Analysis of Nonlinear Delay Differential Equation Models of TCP/AQM Protocols Using Sums of Squares

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***Abstract*— The simplest adequate models for congestion control for the Internet are in the form of deterministic nonlinear delay differential equations. However the absence of efficient, algorithmic methodologies to analyze them at this modelling level usually results in the investigation of their linearizations including delays; or in the analysis of nonlinear yet undelayed models. In this paper we present an algorithmic methodology for efficient stability analysis of network congestion control schemes at the nonlinear delay-differential equation model level, using the Sum of Squares decomposition and SOSTOOLS.**

I. INTRODUCTION

Internet congestion control is an algorithm to allocate available resources to competing sources efficiently so as to avoid congestion collapse. The simplest adequate models are in the form of deterministic nonlinear delay-differential equations [18], [8] but their analysis is difficult and researchers are constrained to the investigation of the properties of their nonlinear undelayed versions, or the linearised delayed ones. Analysis of the linearizations is usually misleading: any result is *local* as nonlinear phenomena are ignored. Also analysis of linearized undelayed versions may result in major pitfalls, as delays are known to usually cause degradation of performance and instabilities. No analysis attempt through exhaustive simulations of the nonlinear models with delays can ever provide a *proof* of the functionality of the protocol.

Stability analysis of time-delay systems (TDS) has been under intense research in the past years [9], [5] and algorithmic analysis procedures were developed for linear TDSs. As far as time-domain procedures are concerned, there are two Lyapunov-based methodologies: using Lyapunov-Krasovskii (L-K) functionals and Lyapunov-Razumikhin (L-R) functions. These Lyapunov certificates are constructed through the solution of Linear Matrix Inequalities — LMIs [2]. Lyapunov-Razumikhin LMI criteria are in general more conservative than the Lyapunov-Krasovskii ones [5].

Work was financially supported by AFOSR MURI, NIH/NIGMS AfCS, the Kitano ERATO Systems Biology Project, and URI.

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Most results on nonlinear stability analysis concentrate on handcrafted Lyapunov-Razumikhin functions, a procedure that was followed in most of the analyses of congestion control schemes for simple network topologies restricted to single bottlenecks [3], [7], [19], [20].

Even in the case of systems described by ODEs, stability analysis has always been a challenging task. An algorithmic methodology was proposed recently [16], [14] that allows analysis of such systems by algorithmically constructing a *Lyapunov function* as a certificate for stability of the zero equilibrium using the Sum of Squares decomposition and SOSTOOLS [17].

This methodology can be extended to the construction of Lyapunov-Krasovskii functionals for nonlinear TDSs and the analysis of network congestion control models algorithmically. The functionals that we use have structures that are similar to the complete functionals used for stability analysis of linear TDSs but they have kernels that are *polynomials*. This allows the use of the Sum of Squares decomposition to check the resulting stability conditions through the solution of LMIs.

In Section II of this paper we present the unified model framework used in congestion control, and the congestion control schemes we wish to analyze. In Section III we present key results on functional differential equations and develop the algorithmic methodology that we propose to use. In Section IV we apply the theory developed to the stability analysis of the network congestion control schemes presented in Section II.

Notation is standard [6]. $\mathbb{R}^n$ is an n-dimensional real Euclidean space with norm $|\cdot|$. For $b > a$ denote $C^n = C([a,b],\mathbb{R}^n)$ the Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. For $\phi \in C^n$ the norm of ϕ is defined as $\|\phi\| = \sup_{a\leq\theta\leq b} |\phi(\theta)|$, where $|\cdot|$ is a norm in $\mathbb{R}^n$. We also denote by C_γ^n the set $\{\phi \in C^n : \|\phi\| < \gamma\}$.

II. CONGESTION CONTROL

Consider a network of L communication links shared by S sources. Define the routing matrix R by:

$$R_{li} = \begin{cases} 1 & \text{if source } i \text{ uses link } l \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Associated with each source i is a transmission rate x_i. All sources whose flow passes through resource l contribute to the *aggregate rate* y_l, the rates being added

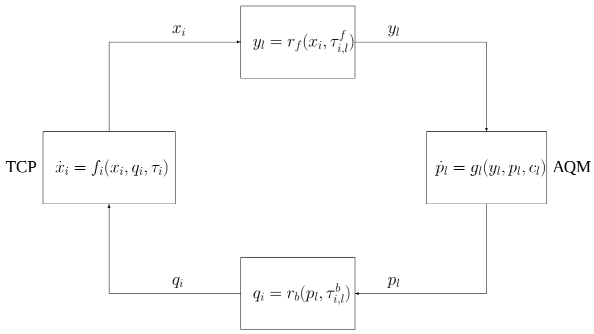

Fig. 1. The internet as an interconnection of sources and links through delays.

with forward time delays $\tau^f_{i,l}$:

$$y_l(t) = \sum_{i=1}^{S} R_{li} x_i(t - \tau^f_{i,l}) \triangleq r_f(x_i, \tau^f_{i,l}) \qquad (2)$$

The resources l react to the aggregate rate y_l by setting a *price* p_l. This is the Active Queue Management (AQM) part of the algorithm. The prices of all the links that source i uses are added to form q_i, the *aggregate price* for source i, again through a delay $\tau^b_{i,l}$:

$$q_i(t) = \sum_{l=1}^{L} R_{li} p_l(t - \tau^b_{i,l}) \triangleq r_b(p_l, \tau^b_{i,l}) \qquad (3)$$

The prices q_i can then be used to set the rate x_i of source i. This is the Transmission Control Protocol (TCP) part of the algorithm, which completes the picture shown in Figure 1. The capacity of link l is denoted by c_l. The forward and backward delays can be combined to yield the Round Trip Time (RTT) for source i, τ_i:

$$\tau_i = \tau^f_{i,l} + \tau^b_{i,l} \qquad (4)$$

This setting is *universal*, and what needs to be specified are two control laws that describe how the ith source reacts to the price signal q_i that it sees

$$\dot{x}_i = f_i(x_i, q_i, \tau_i), \qquad (5)$$

and how the lth router reacts to the signal y_l it observes

$$\dot{p}_l = g_l(y_l, p_l, c_l). \qquad (6)$$

Here f_i models TCP algorithms (e.g. Reno or Vegas) and g_l models AQM algorithms (e.g. RED, REM).

We will be concerned with two congestion control schemes, a dual [11] (i.e. with dynamics only at the links) and a primal-dual [10] (i.e. with dynamics at both sources and links).

A. The dual congestion control scheme

For this congestion control algorithm we have [10]:

$$\dot{p}_l(t) = \begin{cases} \frac{y_l - c_l}{c_l} & \text{if } p_l(t) > 0; \\ \max\{0, \frac{y_l - c_l}{c_l}\} & \text{if } p_l(t) = 0. \end{cases} \triangleq g_l(y_l, c_l) \qquad (7)$$

$$x_i(t) = x_{\max,i} e^{\frac{-\alpha_i q_i(t)}{M_i \tau_i}} \triangleq f_i(q_i, \tau_i) \qquad (8)$$

where M_i is an upper bound on the number of bottleneck links that source i sees in its path, α_i are (positive) source gains, and $x_{\max,i}$ are source constants. Combining (2–8) the system has the following closed loop dynamics:

$$\dot{p}_l(t) = \left\{ \sum_{i=1}^{S} \frac{R_{li}}{c_l} x_{\max,i} e^{\frac{-\alpha_i \sum_{m=1}^{L} R_{mi} p_m (t - \tau^f_{i,l} - \tau^b_{i,m})}{M_i \tau_i}} - 1 \right\} \qquad (9)$$

for $p_l > 0$, and $\dot{p}_l$ is equal to the positive projection of the right hand side of (9) if $p_l = 0$. For the linearisation of system (9) we have:

Theorem 1: [10] If the matrix $\overline{R}$ obtained from eliminating non-bottleneck elements from R is full row rank and $\alpha_i < \pi/2$ then the system described by (1–4) and (7–8) is linearly stable for arbitrary delays and link capacities.

B. The Primal-Dual congestion control scheme

The drawback of the dual control law is that it puts a restriction on the sources' demand curves, as the source law is *static*. A primal-dual congestion control scheme, developed in [11], alleviates this problem. Apart from q_i, y_l and p_l given by (3), (2) and (7) respectively,

$$x_i(t) = x_{m,i} e^{\xi_i} e^{\frac{-\alpha_i q_i(t)}{M_i \tau_i}} \qquad (10)$$

$$\dot{\xi}_i(t) = \frac{\beta_i}{\tau_i} [U_i'(x_i(t)) - q_i(t)] \qquad (11)$$

where $U_i(x_i)$ is the utility function of source i and β_i is a parameter. We have the following result for the stability of the linearised system:

Theorem 2: [11] Assume that for every source i, $\tau_i \leq \tau$. Then the system described by 1–4), (7) and (10–11), with $\alpha_i < \pi/2$ and $z = \frac{\beta_i M_i}{\alpha_i} = \frac{\eta}{\tau}$ for $\eta \in (0, 1)$ small enough depending on $\alpha \geq \alpha_i$ the closed loop system is linearly stable.

III. STABILITY ANALYSIS

Global nonlinear stability analysis of the above network congestion control schemes was performed in [19], [20] in the single bottleneck case based on Lyapunov-Razumikhin (L-R) functions. Other attempts to analyze stability of time delay systems arising from network congestion control include a passivity approach in [4].

Attempts to construct L-R functions for general topology networks can be found in [1], [22]. A Lyapunov-Krasovskii approach can be found in [13].

Here we will be concerned with autonomous Retarded Functional Differential Equations (RFDEs) given by

$$\dot{x}(t) = f(x_t). \tag{12}$$

where $f : \Omega \to \mathbb{R}^n$, $\Omega \subset C^n$, '˙' represents the right-hand derivative and $x_t \in \Omega$, $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$. Stability definitions for the equilibrium x^* of this system satisfying $f(x^*) = 0$, can be found in [6].

Assessing the equilibrium stability properties of (12) can be done using *time-domain* (Lyapunov-based) methodologies. From the two equally important Lyapunov-based procedures, Razumikhin and Krasovskii, we will concentrate on the latter. The L-K theorem can be seen as a generalization of the Lyapunov theorem for systems described by ODEs, in which the existence of a positive definite function $V(x)$ defined in a region of the zero equilibrium with a negative definite derivative proves its asymptotic stability.

More specifically let $\Omega \subset C^n_\gamma$, define $V : \Omega \to \mathbb{R}$ a continuous functional and let $\dot{V}$ denote the *right upper Dini Derivative*. Then we have the following theorem:

Theorem 3: (Lyapunov-Krasovskii) [6] Suppose $V : \Omega \to \mathbb{R}$ is continuous and there exist nonnegative functions $\varphi(s)$, and $\vartheta(s)$ such that $\varphi(s) \to \infty$ as $s \to \infty$, $\varphi(0) = \vartheta(0) = 0$ and

$$\varphi(|\phi(0)|) \le V(\phi), \quad \dot{V}(\phi) \le -\vartheta(|\phi(0)|) \text{ for all } \phi \in \Omega.$$

Then the solution $x = 0$ of (12) is stable. If, in addition, $\vartheta(s)$ is positive definite, then the solution $x = 0$ of (12) is asymptotically stable.

The function $\varphi(s)$ makes the Lyapunov functional *positive definite* in Ω. We define the region $\Omega = C^n_\gamma$ for some $\gamma > 0$. From this condition, a number of constraints can be written on x_{i_t}; for example, one can write

$$g_{1i} \triangleq (x_i(t+\theta) - \gamma)(x_i(t+\theta) + \gamma) \le 0,$$

where $\theta \in [-\tau, 0]$, whose by-products are:

$$\begin{aligned} g_{2i} &\triangleq (x_i(t) - \gamma)(x_i(t) + \gamma) \le 0 \\ g_{3i} &\triangleq (x_i(t-\tau) - \gamma)(x_i(t-\tau) + \gamma) \le 0 \end{aligned}$$

Consider now a time-delay system of the form (12):

$$\dot{x}(t) = f(x_t, p), \tag{13}$$

where $p \in \mathbb{R}^m$ encompasses uncertain parameters. This system is supplemented by a set of equalities and inequalities of the form

$$\begin{aligned} a_{i_1}(x_t, p) &\le 0, \text{ for } i_1 = 1, ..., N_1, && (14) \\ b_{i_2}(x_t, p) &= 0, \text{ for } i_2 = 1, ..., N_2, && (15) \end{aligned}$$

We assume that a_{i_1} and b_{i_2} and $f(x_t, p)$ are polynomial functions in their arguments for all i_1 and i_2. We further require that $f(x_t, p)$ has no singularity in $\tilde{\Omega} \subset C^n \times \mathbb{R}^m$, defined as follows:

$$\tilde{\Omega} = \{(x_t, p) \in C^n \times \mathbb{R}^m \mid a_{i_1}(x_t, p) \le 0, b_{i_2}(x_t, p) = 0, \text{ for all } i_1 \text{ and } i_2\}.$$

The inequalities a_{i_1} can be used to construct $\tilde{\Omega}$ or define the parametric uncertainty region in the p's. The equalities b_{i_2} may result from a change of coordinates, to ensure that the equilibrium is at the origin as the parameters p vary. Without loss of generality, it is assumed that $f(x_t, p) = 0$ for $x_t = 0$ and $p \in P$, where

$$P = \{p \in \mathbb{R}^m | (0, p) \in \tilde{\Omega}\}.$$

We have the following extension of the Lyapunov-Krasovskii stability theorem:

Theorem 4: Suppose that for the system (13–15) there exists a functional $V(x_t, p)$, polynomials $u_{i_1}(x_t, p)$ and $w_{i_2}(x_t, p) \ge 0$ defined in $\tilde{\Omega}$ such that $V(x_t, p)$ is positive definite in $\tilde{\Omega}$. Then

$$\begin{aligned} -\dot{V}(x_t, p) + \sum u_{i_1}(x_t, p) a_{i_1}(x_t, p) + \\ + \sum w_{i_2}(x_t, p) b_{i_2}(x_t, p) \ge 0 \end{aligned} \tag{16}$$

will guarantee that the origin of the state space is a stable equilibrium of the system. Requiring positive definiteness of (16) yields uniform asymptotic stability of the equilibrium.

Proof: The first Lyapunov condition is satisfied, as $V(x_t, p)$ is positive definite in $\tilde{\Omega}$. The second Lyapunov condition is also satisfied, as

$$\begin{aligned} -\dot{V}(x_t, p) \ge -\sum u_{i_1}(x_t, p) a_{i_1}(x_t, p) \\ -\sum w_{i_2}(x_t, p) b_{i_2}(x_t, p) \ge 0 \end{aligned}$$

by virtue of the fact that in $\tilde{\Omega}$ we have $b_{i_2} = 0$ and $a_{i_1} \ge 0$, and we have chosen $u_{i_1} \ge 0$. Hence the equilibrium of the system is stable. If Condition (16) is made positive definite, then uniform asymptotic stability of the equilibrium follows. ■

To use the above theorem, one has to choose a structure for the functional $V(x_t)$ and then construct appropriate Lyapunov conditions. Here we concentrate on functionals of integral form with polynomial kernels. Let us choose the structure

$$V(x_t, p) = V_0(x(t), p) + \int_{-\tau}^{0} V_1(\theta, x(t), x(t+\theta), p) d\theta + \int_{-\tau}^{0} \int_{t+\theta}^{t} V_2(x(\zeta), p) d\zeta d\theta, \tag{17}$$

where V_i are polynomials in their arguments. Then we can write:

Proposition 5: Consider the system given by (13) under the constraints (14–15). Suppose that there exist polynomials $V_0(x(t), p), V_1(\theta, x(t), x(t+\theta), p)$ and $V_2(x(\zeta), p)$ and a positive definite function $\varphi(x(t))$ such that the following conditions hold for all $(x_t, p) \in \tilde{\Omega}$:

1) $V_0(x(t), p) - \varphi(x(t)) \geq 0$,
2) $V_1(\theta, x(t), x(t+\theta), p) \geq 0 \;\forall\; \theta \in [-\tau, 0]$,
3) $V_2(x(\zeta), p) \geq 0$,
4) $V_1(0, x(t), x(t), p) - V_1(-\tau, x(t), x(t-\tau), p) + \frac{\partial V_0}{\partial x(t)} f + \tau V_2(x(t), p) - \tau V_2(x(t+\theta), p) + \tau \frac{\partial V_1}{\partial x(t)} f - \tau \frac{\partial V_1}{\partial \theta} \leq 0, \;\forall\; \theta \in [-\tau, 0]$.

Then the equilibrium 0 of the system given by (13–15) is *robustly stable*.

Proof: Conditions (1-3) in the Proposition above require that $V(x_t, p) \geq \varphi(x(t)) > 0$ in $\tilde{\Omega}$, so the first Lyapunov condition is satisfied. The derivative of V along the system's trajectories is

$$\dot{V}(x_t, p) = \frac{\partial V_0}{\partial x(t)} f + V_1(0, x(t), x(t), p)$$
$$- V_1(-\tau, x(t), x(t-\tau), p) + \int_{-\tau}^{0} \left(\frac{\partial V_1}{\partial x(t)} f - \frac{\partial V_1}{\partial \theta} \right) d\theta$$
$$+ \int_{-\tau}^{0} \left(V_2(x(t), p) - V_2(x(t+\theta), p) \right) d\theta$$
$$= \frac{1}{\tau} \int_{-\tau}^{0} \begin{pmatrix} V_1(0, x(t), x(t), p) + \frac{\partial V_0}{\partial x(t)} f + \tau \frac{\partial V_1}{\partial x(t)} f \\ -V_1(-\tau, x(t), x(t-\tau), p) - \tau \frac{\partial V_1}{\partial \theta} \\ +\tau V_2(x(t), p) - \tau V_2(x(t+\theta), p) \end{pmatrix} d\theta$$

The kernel of this is non-positive by condition (4), hence $\dot{V}(x_t, p) \leq 0$ in $\tilde{\Omega}$ and the equilibrium is robustly stable. ∎

To check the above conditions in an algorithmic way we can use the Sum of Squares (SOS) decomposition and semidefinite programming (Linear Matrix Inequalities), as it was done in the ODE case [14]. A detailed description about SOS and its algorithmic verifiability can be found in [16]. For this the vector field has to be rendered polynomial in the variables $x(t)$, $x(t-\tau)$ as described in [15]. Construction of the semidefinite programme can be cumbersome when the degree of the polynomials is high. For this reason, conversion of SOS conditions to the corresponding semidefinite programme has been automated in SOSTOOLS [17], a software developed for this purpose. This software package was used for solving all the examples in this paper.

We now describe how Proposition 5 can be used in practice. We first construct the polynomials V_0, V_1 and V_2 in SOSTOOLS (respecting the symmetric structure, if there should be one). We construct $\varphi(x(t)) > 0$ as

$$\varphi(x(t)) = \sum_{j=1}^{n} \sum_{i=1}^{m/2} \epsilon_{ij} x_j(t)^{2i}, \qquad \sum_{i=1}^{m/2} \epsilon_{ij} \geq \gamma, \quad (18)$$

for $j = 1, \ldots, n$ with γ a positive number and $\epsilon_{ij} \geq 0$. To impose the conditions $\theta \in [-\tau, 0]$ and the inequalities that arise from constraining the state-space, we use a process similar to the S-procedure. For example, the polynomial $a_1(\theta, x(t), x(t+\theta))$ is required to be a Sum of Squares only when $h = \theta(\theta+\tau) \leq 0$ and the inequalities a_i and equalities b_i are satisfied. We therefore adjoin these constraints to V_1 in the same manner that was done in Theorem 4 using instead of constant positive multipliers (S-procedure), Sum of Squares multipliers for the inequality constraints h and a_i and polynomial multipliers for the equality constraints b_i [14]. Then the four conditions in Proposition 5 will be four SOS constraints in a relevant Sum of Squares programme which can be solved using SOSTOOLS [17].

In a similar manner, other Lyapunov functional structures can be used other than (17), as we will see in the examples to follow. See also [12]. Moreover if asymptotic stability is required, condition (4) in Proposition 5 can be made negative definite by constructing a positive definite $\vartheta(x(t))$ and imposing a similar condition to (1) in Proposition 5, as required by Theorem 3.

IV. Stability of Internet Congestion Control Schemes

In this section we analyze the stability properties of the two congestion control schemes that were described in Section II in simple network topologies.

A. Analysis of instances for the dual control law

1) A single source, single bottleneck: Here we consider a single source and single bottleneck, i.e. $S = L = 1$, $R = 1$ which sets $c = 1$. Under $z = x_{\max} e^{\frac{-\alpha q}{\tau}} - 1$, we have:

$$\dot{z}(t) = -\frac{\alpha}{\tau}[z(t)+1] z(t-\tau), \quad (19)$$

where $-1 \leq z(t) \leq -1 + x_{\max}$.

We assume that $x_{\max} > c$, i.e. the link is a bottleneck. Linearisation about the zero equilibrium gives $\dot{z}(t) = -\frac{\alpha}{\tau} z(t-\tau)$ and so stability is retained locally for $\alpha < \pi/2$ [9]. For the nonlinear version, we attempt to construct the following L-K functional:

$$V(z_t) = V_0(z(t)) +$$
$$+ \int_{-\tau}^{0} \int_{-\tau}^{0} V_1\left(\theta, \xi, z(t), z(t+\theta), z(t+\xi)\right) d\theta d\xi +$$
$$+ \int_{-\tau}^{0} \int_{t+\theta}^{t} V_2(z(\zeta)) d\zeta d\theta + \int_{-\tau}^{0} \int_{t+\xi}^{t} V_2(z(\zeta)) d\zeta d\xi.$$

For $\alpha = 1$ we can construct this V for $|z_t| = 0.42$ when the order of V_0 and V_1 is 2 and V_2 is 4 and V_1 is not a function of θ, ξ. Lyapunov functionals with better properties can be constructed when the kernels are also made functions of θ and ξ, with higher order kernels. In particular, a V was constructed with $-0.99 \leq z_t \leq 1$.

Remark 6: (19) is *Hutchinson's Equation*, a well known FDE [21]. It models single species growth struggling for a common food. This reveals an interesting connection between competition models in ecology and network congestion control. The nonlinear equation (19) has been analyzed in [21] where *global* stability is proven for $\alpha < 37/24 = 1.5417$ and $z_t > -1$, by using properties of the solution (non Lyapunov method).

2) Single bottleneck, many sources: Here we perform the change of coordinates $z_i(t) = \frac{x_{\max,i}}{c} e^{\frac{-\alpha p(t)}{\tau_i}} - \beta_i$ with $\beta_i = 1/S$ to get:

$$\dot{z}_i(t) = -\frac{\alpha}{\tau_i}[z_i(t) + \frac{1}{S}]\dot{p}(t) = -\frac{\alpha}{\tau_i}[z_i(t) + \frac{1}{S}]\sum_{i=1}^{S} z_i(t-\tau_i)$$

for $-\frac{1}{S} \le z_i(t) \le -\frac{1}{S} + \frac{x_{\max,i}}{c}$. Note that this transformation puts a 1-D system in an S-D formulation. There are $S-1$ equality constraints that have to be imposed of the form

$$(Sz_i(t)+1)^{\tau_i} = (Sz_j(t)+1)^{\tau_j}, \quad \forall\ i,j \in S.$$

In the case of two heterogeneous sources, we have:

$$\dot{z_1}(t) = -\frac{\alpha}{\tau_1}[z_1(t) + 0.5][z_1(t-\tau_1) + z_2(t-\tau_2)],$$
$$\dot{z_2}(t) = -\frac{\alpha}{\tau_2}[z_2(t) + 0.5][z_1(t-\tau_1) + z_2(t-\tau_2)],$$
$$(2z_1(t)+1)^{\tau_1} = (2z_2(t)+1)^{\tau_2}$$

where $-0.5 \le z_i(t) \le -0.5 + \frac{x_{\max,i}}{c}$. In case of general τ_1 and τ_2 we approximate $\frac{\tau_1}{\tau_2}$ by a rational number whose numerator and denominator are small integers, and cover the rest in the uncertainty framework developed earlier. This avoids high order terms in the equality constraint.

For the linearisation of these equations about the equilibrium $z_1 = z_2 = 0$ we have the system

$$\dot{z}(t) = -\frac{\alpha}{2\tau_1}z(t-\tau_1) - \frac{\alpha}{2\tau_2}z(t-\tau_2), \quad (20)$$

where $z = z_1$ for which we recall the following result:

Proposition 7: [9] The trivial solution of $\dot{x}(t) = -a_1 x(t-\tau_1) - a_2 x(t-\tau_2)$ is asymptotically stable if $a_1\tau_1 + a_2\tau_2 < \pi/2$.

Therefore a stability condition for the system given by (20) is $\alpha < \pi/2$.

We now analyze the nonlinear case using SOSTOOLS. Since we have a system with two delays, we have to use a different functional. We choose, denoting $z(t) = [z_1(t), z_2(t)]$,

$$V(z_t) = V_0(z(t)) + \sum_{i=1}^{2}\int_{-\tau_i}^{0}\int_{t+\theta_i}^{t} V_{2i}(z(\zeta))d\zeta d\theta_i$$
$$+ \int_{-\tau_1}^{0}\int_{-\tau_2}^{0} V_1(z(t), z(t+\theta_1), z(t+\theta_2))d\theta_1 d\theta_2.$$

In this case the stability analysis was tested for various values of the delay size. For $\tau_1 = 0.2$ and $\tau_2 = 0.3$, we can construct this V with second order V_0, V_1 and 4th order V_{21}, V_{22} for $\alpha = 1$ and $|z_{i_t}| = 0.44$. When $\tau_1 = 0.1$ and $\tau_2 = 0.3$, we can construct this V for $\alpha = 1$ and $-0.49 \le z_{i_t} \le 0.61$.

More complicated topologies can also be analyzed.

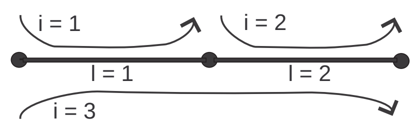

Fig. 2. A simple network.

B. Analysis of simple cases for the primal-dual control law

1) Single source single link case: Consider the case of a single link and flow. The system equations can be simplified to the following:

$$\dot{x}(t) = \frac{K\beta}{\tau} - \frac{\beta}{\tau}q(t)x(t) - \frac{\alpha}{\tau}\left(\frac{x(t)x(t-\tau)}{c} - x(t)\right)$$
$$\dot{q}(t) = \frac{x(t-\tau)}{c} - 1$$

The equilibrium for this system is $x_0 = c$ and $q_0 = \frac{K}{c}$. To avoid numerical ill-conditioning when c is large we scale the state. Define $z_1 = x/c - 1$ and $z_2 = \frac{c}{K}q - 1$ to get:

$$\dot{z}_1 = -\frac{K\beta}{\tau c}(z_1 z_2 + z_1 + z_2) - \frac{\alpha}{\tau}(z_1(t)z_1(t-\tau) + z_1(t-\tau))$$
$$\dot{z}_2 = \frac{c}{K}z_1(t-\tau)$$

For the analysis we use $c = 40$, $\alpha = 1$, $\tau = 0.2$, $\beta = 3.2$ and $K = 20$. We can construct a Lyapunov functional of the form

$$V_1(z_t) = V_0(z_1(t), z_2(t)) + \int_{-\tau}^{0}\int_{t+\theta}^{t} V_2(z_1(\zeta))d\zeta d\theta$$
$$+ \int_{-\tau}^{0} V_1(z_1(t), z_1(t+\theta), z_2(t))d\theta \quad (21)$$

when $|z_{1_t}| \le \gamma_1, z_2 > -1$ for $\alpha = 1$ and $\gamma_1 = 0.75$ with the polynomials V_0 and V_1 second order and V_2 4th order. When their degree is increased by 2, then these become $-1 \le z_{1_t} \le 3.4, z_2 > -1$.

2) A simple network example: Consider the network shown in Figure 2, for which

$$R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

We assume $\tau_{ij} \leq \tau/2$ where τ is a delay overbound. We let $\beta_1 = \beta_2 = \beta_3/2 = \beta$, all the $c_l = c$ and $\alpha_i = \alpha$. Denote $\tilde{K}_1 = \frac{K_1+K_2}{K_1+K_2+K_3}$ and $\tilde{K}_2 = \frac{K_3}{K_1+K_2+K_3}$. The equilibrium for this system is

$$(x_{1,0}, x_{2,0}, x_{3,0}, q_{1,0}, q_{2,0}) = \left(c\tilde{K}_1, c\tilde{K}_1, c\tilde{K}_2, \frac{K_1}{c\tilde{K}_1}, \frac{K_2}{c\tilde{K}_1} \right)$$

We can perform the same manipulation to the closed loop system as before, to get

$$\dot{z}_1(t) = \begin{pmatrix} -\frac{K_1\beta_1}{\tilde{K}_1 c\tau}[z_1(t) + z_4(t) + z_1(t)z_4(t)] \\ -\frac{\alpha}{\tau}[z_1(t) + 1][\tilde{K}_2 z_3(t-\tau) + \tilde{K}_1 z_1(t-\tau)] \end{pmatrix}$$

$$\dot{z}_2(t) = \begin{pmatrix} -\frac{K_2\beta_2}{\tilde{K}_1 c\tau}[z_2(t) + z_5(t) + z_2(t)z_5(t)] + \\ -\frac{\alpha}{\tau}[z_2(t) + 1][\tilde{K}_2 z_3(t-\tau) + \tilde{K}_1 z_2(t-\tau)] \end{pmatrix}$$

$$\dot{z}_3(t) = \begin{pmatrix} -\frac{\beta_3}{\tilde{K}_1 c\tau}[(z_3z_4 + z_3 + z_4)K_1 + (z_3z_5 + z_3 + z_5)K_2] \\ -\frac{\alpha}{2\tau}[z_3(t) + 1] \\ \times[z_1(t-\tau)\tilde{K}_1 + z_2(t-\tau)\tilde{K}_1 + 2z_3(t-\tau)\tilde{K}_2] \end{pmatrix}$$

$$\dot{z}_4(t) = \frac{\tilde{K}_1 c}{K_1}(\tilde{K}_1 z_1(t-\tau) + \tilde{K}_2 z_3(t-\tau))$$

$$\dot{z}_5(t) = \frac{\tilde{K}_1 c}{K_2}((K_1+K_2)z_2(t-\tau) + K_3 z_3(t-\tau))$$

We use the same values for c, α, τ as before. We calculate $\beta = \frac{0.64\alpha}{\tau M_i}$ and we let $K_1 = 15$, $K_2 = 20$, $K_3 = 25$. We can construct a similar Lyapunov functional to (21) with all polynomials V_0, V_1 of second order and V_2 of order 4 for

$$0 \leq x_{1_t} \leq 2.3x_{1,0}, \quad 0 \leq x_{2_t} \leq 2.3x_{2,0},$$
$$0 \leq x_{3_t} \leq 2.3x_{3,0}, \quad q_1 > 0, \quad q_2 > 0.$$

V. Concluding Remarks

We presented a methodology to construct Lyapunov-Krasovskii functionals for time delay systems based on the Sum of Squares decomposition. The construction is entirely algorithmic and is done through the solution of Linear Matrix Inequalities (LMIs). The nonlinear stability of simple topologies of networks employing different congestion control algorithms was analyzed in this way, taking account of the delays present in the feedback mechanism.

This method can be extended to discrete systems with delays and systems with time-varying delays. These two cases have interesting applications to network congestion control. A judicious choice for the structure of the Lyapunov functional would still be required.

Invariant sets in the regions Ω constructed above can also be identified using the Sum of Squares decomposition, as maximal level sets of the Lyapunov functionals that were constructed using SOSTOOLS.

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