Local Stability of FAST TCP

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***Abstract*— We study stability properties of FAST TCP based on a discrete-time model. We prove that FAST TCP is locally asymptotically stable for arbitrary network in the absence of feedback delay. When feedback delay is present, we derive a sufficient condition for local asymptotic stability for the case of a single link. The condition suggests that local stability depends on delays only through their heterogeneity.**

I. Introduction

Congestion control is a distributed feedback algorithm to share network resources among competing users. It is well-known that the current congestion control algorithm in TCP does not scale as the bandwidth-delay product of the Internet grows, e.g., [6], [9]. This has motivated several recent proposals for high speed networks, including HSTCP [4], STCP [8], and FAST TCP [7], [11], [1] (see [7] for extensive references). Preliminary experimental results on FAST TCP are promising and its stability in the absence of feedback delay is proved in [7] for the case of a single link. In this paper, we extend the stability analysis of FAST TCP to the case of a network without feedback delay and a single link with feedback delay.

Most of the stability analysis in the literature is based on the fluid model introduced in [10]. Recently, a new discrete-time link model is proposed in [12], [7] to capture the effect of self-clocking, and detailed experimental and validation has been carried out in [12]. Our study is based on this model, summarized in Section II. We prove in Section III that FAST TCP is always locally asymptotically stable in a network in the absence of feedback delay, and prove in Section IV local asymptotic stability in the presence of feedback delay for the case of a single link. We conclude in Section V.

II. Model

A network consists of a set of L links with finite capacity c_l. The network is shared by a set of N unicast flows, identified by their sources. Let d_i denote the round-trip propagation delay of source i. Let R be the routing matrix where $R_{li} = 1$ if source i uses link l, and 0 otherwise. Our model is a discrete time one, and we take the sampling period as our time unit. All times below represent multiples of the sampling period. Denote the forward delay from source i to link l by τ_{li}^f, and the backward delay from link l to source i as τ_{li}^b. Let $p_l(t)$ denote the queueing delay at link l at (discrete) time t. The end-to-end queueing delay observed by source i at time t is taken to be

$$q_i(t) = \sum_l R_{li} p_l(t - \tau_{li}^b) \qquad (1)$$

and the round-trip time of source i is $T_i(t) := d_i + q_i(t)$. We assume that the round-trip feedback delay is $T_i := \tau_{li}^f + \tau_{li}^b$ for any l in the path of i. There is a subtle assumption in this model. In reality, the feedback delays τ_{li}^f, τ_{li}^b, and T_i depend on the queueing delays $p_l(t)$ and round-trip times $T_i(t)$. Whereas the latter are time-varying, we assume for simplicity that the feedback delays τ_{li}^f, τ_{li}^b, and T_i are constant. We will later analyze linear stability around the system equilibrium in the presence of feedback delay, and hence these feedback delays may be interpreted as determined by the equilibrium values of $p_l(t)$ and $T_i(t)$.

A FAST TCP source adjusts its congestion window in each time period according to:

$$w_i(t+1) = \gamma \left(\frac{d_i w_i(t)}{d_i + q_i(t)} + \alpha_i(w_i(t), q_i(t)) \right) + (1-\gamma) w_i(t) \qquad (2)$$

where $\gamma \in (0, 1]$ and $\alpha_i(w_i, q_i)$ is defined by

$$\alpha_i(w_i, q_i) = \begin{cases} a_i w_i & \text{if } q_i = 0 \\ \alpha_i & \text{otherwise} \end{cases} \qquad (3)$$

Here, we make a key assumption at links. We assume that the source rate $x_i(t) := w_i(t)/T_i(t)$ cannot exceed the throughput that source i receives. This is because of self-clocking: one round-trip time after a congestion window is increased, packet transmission will be clocked at the same rate as the throughput the flow receives. This amounts to assuming that the disturbance in the queues due to window changes settle down quickly compared with the sampling time of the discrete-time model. See [12] for detailed justification and validation experiments. A consequence of this assumption is that the link queueing delay vector $p(t) = (p_l(t)$, for all $l)$ is determined implicitly be the window sizes in a static manner: given $w_i(t), w_i(t-1), \ldots, w_i(t - \max_l \tau_{li}^f)$ for all i, and $p_l(t-1), p_l(t-2), \ldots, p_l(t - \max_{k,i} \tau_{ki}^b)$ for all l, the current delay vector $p(t)$ is given by:

$$\sum_i R_{li} \frac{w_i(t - \tau_{li}^f)}{d_i + q_i(t - \tau_{li}^f)} \quad \begin{cases} = c_l & \text{if } p_l(t) > 0 \\ \leq c_l & \text{if } p_l(t) = 0 \end{cases} \qquad (4)$$

where the q_i is the end-to-end queueing delay given by (1).

In summary, a network of FAST TCP sources is modeled by equations (1)-(4). This generalizes the model in [7] to take into account of feedback delay. Since the equilibrium value of any variable is independent of the time step t, these two system share the same equilibrium. It has been proved in [7] that there is a unique equilibrium for the system when the routing matrix R is full row rank, and the equilibrium rate x^* is α_i-weighted proportionally fair. It is also proved there that the system is exponentially stable in the case of single bottleneck link in the absence of feedback delay. Here we extend the stability result to the case of general network in the absence of feedback delay, and to the case of a single bottleneck in the presence of feedback delay.

Before proceeding, we make another important assumption: we assume equality holds in (4) at all times t. For local stability, this does not lose generality as it amounts to including in R only links l that are bottlenecks, i.e., those links with nonzero equilibrium delays $p_l^* > 0$. This allows us to use implicit function theorem to express $p_l(t)$ in terms of $w_i(t)$ and their previous values.

III. Local stability for general networks without feedback delay

The stability issue of general networks is challenging even when feedback delay is ignored, because of the arbitrary routing matrix R. We will first derive the linear map from $w(t)$ to $w(t+1)$, and then show that its spectral radius is strictly less than 1, implying that FAST TCP is locally stable in the absence of feedback delay.

When the feedback delay is absent, both the forward delay τ_{li}^f and backward delay τ_{li}^b are zero for all l, i. The equation (1) reduces to $q_i(t) = \sum_l R_{li} p_l(t)$, or in vector form $q(t) = R^T p(t)$. Under the assumption that equality always holds in (4), it is simplified to:

$$\sum_i R_{li} \frac{w_i(t)}{d_i + q_i(t)} = c_l \quad \text{for any } l \tag{5}$$

(5) defines a map from $w(t)$ to $q(t)$, as we now show using the implicit function theorem. It is a general relation in the sense that it holds not only at equilibrium but for all $w(t)$.

Denote by $h \in R_+^{N+L}$:

$$h \;:=\; \begin{pmatrix} q \\ p \end{pmatrix}$$

and, for any $w \in R_+^N$ and $h \in R_+^{N+L}$, define the function $G(w, h)$ as

$$G(w,h) \;:=\; \begin{pmatrix} q & - & R^T p \\ \sum_i R_{li} \frac{w_i}{d_i+q_i} & - & c_l \end{pmatrix}$$

Then (1) and (5) imply $G(w, h) = 0$. We have

$$\frac{\partial G}{\partial h} \;=\; \begin{pmatrix} I & -R^T \\ -RB & 0 \end{pmatrix}$$

where

$$B \;:=\; \text{diag}\left(\frac{w_i}{(d_i+q_i)^2} \right) \tag{6}$$

It is easy to check that, for all $w \in R_+^N$ and $(q, p) \in R_+^{N+L}$, we have

$$\begin{pmatrix} I & -R^T \\ -RB & 0 \end{pmatrix} \begin{pmatrix} I & R^T \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ -RB & -RBR^T \end{pmatrix}$$

The matrix block $-RBR^T$ is full rank because it is positive definite for all $(q, p) \in R_+^{N+L}$. Hence the block matrix on the right-hand side of the above equation is always nonsingular. This means that $\partial G/\partial h$ is invertible, and indeed, it can be verified that it is equal to

$$\begin{pmatrix} I & -R^T \\ -RB & 0 \end{pmatrix}^{-1} \tag{7}$$

$$= \begin{pmatrix} I - R^T(RBR^T)^{-1}RB & -R^T(RBR^T)^{-1} \\ -(RBR^T)^{-1}RB & -(RBR^T)^{-1} \end{pmatrix} \tag{8}$$

From the implicit function theorem, the vector h can be uniquely solved in terms of w, and $\partial h/\partial w$ is:

$$\frac{\partial h}{\partial w} \;=\; -\left(\frac{\partial G}{\partial h}\right)^{-1} \frac{\partial G}{\partial w} \tag{9}$$

Define the diagonal matrix D and M as:

$$D := \text{diag}(d_i), \qquad M := \text{diag}\left(\frac{d_i}{d_i + q_i}\right) \tag{10}$$

Then

$$\frac{\partial G}{\partial w} = \begin{pmatrix} 0 \\ RD^{-1}M \end{pmatrix}$$

By implicit function theorem, we have:

$$\begin{aligned} \frac{\partial q}{\partial w} &= (I, 0)\frac{\partial h}{\partial w} \\ &= -(I,0)\left(\frac{\partial G}{\partial h}\right)^{-1}\frac{\partial G}{\partial w} \\ &= -(I,0)\left(\frac{\partial G}{\partial h}\right)^{-1}\begin{pmatrix} 0 \\ RD^{-1}M \end{pmatrix} \end{aligned}$$

Substituting in (8), we get

$$\frac{\partial q}{\partial w} = R^T(RBR^T)^{-1}RD^{-1}M \tag{11}$$

We will write $q(w)$ or $p(w)$ when we want to emphasize the dependence on w. Note that all matrices B, D, M are also functions of window vector w, even though this is not explicit in the notations.

Now we are ready to study the window update algorithm of FAST TCP. Define a vector-valued function F which maps R_+^N onto R_+^N by:

$$F_i(w) := \gamma\left(\frac{d_i w_i}{d_i + q_i(w)} + \alpha_i\right) + (1-\gamma)w_i \tag{12}$$

Then the FAST TCP window update (2) is $w(t+1) = F(w(t))$.

The derivative of F is derived in the following lemma.

Lemma 1. *The derivative of F is*

$$\frac{\partial F}{\partial w} = \gamma(M - DBR^T(RBR^T)^{-1}RD^{-1}M) + (1-\gamma)I \tag{13}$$

where B, D, M are defined in (6) and (10).

Proof: Based on (12), we get

$$\frac{\partial F_i}{\partial w_j} = \begin{cases} -\gamma d_i \frac{w_i}{(d_i+q_i(w))^2}\frac{\partial q_i}{\partial w_j} & i \neq j \\ \gamma\left(\frac{d_i}{d_i+q_i(w)} - \frac{d_i w_i}{(d_i+q_i(w))^2}\frac{\partial q_i}{\partial w_j}\right) + (1-\gamma) \end{cases}$$

By definition of B, D, M, the matrix form of above equation is

$$\frac{\partial F_i}{\partial w_j}(t) = \gamma\left(M - DB\frac{\partial q}{\partial w}(t)\right) + (1-\gamma)I \tag{14}$$

By substituting (11) into the above equation, we have (13). □

Note that Lemma 1 holds at all w, not just the equilibrium point. However, we will focus on local stability in this paper, and will use $\partial F/\partial w$ only at the equilibrium point w. The linearized system of FAST TCP $w(t+1) = F(w(t))$ is

$$\delta w(t+1) = \frac{\partial F}{\partial w}(w)\delta w(t)$$

The necessary and sufficient condition for this linearized system to be asymptotically stable is that the spectral radius of $\partial F/\partial w$, evaluated at the equilibrium w, is less than 1.

Let the eigenvalues of $\partial F/\partial w$ be denoted by $\lambda_i(\gamma)$, $i = 1, \ldots, N$, as a function of $\gamma \in (0, 1]$. We will show in the next two lemmas that when $\gamma = 1$,

$$0 \leq \lambda_i(1) < 1 \quad \text{for all } i \tag{15}$$

Then Lemma 1 implies that, for all $\gamma \in (0, 1]$,

$$0 \leq \lambda_i(\gamma) = \gamma\lambda_i(1) + (1-\gamma) < 1 \quad \text{for all } i$$

and hence the special radius $\rho(\frac{\partial F}{\partial w}) \leq 1$ holds for any given $\gamma \in (0, 1]$. Therefore the system is locally stable.

Let

$$\mu_i = d_i/(d_i + q_i) \tag{16}$$

Define $\mu_{max} := \max_i \mu_i$ and $\mu_{min} = \min_i \mu_i$. Since at equilibrium the end to end queueing delay q_i cannot be zero, $\mu_{max} < 1$. The key observation to prove (15) is that we can bound all the eigenvalues of $\partial F/\partial w$ by μ_{max} when $\gamma = 1$.

Lemma 2. *Suppose* $\gamma = 1$. *Then*

1) *All the eigenvalues of* $\partial F/\partial w$ *are real.*
2) $\lambda_i(1) = 0$, $i = 1, \ldots L$, *where* L *is the number of links, with the corresponding eigenvector as the columns of matrix* $M^{-1}DBR^T$.
3) $\mu_{min} \leq \lambda_j(1) \leq \mu_{max} < 1$, *for nonzero eigenvalues.*

Proof: At $\gamma = 1$:

$$\frac{\partial F}{\partial w} = M - DBR^T(RBR^T)^{-1}RD^{-1}M \tag{17}$$

It is easy to check that:

$$\begin{aligned} \frac{\partial F}{\partial w} M^{-1}DBR^T &= DBR^T - DBR^T \\ &= 0 \end{aligned}$$

Since B, D and M are full rank diagonal matrix and R^T has full column rank, $M^{-1}DBR^T$ is full column rank. Then the column vectors of $M^{-1}DBR^T$ consists of L linearly independent eigenvectors of $\partial F/\partial w$ with corresponding eigenvalue 0.

Suppose that λ is an eigenvalue of $\partial F/\partial w$. Then the matrix A defined as

$$\begin{aligned} A &:= \frac{\partial F}{\partial w} - \lambda I \\ &= (M - \lambda I) - DBR^T(RBR^T)^{-1}RD^{-1}M \end{aligned}$$

is singular.

The following matrix inversion relation is cited from [5]:

$$\begin{aligned} (J &+ EHS)^{-1} \\ &= J^{-1} - J^{-1}E(H^{-1} + SJ^{-1}E)^{-1}SJ^{-1} \end{aligned}$$

Based on this relation, if $J + EHS$ is singular, then either J or $H^{-1} + SJ^{-1}E$ is singular. Define

$$\begin{aligned} J &:= M - \lambda I, & E &:= -DBR^T \\ H &:= (RBR^T)^{-1}, & S &:= RD^{-1}M \end{aligned}$$

Since $A = J + EHS$ is singular, then either $J = M - \lambda I$ or

$$\begin{aligned} H^{-1} + SJ^{-1}E &= RBR^T - RD^{-1}M(M-\lambda I)^{-1}DBR^T \\ &= R(B - M(M-\lambda I)^{-1}B)R^T \end{aligned}$$

is singular.

Case 1: $M - \lambda I$ is singular. Since M is diagonal,

$$\lambda = \frac{d_i}{d_i + q_i} = \mu_i$$

So the eigenvalue λ is real, and $\mu_{min} \leq \lambda \leq \mu_{max}$

Case 2: $R(B - M(M-\lambda I)^{-1}B)R^T$ is singular.

We have

$$\begin{aligned} & B - M(M-\lambda I)^{-1}B \\ =\ & (I - M(M-\lambda I)^{-1})B \\ =\ & \text{diag}(1 - \mu_i(\mu_i - \lambda)^{-1})B \\ =\ & -\lambda B\Lambda \end{aligned}$$

where

$$\Lambda := \text{diag}(\frac{1}{\mu_i - \lambda})$$

then $\lambda RB\Lambda R^T$ is singular for $\lambda \neq 0$.

In order for $RB\Lambda R^T$ to be singular, λ must be real. Suppose that this is not true, and that $\lambda = a + jb$ with $b \neq 0$. Then

$$\frac{1}{\mu_i - \lambda} = \frac{1}{\mu_i - a - jb} = \frac{\mu_i - a + jb}{(\mu_i - a)^2 + b^2}$$

The matrix Λ is:

$$\Lambda = \text{diag}\left(\frac{\mu_i - a}{(\mu_i - a)^2 + b^2}\right) + j\text{diag}\left(\frac{b}{(\mu_i - a)^2 + b^2}\right)$$

The real part of the matrix $RB\Lambda R^T$ is symmetric, and the imaginary part

$$RB\,\text{diag}\left(\frac{b}{(\mu_i - a)^2 + b^2}\right)R^T$$

which is positive definite or negative definite depending on the sign of b. From Lemma 3 below, the matrix $RB\Lambda R^T$ cannot be singular, a contradiction.

Therefore λ has to be real. If $\lambda > \mu_{max}$, $\Lambda = \text{diag}(\frac{1}{\mu_i - \lambda})$ is negative definite, and $RB\Lambda R^T$ is negative definite contradicting to the fact that it is singular. Similarly if $\lambda < \mu_{min}$, $RB\Lambda R^T$ is positive definite. Hence for $RB\Lambda R^T$ to be singular, λ has to be real and in the range $[\mu_{min}, \mu_{max}]$. □

Lemma 3. *If the real part of a complex matrix is symmetric, and the imaginary part is positive definite or negative definite, then the matrix is nonsingular.*

Proof: Suppose that $A = A_r + jA_i$ where $A_r = A_r^T$ and A_i is positive definite. If A is singular, there exists a nonzero vector v such that $Av = 0$. Suppose that $v = \alpha + j\beta$. Then $Av = 0$ gives

$$A_r\alpha - A_i\beta = 0 \tag{18}$$

$$A_r\beta + A_i\alpha = 0 \tag{19}$$

Multiplying β^T to equation (18), we have:

$$\beta^T A_r\alpha = \beta^T A_i\beta \geq 0 \tag{20}$$

Multiplying α^T to equation (19), we have

$$\alpha^T A_r\beta = -\alpha^T A_i\alpha \leq 0 \tag{21}$$

Since $\beta^T A_r\alpha = \alpha^T A_r^T\beta = \alpha^T A_r\beta$, both (20) and (21) must hold with equality. This means that both α and β are zero, contradicting the fact that v is not zero. The proof for A_i to be negative definite is the same. □

We summarize this section in the following results.

Theorem 4. *FAST TCP, represented by (5) and $w(t+1) = F(w(t))$ with F defined in (12), is locally asymptotically stable in the absence of feedback delay for all $\gamma \in [0, 1)$.*

IV. Local stability for single link with feedback delay

In this section we will apply Z-transform to the linearized system with feedback delay, and use the generalized Nyquist criterion for discrete systems to derive a sufficient condition for local asymptotic stability for the case of a single bottleneck link.

Suppose that $W(z)$, $Q(z)$ and $P(z)$ denote the corresponding Z-transform of linearized system $\delta w(t)$, $\delta q(t)$ and $\delta p(t)$ respectively. The queue model (4) can be rewritten as :

$$\sum_i R_{li}\frac{w_i(t-\tau_{li}^f)}{d_i + q_i(t-\tau_{li}^f)} = c_l \tag{22}$$

Let q_i and w_i denote the end-to-end queueing delay and congestion window for source i at the equilibrium. Then (22) can be linearized to:

$$\sum_i (\delta w_i(t-\tau_{li}^f)\frac{R_{li}}{d_i+q_i} - \delta q_i(t-\tau_{li}^f)\frac{w_i R_{li}}{(d_i+q_i)^2}) = 0 \tag{23}$$

The Z-transform of (23) is:

$$z^{-\tau_{li}^f}R_{li}W_i(z)\frac{1}{d_i+q_i} - z^{-\tau_{li}^f}R_{li}Q_i(z)\frac{w_i}{(d_i+q_i)^2} = 0 \tag{24}$$

Define matrix $R_f(z)$ as:

$$(R_f(z))_{li} := \begin{cases} z^{-\tau_{li}^f} & \text{if } R_{li} = 1 \\ 0 & \text{if } R_{li} = 0 \end{cases} \tag{25}$$

The matrix form of (24) is:

$$R_f(z)D^{-1}MW(z) - R_f(z)BQ(z) = 0 \tag{26}$$

where matrix B, D and M are diagonal matrices defined in the previous section evaluated at equilibrium.

The relation $q_i(t) = \sum_l R_{li}p_l(t-\tau_{li}^b)$ is already linear, and the corresponding Z-transform is:

$$Q_i(z) = \sum_l R_{li}z^{-\tau_{li}^b}P_l(z) \tag{27}$$

Define matrix $R_b(z)$ as:

$$(R_b(z))_{li} := \begin{cases} z^{-\tau_{li}^b} & \text{if } R_{li} = 1 \\ 0 & \text{if } R_{li} = 0 \end{cases} \tag{28}$$

The matrix form for (27) is:

$$Q(z) = R_b(z)^T P(z) \tag{29}$$

From (26) and (29), we have:

$$\begin{pmatrix} I & -R_b^T(z) \\ R_f(z)B & 0 \end{pmatrix}\begin{pmatrix} Q(z) \\ P(z) \end{pmatrix} = \begin{pmatrix} 0 \\ R_f(z)D^{-1}M \end{pmatrix} W(z)$$

By solving the above linear equation, we get the transfer function from $W(z)$ to $Q(z)$:

$$\frac{Q(z)}{W(z)} = R_b^T(z)(R_f(z)BR_b^T(z))^{-1}R_f(z)D^{-1}M \tag{30}$$

Applying Z-transform to the time-delayed version of (14), we get

$$\frac{\partial F}{\partial w}(z) = \gamma\left(M - DB\frac{Q(z)}{W(z)}\right) + (1-\gamma)I$$

Hence the open loop transfer function from δw to δw is

$$L(z) = -z^{-1}(\gamma(M - DBR_b^T(z)(R_f(z)BR_b^T(z))^{-1} R_f(z)D^{-1}M) + (1-\gamma)I)$$

According to [3], the Nyquist criterion for discrete time systems is almost identical to that for the continuous time systems [2], except that the closed path is the unit circle. Hence if we can show that the eigenvalues of $L(e^{jw})$ does not enclose -1 for $\omega \in (0, 2\pi)$, then the closed loop system is stable. We will derive a sufficient condition for the spectral radius of $L(e^{jw})$ to be strictly less than 1 for $\omega \in (0, 2\pi)$. Then no eigenvalues of $L(e^{jw})$ can enclose -1 and the system is stable.

When $z = e^{jw}$, the spectral radius of $L(z)$ and $-zL(z)$ are the same. The eigenvalues of $-zL(z)$ are dependent on γ. Similarly to the techniques used in the previous section, if we can show that the spectral radius is less than 1 when $\gamma = 1$, then the spectral radius of $-zL(z)$ will be less than 1 for $\gamma \in (0, 1]$. Define

$$J(z) = M - DBR_b^T(z)(R_f(z)BR_b^T(z))^{-1}R_f(z)D^{-1}M$$

which is just $-zL(z)$ at $\gamma = 1$.

The next theorem provides a sufficient condition for $\rho(J(j\omega)) < 1$ for $\omega \in (0, 2\pi)$, for the case of a single link. Surprisingly, it implies that the local stability of the system depends on the feedback delays on through their heterogeneity. In particular, a system of homogeneous sources is locally stable for arbitrary delay.

Theorem 5. *Suppose there is only a single link. The FAST TCP system is locally stable if $d_{\max} - d_{\min} < 1/4$ where $d_{\max}$ and $d_{\min}$ are the largest and smallest round-trip propagation delays respectively.*

Proof: Suppose that λ is an eigenvalue of $J(z)$ and $z = j\omega$. Define

$$A(z) := M - \lambda I - DBR_b^T(z)(R_f(z)BR_b^T(z))^{-1}R_f(z)D^{-1}M$$

Then matrix A is singular. If $M - \lambda I$ is singular, then $\lambda = \mu_i < 1$ and hence λ is inside the unit circle. Otherwise by using the matrix inverse theorem, $A(z)$ is singular if and only the following matrix singular:

$$R_f(z)BR_b^T(z) - R_f(z)M(M - \lambda I)^{-1}BR_b^T(z)$$

When there is only one bottleneck, the above matrix reduces to a scalar. Define

$$\beta_i := \frac{w_i}{(d_i + q)^2}$$

We have:

$$\begin{aligned} 0 &= R_f(z)BR_b^T(z) - R_f(z)M(M - \lambda I)^{-1}BR_b^T(z) \\ &= R_f(z)B(1 - M(M - \lambda I)^{-1})R_b^T(z) \\ &= R_f(z)\text{diag}(\beta_i(1 - \frac{\mu_i}{\mu_i - \lambda})R_b^T(z) \\ &= -\lambda R_f(z)\text{diag}(\frac{\beta_i}{\mu_i - \lambda})R_b^T(z) \\ &= \lambda \sum_i z^{-\tau_i^f} \frac{\beta_i}{\lambda - \mu_i} z^{-\tau_i^b} \\ &= \lambda \sum_i \frac{1}{\lambda - \mu_i} z^{-(d_i+q)} \beta_i \\ &= \lambda z^{(q+d_{\min})} \sum_i \frac{\beta_i}{\lambda - \mu_i} z^{-(d_i - d_{\min})} \end{aligned}$$

From above equations, either λ is 0 or it satisfies the following equation:

$$\sum_i \frac{\beta_i}{\lambda - \mu_i} z^{-(d_i - d_{\min})} = 0 \tag{31}$$

When $z = e^{jw}$ and $\omega \in (0, 2\pi)$ we have

$$\sum_i \frac{\beta_i}{\lambda - \mu_i} e^{-j\omega(d_i - d_{\min})} = 0 \tag{32}$$

From corollary 7 below, if $\omega(d_{\max} - d_{\min}) < \pi/2$, then all eigenvalue λ will be in the unit circle. Since $\omega \in (0, 2\pi)$, then if $d_{max} - d_{min}$ is less than $1/4$, the spectral radius of $J(e^{-jw})$ is less than 1. This means the system is locally stable. □

The proof of Theorem 5 is complete after the next two results.

Lemma 6. *Suppose $0 < \mu_i < 1$, if $0 \le \theta_i < \pi - \arcsin \mu_{\max}$, where $\mu_{\max} = \max_i \mu_i$, then $|\lambda| < 1$, where λ solves the following equation:*

$$\sum_i \frac{\beta_i}{\lambda - \mu_i} e^{-j\theta_i} = 0 \tag{33}$$

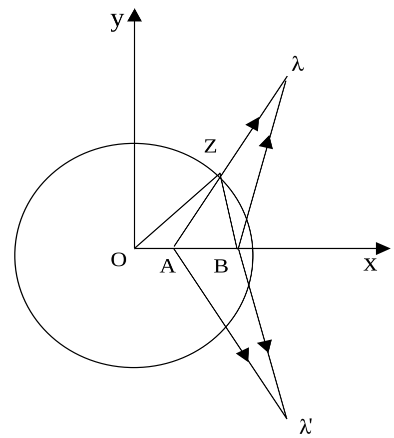

Fig. 1. Proof of Lemma 6.

Proof: See Figure 1, where the x-coordinates for points A and B are μ_{min} and μ_{max} , respectively. We prove the lemma in four steps.

Step 1: If $\theta_i < \pi - \angle A\lambda B$, all solutions to (33) are within the unit circle. Suppose for the sake of contradiction that λ solves the equation and it is outside the unit circle. Let λ' denote the point whose real part is the same as λ and whose imaginary part is negative of that of λ. Since $\mu_{\min} < \mu_i < \mu_{\max}$, all vectors $1/(\lambda - \mu_i)$ lie in between $A\lambda'$ to $B\lambda'$. The factor $e^{-j\theta_i}$ rotates the vector $1/(\lambda - \mu_i)$ clockwise by θ_i. Then the minimal θ that is required to make the sum of the rotated vectors zero is $\pi - \angle A\lambda' B$. To see this, consider any θ that is less than $\pi - \angle A\lambda' B$, then all vectors after rotation will be in the region $A\lambda'$ to $-A\lambda'$ clockwise, hence their sum cannot be zero since they are all pointing to the same half plane. Noticing $\angle A\lambda B = \angle A\lambda' B$, we conclude that $\theta_i < \pi - \angle A\lambda B$, a contradiction.

Step 2: If $\theta_i < \pi - \angle AZB$, all solutions to (33) are within unit circle. This is clear since $\angle AZB = \angle A\lambda B + \angle ZB\lambda \ge \angle A\lambda B$.

Step 3: If $\theta_i < \pi - \angle OZB$, all solutions to (33) are within unit circle. This is clear since $\angle OZB = \angle OZA + \angle AZB \ge \angle AZB$.

Step 4: $\max_Z \angle OZB = \arcsin \mu_{max}$. Consider $\triangle OZB$, by law of cosine, we have

$$\cos(\angle OZB) = \frac{1 + (x - \mu_{max})^2 + y^2 - (\mu_{max})^2}{2\sqrt{(x - \mu_{max})^2 + y^2}} \tag{34}$$

where (x, y) are coordinates of z. Since z is on the unit circle, we have $x^2 + y^2 = 1$. This gives

$$\cos(\angle OZB) = \frac{1 - x\mu_{max}}{\sqrt{1 + (\mu_{max})^2 - 2x\mu_{max}}} \tag{35}$$

It is easy to check that $\cos(\angle OZB)$ reaches its minimum when $x = \mu_{max}$ and the corresponding $\angle OZB = \arcsin \mu_{max}$.

Combining step 3 and step 4 completes the proof. □

Corollary 7. *Suppose $0 < \mu_i < 1$, if $\theta_i < \pi/2$, then $|\lambda| < 1$, where λ solves (33).*

Proof: Since $\mu_{max} < 1$, we have $\arcsin \mu_{max} < \pi/2$. By Lemma 6, when $\theta < \pi - \arcsin \mu_{max} < \pi/2$, we have $|\lambda| < 1$, where λ solves (33). □

V. CONCLUSION

In this paper, we have studied local stability of FAST TCP based on a discrete model. We prove that the system is always locally asymptotically stable for general networks in the absence of feedback delay. We provide a sufficient condition for local asymptotic stability in the presence of feedback delay for the case of a single link. The condition says that the system is locally asymptotically stable if the difference among delays of different sources is small.

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