Local Stability of REM Algorithm With Time-Varying Delays

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***Abstract*—In this letter, we investigate the local stability in equilibrium for Internet congestion control algorithm proposed by Low (IEEE/ACM TRANSACTIONS ON NETWORKING, 1999). The network consists of multisource and one-bottleneck link with heterogenous time-varying propagation delays. Linear matrix inequality (LMI) stability criteria is presented for discrete congestion control algorithm of TCP/REM dual model, which can be efficiently and easily solved by the LMI toolbox provided by Matlab software. An important feature is to acquire the maximum network delays to guarantee the stability of congestion control algorithm, i.e., the scale stability domain of REM algorithm.**

***Index Terms*—Congestion control, linear matrix inequality (LMI), Local stability, REM.**

I. INTRODUCTION

CONGESTION control algorithms, first proposed by Jacobson, have been extensively studied. This algorithm uses the additive-increase/multiplicative-decrease (AIMD) rate control mechanism and plays an important role in guaranteeing the stability of the Internet to date. Lacking in network information, the simple AIMD control mechanism cannot adapt to the requirement, which can lead to performance degradation and even congestion collapse. Such a case was resulted by the dramatic growth in multimedia and peer-to-peer application in the Internet. As a result, there has been a surge of interest in designing best-effort service networks that can deliver low loss and delay service by encouraging the users to adapt to network congestion, and by using minimal information from the network in the framework of end-to-end arguments [6], [8]. Many AQM schemes detecting the onset of congestion intelligently are proposed [1], [2], [4], [7]. REM has an attractive feature to decouple the equilibrium value of congestion measure and that of performance measure so as to achieve high utilization with low loss and delay in equilibrium. So it is worthy of serious exploration for the design of rationale behind REM [1]. Herein, the stability analysis for the REM discrete algorithm is concerned.

Recently, Kelly *et al.* [6] has developed an optimization framework for TCP rate control that allows theoretical analysis of performance, stability and robustness of protocols. The attractive feature of these methods is that they can be viewed as a decentralized algorithm to maximize an aggregate utility function across all sources, and are subject to link capacity constraints which can be solved by a convex optimization problem. The first objective of the optimization algorithm is its convergence and stability. As far as the congestion control model is proposed by Kelly *et al.* [6], there are fruitful results with/without considering communication delays, e.g., the stability analysis and the proof of convergence in the absence of delay or with constant delay have been investigated [5], [6], [9]. A related work by Low *et al.* [8] has been developed, which is closer to Kelly's dual algorithm by using the gradient projection algorithm to solve easily the dual problem. Because the first-order algorithm is such that prices are proportional to link backlogs, and thus, the same to the drawback of RED algorithm, the equilibrium can have large backlogs. This has motivated the REM algorithm proposed in [1], which can decouple congestion measure from performance measure such as loss, queue length of delay. The global asymptotic stability proof of the TCP/REM dual continuous model in absence of communication delays is presented in [10]. The convergence of discrete model at a single link without considering the communication delay is illustrated in [11]. The communication delays have an important role on the stability in equilibrium and its dynamic of TCP/REM model, but it is scarcely exploited to date, only seen in [12],[1] where local stability results with one or two-step constant identical delay are presented. However, the time-varying roundtrip delays case, does exist in today's Internet for the jitter of queue and the uncertainty of processor time, is to be further studied.

The aim of this letter is to give a stability analysis of REM algorithm with presence of time-varying delays. Here we consider the discrete algorithm with one-source and one-bottleneck link network topology as well as multi-source and one-bottleneck link network models with heterogenous delays.

II. MODEL AND RESULTS

A. One Resource and One Route Case

Based on the dual model proposed by [8], the network model can be given as the following nonlinear discrete time-delay systems in the case:

$$(\Theta_1): \begin{cases} y(t) = x\left(t - d^{\rightarrow}(t)\right) \\ q(t) = p\left(t - d^{\leftarrow}(t)\right) \\ x(t) = f\left(q(t)\right) \\ p(t+1) = \left[p(t) + \gamma\left(\alpha\left(b(t) - b^*\right) + y(t) - c\right)\right]^+ \\ b(t+1) = \left[b(t) + y(t) - c\right]^+ \end{cases}.$$

[1][Online]. Available: http://netlab.caltech.edu

Manuscript received September 15, 2002. The associate editor coordinating the review of this letter and approving it for publication was Prof. D. Petr. This work was supported in part by the National Science Foundation of China under Grant 60174010 and Grant 60274023.

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Digital Object Identifier 10.1109/LCOMM.2003.810070

Here, we assume that $d^{\rightarrow}(t) + d^{\leftarrow}(t) = d(t)$ and $d(t)$ has an upper bound m. Obviously, the nonlinear term of the above model is $y(t) = f(p(t - d(t)))$. We linearize it about the equilibrium point to obtain

$$y(t) = y^* + f'(p^*)\left(p\left(t - d(t)\right) - p^*\right)$$

In equilibrium $y^* = c$ and $p^* > 0$, then the nonlinear model can be linearized as the following compact form of linear discrete time-delay systems:

$$\chi(t+1) = \mathcal{A}\chi(t) + \mathcal{B}\chi\left(t - d(t)\right) \tag{1}$$

Where

$$\chi(t) = \left[\delta p(t) \quad \delta b(t)\right]^T, \ \eta = \gamma\alpha, \ \delta p = p - p^*$$
$$\mathcal{A} = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}, \ \mathcal{B} = \begin{pmatrix} \gamma f'(p^*) & 0 \\ f'(p^*) & 0 \end{pmatrix},$$
$$\delta b = b - b^*. \tag{2}$$

Theorem II.1: Suppose the delay is upper-bounded with m. The equilibrium of REM is said to be locally asymptotically stable if there exist the positive scalars γ, ε, δ and $0 < \eta < 1$ such that the following LMI holds:

$$\begin{pmatrix} (\varepsilon - 1)I_{2\times 2} & 0 & \mathcal{A}^T \\ 0 & -\delta I_{2\times 2} & \mathcal{B}^T \\ \mathcal{A} & \mathcal{B} & -I_{2\times 2} \end{pmatrix} < 0. \tag{3}$$

Moreover, if the feasible solutions $(\eta^*, \gamma^*, \varepsilon^*, \delta^*)$ of the above LMI exist, then $\alpha = \eta^*/\gamma^*$,$m = \varepsilon^*/\delta^*$.

Remark 1: Theorem II.1 exhibits the nonlinear relation between network parameter m and AQM controller parameters pair (α, γ) for the sufficient condition of locally asymptotically stable in equilibrium. An appeal characteristic of our result is the decouple of the nonlinear relation between the network parameter and AQM controller parameters pair, which is based on a mild transformation and it can be solved easily by LMI toolbox [3].

Another important argument of our LMI method can answer to the robust stability domain of REM algorithm, i.e., the maximal network delays can be born to guarantee the REM algorithm stability which can be solved by the GEVP (generalized eigenvalue problem) of LMI toolbox as the following result in Theorem II.2.

B. Multi-Source With Heterogeneous Time-Varying Delays Case

The network model about this network topology can be expressed as follows:

$$(\Theta_2): \begin{cases} y(t) = \sum\limits_{S=1}^{S} x\left(t - d_s^{\rightarrow}(t)\right) \\ q_s(t) = p\left(t - d_s^{\leftarrow}(t)\right) \\ x_s(t) = f_s\left(q_s(t)\right) \\ p(t+1) = \left[p(t) + \gamma\left(\alpha\left(b(t) - b^*\right) + y(t) - c\right)\right]^+ \\ b(t+1) = \left[b(t) + y(t) - c\right]^+ \end{cases}$$

and we assume that $d_s(t) = d_s^{\rightarrow}(t) + d_s^{\leftarrow}(t), \forall s \in S$, which has an upper bound m_s.

We linearize the nonlinear model in equilibrium as the above one-source and one-bottle-node case to obtain the following linear discrete delay system with multiple delays:

$$\chi(t+1) = \mathcal{A}\chi(t) + \sum_{i=1}^{S}\mathcal{B}_i\chi\left(t - d_i(t)\right) \tag{4}$$

where $\mathcal{A}$, $\chi(t)$ are in (2) and

$$\mathcal{B}_i = \begin{pmatrix} \gamma f_i'(p^*) & 0 \\ f_i'(p^*) & 0 \end{pmatrix}.$$

Theorem II.2: If there exists the feasible solution set $\Omega = \{(\gamma, \delta, \eta) | \gamma > 0, \delta > 0, 0 < \eta < 1\}$ satisfying the following LMI:

$$\begin{pmatrix} (\varepsilon - 1)I & 0 & \mathcal{A}^T \\ 0 & -\delta I & \widetilde{\mathcal{B}}^T \\ \mathcal{A} & \widetilde{\mathcal{B}} & -I \end{pmatrix} < 0 \tag{5}$$

where $\widetilde{\mathcal{B}} = [\mathcal{B}_1 \ \mathcal{B}_2 \ \cdots \ \mathcal{B}_S]$ and $\varepsilon = \sum_{i=1}^{S} m_i\delta$, then the REM algorithm with multisource and one-bottle-node network model is said to be locally asymptotically stable. Furthermore, if the following **GEVP** holds:

$$\begin{cases} \widetilde{\mathcal{C}}(\vartheta) < \left(\frac{1}{\sum_{i=1}^{S} m_i}\right)\widetilde{\mathcal{D}}(\vartheta) \\ \widetilde{\mathcal{D}}(\vartheta) > 0 \end{cases}$$

where

$$\widetilde{\mathcal{C}}(\vartheta) = \begin{pmatrix} \delta I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \widetilde{\mathcal{D}}(\vartheta) = \begin{pmatrix} I & 0 & -\mathcal{A}^T \\ 0 & \delta I & -\widetilde{\mathcal{B}}^T \\ -\mathcal{A} & -\widetilde{\mathcal{B}} & I \end{pmatrix}$$

then the maximum of the sum of heterogenous delays can be no larger than the optimal solution $M = \sum_{i=1}^{S} m_i$ of the above GEVP to guarantee the stability of REM algorithm with AQM controller parameters pair $((\alpha, \gamma)$.

C. Proofs

Lemma 1: Consider the delay-difference equation

$$x(t+1) = Ax(t) + Bx\left(t - d(t)\right). \tag{6}$$

We assume that the time-varying delay $d(t)$ is upper-bounded, i.e., $0 < d(t) \le m, \forall t \ge 0$. The initial condition $x(t) = \phi(t)$, $t \in \{-m, \cdots, 0\}$ for some continuous function $\Phi(t)$. Suppose that there exist positive definite matrices P and Q such that the following matrix inequality holds:

$$-P + A^TPA + A^TPB\overline{Q}^{-1}B^TPA + mQ < 0 \tag{7}$$

where $\overline{Q} = Q - B^TPB > 0$, Then the system (6) is uniformly asymptotically stable.

Proof: Define a Lyapunov functional V_t as follows:

$$V_t = x_t^TPx_t + \sum_{i=1}^{m}\sum_{j=t-i}^{t-1} x_j^TQx_j$$

where P and Q are positive-definite matrices. The system (6) can be rewritten as follows in virtue of the upper bound of time-

$$\Delta V_t = \chi_{t+1}^T P X_{t+1} - \chi_t^T P X_t + \sum_{i=1}^{S}\sum_{j=1}^{m_i}\left\{\chi_t^T Q \chi_t - \chi_{t-j}^T Q \chi_{t-j}\right\} = \chi_t^T \Pi \chi_t$$

where

$$\chi = \begin{pmatrix} \chi(t)^T & \chi(t-d_1(t))^T & \cdots & \chi(t-d_s(t))^T \end{pmatrix}^T$$

$$\Pi = \begin{pmatrix} -P + \mathcal{A}^T P \mathcal{A} + \Sigma_{i=1}^{S} m_i Q & \mathcal{A}^T P \mathcal{B}_1 & \cdots & \mathcal{A}^T P \mathcal{B}_S \\ \mathcal{B}_1^T P \mathcal{A} & -Q + \mathcal{B}_1^T P \mathcal{B}_1 & \cdots & \mathcal{B}_1^T P \mathcal{B}_s \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{B}_S^T P \mathcal{A} & \mathcal{B}_S^T P \mathcal{B}_1 & \cdots & -Q + \mathcal{B}_S^T P \mathcal{B}_S \end{pmatrix}$$

varying delay $d(t)$:

$$x(t+1) = Ax(t) + B\sum_{i=1}^{m}\delta\left(d(t)-i\right)x(t-i)$$

where

$$\delta(n) = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \end{cases}$$

then

$$\begin{aligned} \Delta V_t &= x_{t+1}^T P x_{t+1} - x_t^T P X_t + \sum_{i=1}^{m}\left\{x_t^T Q x_t - x_{t-i}^T Q x_{t-i}\right\} \\ &= x_t^T\{A^T P A - P + mQ\}x_t + 2x_t^T A^T P \sum_{i=1}^{m} B_i x_{t-i} \\ &\quad + \sum_{i=1}^{m}\sum_{j=1}^{m}\left\{x_{t-i}^T B_i^T P B_j x_{t-j}\right\} - \sum_{i=1}^{m}\left\{x_{t-i}^T Q x_{t-i}\right\} \end{aligned}$$

where $B_i = \delta(d(t)-i)B$, $B_j = \delta(d(t)-j)B$.

Since

$$\sum_{i=1}^{m}\left\{x_{t-i}^T Q x_{t-i}\right\} \geq \sum_{i=1}^{m}\left\{\delta\left(d(t)-i\right)x_{t-i}^T Q x_{t-i}\right\}$$

Then

$$\begin{aligned} \Delta V_t &\leq x_t^T\{-P + A^T P A + A^T P B \overline{Q}^{-1} B^T P A + mQ\}x_t \\ &< 0, \qquad \forall t \geq 0 \end{aligned}$$

Therefore, the system (6) is uniformly asymptotically stable.

Based on Lemma 1 and Schur complements Lemma [3], the proof of Theorem 1 is easy with some matrix manipulation, thus omitted here. The following we give the sketch procedure of the proof of Theorem 2.

Proof (Proof of Theroem II.2): The linearization system in equilibrium of system (Θ_2) is a multidelays system. Therefore, we can define the following Lyapunov functional:

$$V_t = \chi_t^T P \chi_t + \sum_{i=1}^{S}\sum_{j=1}^{m_i}\sum_{k=t-j}^{t-1}\chi_k^T Q \chi_k$$

where P and Q are positive-definite matrices. Taking the forward difference, we can obtain the equation shown at the top of page. Based on the proof of Lemma 1 and Schur Complements, if the LMI (5) holds, then $\Delta V_t < 0$, Therefore, the system (Θ_2) in equilibrium is uniformly asymptotically stable.

III. Conclusion

The letter has considered the stability of network rate control in the presence of time-varying communication delays. Stability conditions are given for both single-resource and multi-resource with diverse delays, which have diverse upper bounds. The stability domain of maximum network delays, i.e., the scale ability of REM algorithm specified by AQM controller parameters α and γ to guarantee stability, can be solved by numerically using our method. The future work is to exploit the stability condition for discrete congestion control algorithm with arbitrary network topology and communication delays.

Acknowledgment

The authors would like to thank Prof. S. H. Low and the anonymous reviewers for their very helpful comments on the article.

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