g_{pq} - \mathbf{i}b_{pq}) & \text{if } (i, j) = (q, p) \\ 0 & \text{otherwise,} \end{cases} \quad (19)$$

$$[T^{pq}]_{ij} = \begin{cases} g_{pq} & \text{if } i = j = p \text{ or } i = j = q \\ -g_{pq} & \text{if } (i, j) = (p, q) \text{ or } (i, j) = (q, p) \\ 0 & \text{otherwise.} \end{cases} \quad (20)$$

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