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Fast Algorithms and Performance Bounds for Sum Rate Maximization in Wireless Networks

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***Abstract*— Sum rate maximization by power control is an important, challenging, and extensively studied problem in wireless networks. It is a nonconvex optimization problem and achieves a rate region that is in general nonconvex. We derive approximation ratios to the sum rate objective by studying the solutions to two related problems, sum rate maximization using an SIR approximation and max-min weighted SIR optimization. We also show that these two problems can be solved very efficiently, using much faster algorithms than the existing ones in the literature. Furthermore, using a new parameterization of the sum rate maximization problem, we obtain a characterization of the power controlled rate region and its convexity property in various asymptotic regimes. Engineering implications are discussed for IEEE 802.11 networks.**

Index Terms— Duality, Distributed algorithm, Power control, Weighted sum rate maximization, Nonnegative matrices and applications, Nonconvex optimization, Wireless networks.

I. Introduction

The following problem of sum rate maximization through power control has been extensively studied in wireless and DSL network design (e.g., a partial list of recent work include [1], [2], [3], [4], [5], [6], [7], [8], [9], [10]):

$$\begin{array}{ll} \text{maximize} & \sum_l w_l \log(1 + \mathsf{SIR}_l(\mathbf{p})) \\ \text{subject to} & 0 \leq p_l \leq \bar{p}_l, \quad \forall\, l \\ \text{variables} & p_l, \quad \forall\, l, \end{array} \tag{1}$$

where p_l is the transmit power, $\bar{p}_l$ is the maximum power, w_l is some positive weight assigned by the network to the lth link (to reflect some long-term priority) and $\mathsf{SIR}_l(\mathbf{p})$ is the signal-to-interference noise ratio. Without loss of generality, we assume that $\mathbf{w}$ is a probability vector. Denote the optimal solution to (1) by $\mathbf{p}^\star$.

This fundamental problem is difficult to solve or to analyze: (1) is a nonconvex optimization problem and the resulting rate region (obtained by varying $\mathbf{w}$ and solving for $\mathbf{p}^\star$ accordingly) is in general a nonconvex set. Moreover, (1) may even be hard to approximate [9]. This paper aims at answering the following questions, interesting in their own right as well as for their importance in understanding cross-layer optimization using transmit powers:

- Can some related efficiently-solvable problems provide provable approximation ratios to (1)? We show how to solve (1) by (i) approximating the function that describes rate as a function of SIR and (ii) by solving the max-min SIR problem in Section III (Theorems 1, 3, 4). We derive algorithms that are much faster than the existing algorithms for these two extensively studied problems, and then quantify their approximation ratios with respect to $\mathbf{p}^\star$ of (1) in Section IV (Theorems 7,8).
- Can we completely characterize the resulting rate region obtained by solving (1)? We provide the answer in a closed-form expression in Section V (Theorem 10), which quantifies the intuition that power-controlled rate region is convex for sufficiently weak interference channels or sufficiently small maximum powers.

Overall, the contributions of the paper are as follows:

1) We start with the weighted sum rate maximization problem. Then, as in past work based on Geometric Programming (GP) [1], we convert it to a convex problem by approximating $\log(1+\mathsf{SIR})$ by $\log \mathsf{SIR}$, followed by a change of variables. At this point, we make a departure and show that there exists a faster algorithm than the traditional gradient algorithm to solve this SIR approximation power control (SAPC) problem.
2) Then, we consider the max-min weighted SIR problem. During the process of convexifying the problem, we discover an unexpected connection to the SAPC problem, which allows to come up with a fast two time-scale algorithm. In addition, we derive a closed-form expression for the optimal power levels which is of independent theoretical interest and also of practical use in small networks. In the equal maximum power case, we also derive a new algorithm for the max-min weighted SIR problem by exploiting a connection to the nonlinear Perron-Frobenius theory.
3) We derive a condition (a nonnegative matrix $\tilde{\mathbf{B}}$ exists) under which we are able to bound the ratio of the optimal weighted sum rate to the sum rate obtained under SAPC and max-min weighted SIR problem.
4) For joint power control and scheduling, we first study the geometry of the rate region. If the rate region is convex, then time-sharing is not necessary; otherwise, it is. We obtain a closed-form expression for the capacity region without time-sharing and show that if the interference channel gains are small or if the maximum powers are small, then the region is convex. Otherwise, it may not be.
5) When the capacity region without time-sharing is not convex, then it is interesting to understand if power

control is useful at all. For this purpose, we assume that time-sharing is performed by an IEEE 802.11 protocol and we study the impact of solving the weighted sum rate maximization (the same as the *maxweight problem* (e.g., in [11], [12]) if queue lengths are weights) using SAPC. Using standard IEEE 802.11 values for RTS and CTS SIR thresholds, we numerically show that $\tilde{\mathbf{B}}$ exists after the RTS/CTS protocol is executed, and the approximation ratio of our algorithm is close to 1. Further, the sum-rate with SAPC is significantly higher than sum rate without power control.

We sharpen and apply a variety of tools from nonnegative matrix theory. An outline of the proof of Theorem 2 is provided in Section III to aid the flow of that section, and the proof of Theorem 4 in the Appendix. Due to space constraint in this paper, all other proofs can be found in [13].

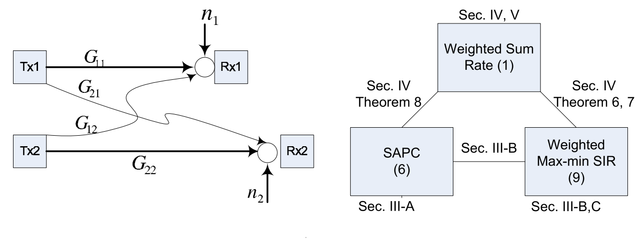
The following notation is used. Boldface uppercase letters denote matrices, boldface lowercase letters denote column vectors, italics denote scalars, and $\mathbf{u} \geq \mathbf{v}$ denotes componentwise inequality between vectors $\mathbf{u}$ and $\mathbf{v}$. We also let $(\mathbf{By})_l$ denote the lth element of $\mathbf{By}$. Let $\mathbf{x} \circ \mathbf{y}$ denote the Schur product of the vectors $\mathbf{x}$ and $\mathbf{y}$, i.e., $\mathbf{x} \circ \mathbf{y} = [x_1 y_1, \ldots, x_L y_L]^\top$. Let $\| \cdot \|_\infty^{\mathbf{x}}$ be the weighted maximum norm of the vector $\mathbf{w}$ with respect to the weight $\mathbf{x}$, i.e., $\|\mathbf{w}\|_\infty^{\mathbf{x}} = \max_l w_l/x_l$, $\mathbf{x} > \mathbf{0}$. We write $\mathbf{B} \geq \mathbf{F}$ if $B_{ij} \geq F_{ij}$ for all i, j. The Perron-Frobenius eigenvalue of a nonnegative matrix $\mathbf{F}$ is denoted as $\rho(\mathbf{F})$, and the Perron (right) and left eigenvector of $\mathbf{F}$ associated with $\rho(\mathbf{F})$ are denoted by $\mathbf{x}(\mathbf{F}) \geq \mathbf{0}$ and $\mathbf{y}(\mathbf{F}) \geq \mathbf{0}$ (or, simply $\mathbf{x}$ and $\mathbf{y}$, when the context is clear) respectively. Recall that the Perron-Frobenius eigenvalue of $\mathbf{F}$ is the eigenvalue with the largest absolute value. Assume that $\mathbf{F}$ is a nonnegative irreducible matrix. Then $\rho(\mathbf{F})$ is simple and positive, and $\mathbf{x}(\mathbf{F}), \mathbf{y}(\mathbf{F}) > \mathbf{0}$ [14]. We will assume the normalization: $\mathbf{x}(\mathbf{F}) \circ \mathbf{y}(\mathbf{F})$ is a probability vector. The super-script $(\cdot)^\top$ denotes transpose. We denote $\mathbf{e}_l$ as the lth unit coordinate vector and $\mathbf{I}$ as the identity matrix.

II. System Model

We consider an infrastructure-less wireless network where the channel is interference-limited, and all the L links (equivalently, transceiver pairs) treat interference as white noise and do not use multiuser detection. Such a model of communication has also been previously used (in, e.g., [8], [1], [2], [7], [5], [15]) for cellular and ad hoc networks. To clearly express the fact that we consider networks without multihop transmissions, we call our model a peer-to-peer wireless networks, i.e., one where each transmitter directly communicates with its receiver without using an intermediate relay. Further, we assume that each node is either a transmitter or a receiver, but not both. Finally, each transmitter is associated with a unique receiver and vice versa. Suppose that a node is equipped with multiple transceivers, then each transceiver associated with the node can be thought of as a virtual link and the above model still captures such scenarios including multiple access and broadcast. Figure 1(a) shows the model for a 2-user interference-limited channel.

The transmit power for the lth link is denoted by p_l for all l. Assuming a matched-filter receiver, the SIR for the lth receiver is given by

$$\mathsf{SIR}_l(\mathbf{p}) = \frac{G_{ll} p_l}{\sum_{j \neq l} G_{lj} p_j + n_l}, \tag{2}$$



(a) (b)

Fig. 1. (a) A model for a 2-user interference-limited channel. (b) Overview of the connection between the three main optimization problems in the respective sections: i) Weighted sum rate maximization in (1), ii) SAPC in (6), iii) Max-min weighted SIR in (9). The theorems connecting the optimization problems are also indicated.

where G_{lj} are the channel gains from transmitter j to receiver l and n_l is the additive white Gaussian noise (AWGN) power for the lth receiver. The channel gain matrix $\mathbf{G}$ takes into account propagation loss, spreading loss and other transmission modulation factors. Assuming a fixed bit error rate (BER) at the receiver, the Shannon capacity formula can be used to deduce the achievable data rate of the lth user as [16]:

$$\log \left(1 + \frac{\mathsf{SIR}_l(\mathbf{p})}{\Gamma} \right) \quad \text{nats/symbol}, \tag{3}$$

where Γ is the gap to capacity, which is always greater than 1. We absorb $(1/\Gamma)$ into G_{ll} for all l, and write the achievable data rate as $\log(1 + \mathsf{SIR}_l(\mathbf{p}))$. The lth user has signal-to-noise ratio (SNR) $G_{ll} p_l / n_l$.

Next, we define

$$F_{lj} = \begin{cases} 0, & \text{if } l = j \\ \frac{G_{lj}}{G_{ll}}, & \text{if } l \neq j \end{cases} \tag{4}$$

and

$$\mathbf{v} = \left(\frac{n_1}{G_{11}}, \frac{n_2}{G_{22}}, \ldots, \frac{n_L}{G_{LL}} \right)^\top. \tag{5}$$

Clearly, $\mathbf{F}$ is nonnegative. Moreover, we assume that $\mathbf{F}$ is irreducible, i.e., each link has at least an interferer. $\mathsf{SIR}_l(\mathbf{p})$ can be compactly written as $p_l / ((\mathbf{Fp})_l + v_l)$.

III. Fast power control algorithms

In this section, we consider two widely studied power control problems. The first one we consider is a direct approximation to (1), i.e., maximize a weighted sum of link rates by first approximating the link rate expression $\log(1 + \mathsf{SIR})$ by $\log(\mathsf{SIR})$, perform a change of variables to convexify the problem, and then use a gradient projection technique to solve the problem [1], [8], [17]. We call such an approximation method SAPC. Here, we show that the gradient approach can be replaced by an alternative procedure that leads to faster convergence. Specifically, we show that the approximation to the link rate expression can be viewed in terms of standard interference function introduced in [18], which allows us to use *greedy* algorithm. Both theoretical analysis and numerical results confirm that this approach leads to much faster convergence than the gradient approach.

Another problem we consider in this section is the widely-studied max-min weighted SIR power control problem where the goal is to maximize the minimum weighted SIR at any receiver in the network. By studying its Lagrangian dual problem, we observe an interesting connection between the max-min weighted SIR problem and SAPC, which allows us to use the interference function approach mentioned earlier as an intermediate step to solve the max-min weighted SIR problem as well. In the special case when the weights in the max-min SIR problem are equal and when all nodes in the network have the same maximum transmit power constraint, we develop an even faster power-control algorithm.

We also derive theoretical results which are of interest in their own right. We first derive a closed-form expression for the max-min weighted power control problem and then derive conditions under which the max-min SIR problem and SAPC have the same solution. These results are shown by exploiting a connection between power-control problems and nonnegative matrix theory. Figure 1(b) overviews the connections between these main optimization problems in the respective sections.

A. SIR *approximation power control (SAPC)*

In this section, we consider the following SIR approximation to (1):

$$\begin{array}{ll} \text{maximize} & \sum_l w_l \log \mathsf{SIR}_l(\mathbf{p}) \\ \text{subject to} & 0 \le p_l \le \bar{p}_l, \quad \forall\, l \\ \text{variables} & p_l, \quad \forall\, l. \end{array} \tag{6}$$

It is well-known that problem (6) can be turned into a GP [1], [8], [17]. We denote $\mathbf{p}'$ as the optimal solution to (6). In particular, by making a change of variable, i.e., $\tilde{p}_l = \log p_l$ for all l, (6) is convex in $\tilde{\mathbf{p}}$ and thus $\tilde{p}'_l = \log p'_l$ for all l [8]. To solve (6) optimally, there are gradient-based algorithms (requiring step size tuning) in [8], [17], which are based on the dynamical system $\dot{p}_l = f(p_l)$ for some function f after changing the variables in $\tilde{\mathbf{p}}$ to $\mathbf{p}$.

A different approach is to derive a fixed point iteration, i.e., $\mathbf{p} = g(\mathbf{p})$ for some function g based on the Karush-Kuhn-Tucker (KKT) optimality conditions (see [19]). Now, it can be verified that the KKT conditions of the convex form of (6) in the variables $\tilde{\mathbf{p}}$ can be rewritten in terms of $\mathbf{p}$ as $\mathbf{p} = \min(I(\mathbf{p}), \bar{\mathbf{p}})$ for some vector function I. Furthermore, it can be shown that $I(\mathbf{p})$ is a standard interference function [18]. By leveraging the interference function results in [18], this leads us to propose the following (step size free) algorithm that converges to the optimal solution of (6), and geometrically fast when the initial point is $\bar{\mathbf{p}}$. This algorithm is also used as a building block for the max-min weighted SIR power control later.

Algorithm 1 (SAPC Algorithm):

1) Update $\mathbf{p}(k+1)$:

$$p_l(k+1) = \min\left\{ w_l \Big/ \left(\sum_{j \neq l} \frac{w_j F_{jl} \mathsf{SIR}_j(\mathbf{p}(k))}{p_j} \right), \bar{p}_l \right\}, \tag{7}$$

for all l, where k indexes discrete time slots.

Remark 1: The information required for computation in (7) can be obtained by distributed message passing: For $j \neq l$, the jth user first computes $w_j \mathsf{SIR}_j(\mathbf{p}(k))/(G_{jj}p_j)$ and measures G_{jl} by pilot signal transmitted from the lth user. Then, $G_{jl} w_j \mathsf{SIR}_j(\mathbf{p}(k))/(G_{jj}p_j)$ is broadcasted to the lth user for computation.

Theorem 1: Starting from any initial point $\mathbf{p}(0)$, $\mathbf{p}(k)$ in Algorithm SAPC converges to $\mathbf{p}'$ asymptotically under both synchronous and asynchronous updates.

Corollary 1: Starting from the initial point $\mathbf{p}(0) = \bar{\mathbf{p}}$, $\mathbf{p}(k)$ in Algorithm SAPC converges synchronously at a geometric rate to $\mathbf{p}'$.

Remark 2: The choice of $\mathbf{p}(0) = \bar{\mathbf{p}}$ does not limit the application of Corollary 1, because it can be shown that, at optimality of (6), some users transmit at maximum power. Hence, some users will not need further power update since they are already optimal at the initial step.

In the following example, we compute the geometric convergence rate in Corollary 1 when $L = 2$.

Example 1: When $L = 2$, we can rewrite (7) as

$$p_l(k+1) = \min\left\{ \frac{(\frac{w_l}{w_j F_{jl}}) p_j(k)}{\mathsf{SIR}_j(\mathbf{p}(k))}, \ \bar{p}_l \right\}, \quad l \neq j \tag{8}$$

for $l, j = 1, 2$, which has a structure similar to the well known Distributed Power Control (DPC) algorithm [20] (with the virtual uplink receivers interchanged), where $w_2/(w_1 F_{12})$ and $w_1/(w_2 F_{21})$ can be treated as a SIR target used in the DPC algorithm for the 1st and 2nd user, respectively. This allows a 1-bit message passing scheme to be implemented, where the 1st receiver instructs the 2nd transmitter to increase or decrease its power by comparing the received SIR with the virtual SIR target at each iteration. If $w_l < w_j, l \neq j$ for $l, j = 1, 2$, then w_l/w_j and 1 are the geometric convergence rate for $p_l(k+1)$ and $p_j(k+1)$ respectively. In other words, the user with a larger weight always transmits at maximum power. If $w_1 = w_2$, then the geometric convergence rate is simply 1. To be exact, both links always transmit at maximum power.

Example 2: We evaluate the performance of Algorithm SAPC and the subgradient algorithm in [8]. We adopt the path loss model with a path loss exponent of 3.7, log-normal shadowing with standard deviation of 8.9dB, and we assume slow fading. Ten users are distributed uniformly in a cell. All users have the same maximum power of 33mW. There are many standard choices of step size $\alpha(k)$ used in the subgradient method [21]. We use a (sufficiently small) constant stepsize and $\alpha(k) = (m+1)/(m+k)$ for $m = 5, 500$ [21] for the subgradient approach in [8]. Figure 2 shows the evolution of SIR_1, which illustrate that Algorithm SAPC converges much faster than the subgradient algorithm in [8], regardless of the subgradient stepsizes. In most of our simulations,

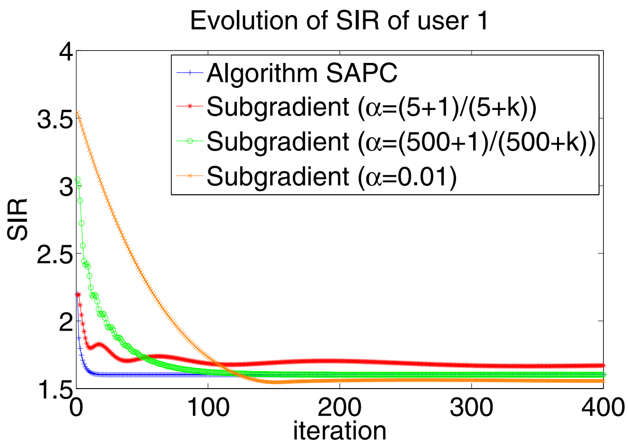


Fig. 2. Performance comparison of Algorithm SAPC and the subgradient approach in [8] that uses different stepsizes α of i) $\alpha(k) = (5+1)/(5+k)$, ii) $\alpha(k) = (500+1)/(500+k)$ and iiii) $\alpha(k) = 0.01$. Evolution of SIR_1.

the convergence speed of Algorithm SAPC is several orders magnitude faster than the subgradient algorithm.

B. Max-min weighted SIR *power control*

Let $\boldsymbol{\beta}$ be a priority vector similar to $\mathbf{w}$ in (6). Consider the following constrained max-min weighted SIR problem:

$$\begin{array}{ll} \text{maximize} & \min_l \dfrac{\mathsf{SIR}_l(\mathbf{p})}{\beta_l} \\ \text{subject to} & \mathbf{p} \leq \bar{\mathbf{p}} \\ \text{variables} & \mathbf{p}. \end{array} \quad (9)$$

By exploiting a connection between nonnegative matrix theory and the algebriac structure of max-min SIR power control, we can give a closed form solution to (9).

Theorem 2: Consider the constrained Max-min SIR problem in (9). An optimal solution is such that the weighted SIR for all users are equal. This weighted SIR is given by

$$\gamma^* = \frac{1}{\rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top))}, \quad (10)$$

where

$$i = \arg\min_l \frac{1}{\rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top))}. \quad (11)$$

Further, all links i that achieve the minimum in (11) transmit at maximum power $\bar{p}_i$ and the rest do not. Further, the optimal $\mathbf{p}$, denoted by $\mathbf{p}^*$, is $t\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top))$ for a constant $t = \bar{p}_i/x_i$.

Remark 3: Suppose $\boldsymbol{\beta} = \mathbf{1}$ and we let $\bar{p}_l \to \infty$ for all l or let $\mathbf{v} = \mathbf{0}$ in Theorem 2, we obtain as a special case the result in [22], where the additive white Gaussian noise and maximum power constraints are assumed not to be present.

Next, we sketch an outline of the proof for Theorem 2, which is based on Lagrange duality [21]. Introducing an auxiliary variable τ, we can rewrite (9) as

$$\begin{array}{ll} \text{maximize} & \tau \\ \text{subject to} & \tau \leq \frac{\mathsf{SIR}_l(\mathbf{p})}{\beta_l}, \ \ \forall\, l \\ & \mathbf{p} \leq \bar{\mathbf{p}} \\ \text{variables} & \tau,\ \mathbf{p}. \end{array} \quad (12)$$

Making a change of variables: $\tilde{\tau} = \log \tau$ and $\tilde{p}_l = \log p_l$ for all l, (12) is equivalent to the following convex problem:

$$\begin{array}{ll} \text{maximize} & \tilde{\tau} \\ \text{subject to} & \tilde{\tau} \leq \log(\mathsf{SIR}_l(\tilde{\mathbf{p}})/\beta_l), \ \ \forall\, l \\ & \tilde{p}_l \leq \log \bar{p}_l \ \ \forall\, l, \\ \text{variables} & \tilde{\tau},\ \tilde{\mathbf{p}}. \end{array} \quad (13)$$

Next, introducing the dual variables $\boldsymbol{\lambda}$, the partial Lagrangian of (13) is given by

$$L(\{\tilde{\tau}, \tilde{\mathbf{p}}\}, \{\boldsymbol{\lambda}\}) = \tilde{\tau}(1 - \sum_l \lambda_l) + \sum_l \lambda_l \log(\mathsf{SIR}_l(\tilde{\mathbf{p}})/\beta_l). \quad (14)$$

In order for (14) to be bounded, we must have $\sum_l \lambda_l = 1$. Hence, the dual problem of (13) is given by

$$\begin{array}{ll} \text{minimize} & \max_{\tilde{\tau}, \tilde{p}_l \leq \log \bar{p}_l \ \forall\, l} L(\{\tilde{\tau}, \tilde{\mathbf{p}}\}, \{\boldsymbol{\lambda}\}) \\ \text{subject to} & \mathbf{1}^\top \boldsymbol{\lambda} = 1,\ \boldsymbol{\lambda} \geq \mathbf{0} \\ \text{variables} & \boldsymbol{\lambda}. \end{array} \quad (15)$$

At optimality, the optimal objective to (13) is equal to the optimal dual function of (15) and is given by

$$\begin{aligned} \tilde{\tau}^\star &= \max_{\tilde{p}_l \leq \log \bar{p}_l \ \forall\, l} L(\{\tilde{\tau}, \tilde{\mathbf{p}}\}, \{\boldsymbol{\lambda}^\star\}) \\ &= \textstyle\sum_l \lambda_l^\star \log(\mathsf{SIR}_l(\mathbf{p}(\boldsymbol{\lambda}^\star))/\beta_l), \end{aligned} \quad (16)$$

where

$$\mathbf{p}(\boldsymbol{\lambda}) = \arg \max_{p_l \leq \bar{p}_l \ \forall\, l} L(\{\tilde{\tau}, \mathbf{p}\}, \{\boldsymbol{\lambda}\}). \quad (17)$$

A final key step is to show that $\tilde{\tau}$ in (16) can be upper bounded using the Friedland-Karlin inequalities in [23] (see [24] for its extensions), and this upper bound is in turn achieved by a feasible $\mathbf{p} = t\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top))$ for some constant $t > 0$, and moreover $tx_i = \bar{p}_i$ [13].

While the closed form solution in Theorem 2 is only useful when the eigenvector $\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top))$ can be computed centrally, we next discuss how to solve (9) for any number of users distributively. From (16) and (17), we observe that the optimal dual function has a form similar to the SAPC problem in (6), which allows us to evaluate this dual function, possibly asynchronously, with fast convergence (cf. Theorem 1 and Corollary 1). This leads us to solve (9) using the following max-min weighted SIR Algorithm.

Algorithm 2 (Max-min Weighted SIR *Algorithm):*

1) Initialize an arbitrarily positive $\mathbf{w}(t)$ and small $\epsilon, \alpha(1)$.
2) Set $\mathbf{p}(0) = \bar{\mathbf{p}}$. Repeat

$$p_l(k+1) = \min\left\{ w_l(t) \Big/ \left(\sum_{j \neq l} \frac{w_j(t) F_{jl} \mathsf{SIR}_j(\mathbf{p}(k))}{p_j(k))} \right), \bar{p}_l \right\}$$

until $\|\mathbf{p}(k+1) - \mathbf{p}(k)\| \leq \epsilon$.

3) Compute $w_l(t+1) = \max\{w_l(t) + \alpha(t)(\sum_j w_j(t) \log(\mathsf{SIR}_j(\mathbf{p}(k+1))/\beta_j) - \log(\mathsf{SIR}_l(\mathbf{p}(k+1))/\beta_l)), 0\}$ for all l, where t indexes discrete time slots much larger than k.
4) Normalize $\mathbf{w}(t+1)$ so that $\mathbf{1}^\top \mathbf{w}(t+1) = 1$. Go to Step 2.

Theorem 3: Starting from any initial point $\mathbf{w}(0)$ and small ϵ, if the positive step size $\alpha(t)$ is strictly less than

$$2(-\log\rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F}+(1/\bar{p}_i))\mathbf{v}\mathbf{e}_i^\top)- \sum_l w_l(t)\log(\mathsf{SIR}_l(\mathbf{p}(\mathbf{w}(t)))/\beta_l))/\|\mathbf{g}(t)\|^2,$$

where

$$i = \arg\min_l \frac{1}{\rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F}+(1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top))}. \quad (18)$$

and $(\mathbf{g}(t))_l = \sum_j w_j(t)\log(\mathsf{SIR}_j(\mathbf{p}(\mathbf{w}(t)))/\beta_j) - \log\frac{\mathsf{SIR}_l(\mathbf{p}(\mathbf{w}(t)))}{\beta_l}$, $\mathbf{p}(k)$ in Algorithm 2 converges synchronously at a geometric rate and asynchronously to $\mathbf{p} = \mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F}+(1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top))$ (unique up to a scaling constant), and $\mathbf{w}(t)$ converges to $\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F}+(1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top))\circ\mathbf{y}(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F}+(1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top))$.

Remark 4: For a given $\mathbf{w}(t)$ at Step 2 of the Max-min SIR Algorithm, Algorithm SAPC is used as an intermediate step in the iterative method, whose geometrical convergence rate is guaranteed by Corollary 1.

Next, we particularize (9) to the special case when β_l's are the same for all l, and consider the constrained max-min SIR problem:

$$\begin{array}{ll} \text{maximize} & \min_l \mathsf{SIR}_l(\mathbf{p}) \\ \text{subject to} & \mathbf{p} \leq \bar{\mathbf{p}} \\ \text{variables} & \mathbf{p}. \end{array} \quad (19)$$

As a side note, the following result illustrates that if the weight vector $\mathbf{w}$ in (6) is chosen in a particular form, then the solution obtained by Algorithm SAPC is the same as that obtained by Max-min SIR Algorithm.

Corollary 2 (Relating Max-min SIR *and SAPC):* Let $\mathbf{x}$ and $\mathbf{y}$ be the Perron and left eigenvectors of $\mathbf{F}+(1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top$ respectively, where i is defined in (11). Recall that $t\mathbf{x}$ is also the optimal power vector for the Max-min SIR problem for some $t>0$. Now consider the SAPC with $\mathbf{w}=\mathbf{x}\circ\mathbf{y}$. Then, $t\mathbf{x}$ is also the optimal power vector for the SAPC.

From Corollary 2, we deduce that the optimal SIR allocation in (6) is a weighted geometric mean of the optimal SIR in (19), where the weights are the Schur product of the Perron and left eigenvectors of $\mathbf{F}+(1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top$:

$$\prod_l(\mathsf{SIR}_l(t\mathbf{x}))^{x_l y_l} = 1/\rho(\mathbf{F}+(1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top), \quad (20)$$

where i is defined in (11) and $t=\bar{p}_i/x_i$.

Remark 5: Note that all the links in the network are activated using Algorithm SAPC (and also the max-min SIR being a special case). As $\bar{p}_l\to\infty$ for all l, using Theorem 3, the solution computed by Algorithm SAPC tends towards that of the unconstrained max-min SIR problem with no noise, i.e., (19) without the maximum power constraints and with $\mathbf{v}=\mathbf{0}$ as $1/\rho(\mathbf{F}+(1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)\to 1/\rho(\mathbf{F})$ in Theorem 2.

C. A faster max-min weighted SIR *Algorithm in special case*

Algorithm (2) is a (two-timescale) primal-dual algorithm that solves (9) for any maximum power constraint, but requires a step size that is adapted iteratively. Under the special case when $\bar{p}_l = \bar{p}$ for all l, we give the following faster (single timescale and step size free) algorithm to solve (9), also with the added advantage of backward compatibility with existing CDMA power control.

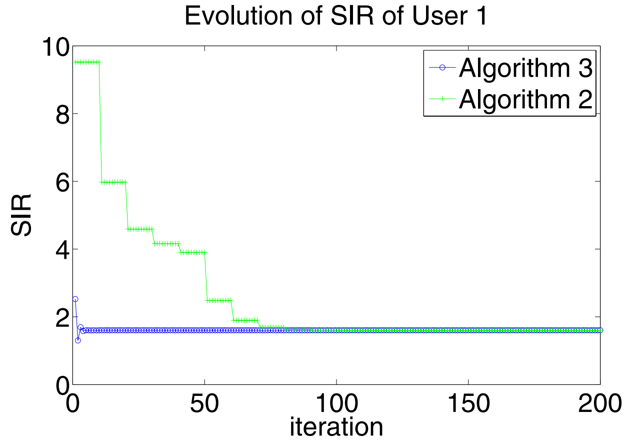


Fig. 3. Performance comparison of the max-min SIR Algorithm 2 and 3 in a typical numerical example of ten users with equal power constraint. Evolution of SIR_1.

Algorithm 3 (Equal Power Max-min Weighted SIR*):*

1) Update power $\mathbf{p}(k+1)$:

$$p_l(k+1) = \frac{\beta_l}{\mathsf{SIR}_l(\mathbf{p}(k))}p_l(k), \;\; \forall\, l.$$

2) Normalize $\mathbf{p}(k+1)$:

$$p_l(k+1) \leftarrow p_l(k+1)\cdot\bar{p}/\max_j p_j(k+1), \;\; \forall\, l.$$

Theorem 4: Starting from any initial point $\mathbf{p}(0)$, $\mathbf{p}(k)$ in Algorithm 3 converges geometrically fast to $\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F}+(1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top))$ (unique up to a scaling constant).

Remark 6: Interestingly, Step 1 of Algorithm 3 is simply the DPC algorithm in [20], where the lth user has a virtual SIR threshold of β_l. DPC algorithm can be implemented using a local feedback from the lth receiver to the lth transmitter after a local SIR measurement. The only global coordination is for computing $\max_j p_j(k+1)$ in Step 2.

Using the numerical simulation model in Example 2, Figure 3 shows that Algorithm 3 converges much faster than Algorithm 2. Algorithm 2 however applies to the case where individual power constraint can differ and is thus more general.

IV. Approximability and Performance Guarantees

The two problems (6,9) in the last section can be solved efficiently and sometimes distributedly. But how well do they approximate the difficult problem of sum rate maximization (1)? In this section, we will derive a set of sufficient conditions, under which we can compute the approximation ratio of Algorithm 1. This technique is also applied to derive the approximation ratio of the max-min weighted SIR power control (using Algorithm 2 or Algorithm 3). Hence, the fast algorithms in Section III provide a performance guarantee. Conditions under which the algorithms solve (1) optimally are also given.

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A. Approximation ratio

First, we rewrite (1) in matrix form as

$$\begin{array}{ll} \text{maximize} & \sum_l w_l \log\left(\left((\mathbf{I}+\mathbf{F})\,\mathbf{p}+\mathbf{v}\right)_l / \left(\mathbf{F}\mathbf{p}+\mathbf{v}\right)_l\right) \\ \text{subject to} & 0 \le \mathbf{p} \le \bar{\mathbf{p}} \\ \text{variables} & \mathbf{p}. \end{array} \tag{21}$$

Next, we consider a relaxed (but still nonconvex) version of (21):

$$\begin{array}{ll} \text{maximize} & \sum_l w_l \log\left(\left((\mathbf{I}+\mathbf{F})\,\mathbf{p}+\mathbf{v}\right)_l / \left(\mathbf{F}\mathbf{p}+\mathbf{v}\right)_l\right) \\ \text{subject to} & \mathbf{1}^\top \mathbf{p} \le \mathbf{1}^\top \bar{\mathbf{p}} \\ \text{variables} & \mathbf{p}. \end{array} \tag{22}$$

Clearly, the optimal objective in (22) upper bounds that in (1), and if the optimizer of (22) satisfies the constraint set of (1), then it also solves (1) optimally. Next, we result characterize the solution of (22).

Lemma 1: The constraint in (22) is tight at optimality.

Consider solving (22), and denote its optimal solution by $\widehat{\mathbf{p}}$. Let $\mathbf{z} = (\mathbf{I}+\mathbf{B})\,\mathbf{p}$ where $\mathbf{B}$ is the matrix:

$$\mathbf{B} = \mathbf{F} + \sum_l \frac{1}{\mathbf{1}^\top \bar{\mathbf{p}}} \mathbf{v}\mathbf{e}_l^\top. \tag{23}$$

To proceed further, we need to introduce the notion of quasi-invertibility of a nonnegative matrix in [25], particularly of $\mathbf{B}$ in (23), which will be useful in solving (22) optimally.

Definition 1 (Quasi-invertibility): A square nonnegative matrix $\mathbf{B}$ is a quasi-inverse of a square nonnegative matrix $\tilde{\mathbf{B}}$ if $\mathbf{B} - \tilde{\mathbf{B}} = \mathbf{B}\tilde{\mathbf{B}} = \tilde{\mathbf{B}}\mathbf{B}$. Furthermore, $(\mathbf{I} - \tilde{\mathbf{B}})^{-1} = \mathbf{I} + \mathbf{B}$ [25].

Lemma 2: If $\tilde{\mathbf{B}}$ exists, then $\tilde{\mathbf{B}}$ has the Perron-Frobenius eigenvalue

$$\rho(\tilde{\mathbf{B}}) = \frac{\rho(\mathbf{B})}{1+\rho(\mathbf{B})}, \tag{24}$$

with the corresponding left and Perron eigenvectors of $\mathbf{B}$.

We next study the existence of $\tilde{\mathbf{B}}$ in different SNR regimes. First is a negative result. In the case where the maximum power of any user is very large, e.g., $\bar{p}_l \to \infty$ for some l (high SNR regime) or when interference (off-diagonals of $\mathbf{F}$) is very large, $\tilde{\mathbf{B}}$ does not exist.

Lemma 3: $\tilde{\mathbf{B}}$ does not exist when $\mathbf{B} = \mathbf{F}$, where $F_{lj} > 0$ for all l, j and $l \neq j$.

On the other hand, when $\mathbf{F} = \mathbf{0}$ (no interference) such that $\mathbf{B} = \mathbf{v}\mathbf{1}^\top/(\mathbf{1}^\top \bar{\mathbf{p}})$, or when $\mathbf{1}^\top \bar{p}$ is sufficiently small (low SNR regime) such that $\mathbf{B} \approx \mathbf{v}\mathbf{1}^\top/(\mathbf{1}^\top \bar{\mathbf{p}})$, then $\tilde{\mathbf{B}}$ always exists.

Lemma 4: For any nonnegative vector $\mathbf{v}$, $\tilde{\mathbf{B}} = 1/(1 + \mathbf{1}^\top \mathbf{v})\mathbf{v}\mathbf{1}^\top$ when $\mathbf{B} = \mathbf{v}\mathbf{1}^\top$.

In the following, we assume that $\tilde{\mathbf{B}}$ exists. By using Lemma 1, we can rewrite (22) as

$$\begin{array}{ll} \text{maximize} & \sum_l w_l \log\left(z_l/(\tilde{\mathbf{B}}\mathbf{z})_l\right) \\ \text{subject to} & (\mathbf{I} - \tilde{\mathbf{B}})\mathbf{z} \ge \mathbf{0} \\ \text{variables} & \mathbf{z}. \end{array} \tag{25}$$

We note that (25) is equivalent to minimizing

$$\prod_l \left((\tilde{\mathbf{B}}\mathbf{z})_l/z_l\right)^{w_l} \tag{26}$$

subject to $(\mathbf{I}-\tilde{\mathbf{B}})\mathbf{z} \ge \mathbf{0}$. Since (26) as a function of $\mathbf{z}$ is plainly homogeneous of degree 0, we can restrict our attention to the set of $\mathbf{z}$ for which $\mathbf{1}^\top \mathbf{z} + \mathbf{1}^\top \tilde{\mathbf{B}}\mathbf{z} = \mathbf{1}^\top \bar{\mathbf{p}}$.

Now, the key step is to exploit the eigenspace of the quasi-inverse $\tilde{\mathbf{B}}$ and connect (26) with the Friedland-Karlin inequalities in [23], which leads to the following result [13].

Theorem 5: Suppose $\tilde{\mathbf{B}}$ exists. Then,

$$\sum_l w_l \log(1 + \mathsf{SIR}_l(\widehat{\mathbf{p}})) \le \|\mathbf{w}\|_\infty^{\mathbf{x}(\mathbf{B})\circ\mathbf{y}(\mathbf{B})} \log\left(1 + 1/\rho(\mathbf{B})\right), \tag{27}$$

where $\mathbf{x}(\mathbf{B}), \mathbf{y}(\mathbf{B})$ are the left and Perron eigenvectors of $\mathbf{B}$ respectively, and $\mathbf{x} \circ \mathbf{y}$ is the Schur product of $\mathbf{x}$ and $\mathbf{y}$.

We have equality in (27) if and only if $\mathbf{w} = \mathbf{x}(\mathbf{B}) \circ \mathbf{y}(\mathbf{B})$. In this case, $\widehat{\mathbf{p}} = (\mathbf{1}^\top \bar{\mathbf{p}}(1+\rho(\mathbf{B}))/\mathbf{1}^\top \mathbf{x}(\mathbf{B}))(\mathbf{I}+\mathbf{B})^{-1}\mathbf{x}(\mathbf{B})$.

Interestingly, the upper bound in (27) can be interpreted as a positive multiple scaling of a rate obtained at a particular SIR, $\gamma_l^\star = 1/\rho(\mathbf{B})$ for all l. More precisely, $\gamma_l^\star = 1/\rho(\mathbf{B})$ is in fact the total power constrained max-min SIR whose derivation and the associated power vector are given in the following result.[1]

Lemma 5: The optimal objective and solution of:

$$\begin{array}{ll} \text{maximize} & \min_l \mathsf{SIR}_l(\mathbf{p}) \\ \text{subject to} & \mathbf{1}^\top \mathbf{p} \le \mathbf{1}^\top \bar{\mathbf{p}} \\ \text{variables} & \mathbf{p} \end{array} \tag{28}$$

is given by $1/\rho(\mathbf{B})$ and $(\mathbf{1}^\top \bar{\mathbf{p}}/\mathbf{1}^\top \mathbf{x}(\mathbf{B}))\mathbf{x}(\mathbf{B})$ respectively.

Since $\log(1 + 1/\rho(\mathbf{B})) \le \sum_l w_l \log(1 + \mathsf{SIR}_l(\widehat{\mathbf{p}}))$, $1/\|\mathbf{w}\|_\infty^{\mathbf{x}(\mathbf{B})\circ\mathbf{y}(\mathbf{B})}$ in Theorem 5 can be viewed as an approximation ratio of the total power constrained max-min SIR method, i.e., (28) achieves at least a fraction $1/\|\mathbf{w}\|_\infty^{\mathbf{x}(\mathbf{B})\circ\mathbf{y}(\mathbf{B})}$ of the global optimal solution of (22).

We next state our main result in the following theorem.

Theorem 6: If $\tilde{\mathbf{B}}$ exists, then

$$\sum_l w_l \log(1 + \mathsf{SIR}_l(\mathbf{p}^\star)) \le \|\mathbf{w}\|_\infty^{\mathbf{x}\circ\mathbf{y}} \log\left(1 + 1/\rho(\mathbf{B})\right), \tag{29}$$

where $\mathbf{x}, \mathbf{y}$ are the Perron and left eigenvectors of $\mathbf{B}$ respectively.

Equality is achieved if the parameters of all users are symmetric, i.e., $\mathbf{F} = \mathbf{F}^\top$, and $\bar{p}_l = \bar{p}_j = \bar{p}$ and $v_l = v_j$ for all $l \neq j$ and $\mathbf{w} = \mathbf{x} \circ \mathbf{x}$. In this case, $\mathbf{p}^\star = \mathbf{x}(\mathbf{F} + (1/\bar{p})\mathbf{v}\mathbf{e}_i^\top)$ (unique up to a scaling constant).

Remark 7: The achievability part in Theorem 6 illustrates a sufficient condition under which the max-min SIR in (19) optimally solves (1).

Based on Theorem 6, we now state the approximation ratio provided by solving the max-min weighted SIR problem (with $\boldsymbol{\beta} = \mathbf{w}$) using Algorithm 2.

Theorem 7: Suppose $\tilde{\mathbf{B}}$ exists. Let

$$\eta = \frac{\sum_l w_l \log(1 + w_l/\rho(\mathrm{diag}(\mathbf{w})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)))}{\|\mathbf{w}\|_\infty^{\mathbf{x}(\mathbf{B})\circ\mathbf{y}(\mathbf{B})} \log(1 + 1/\rho(\mathbf{B}))}, \tag{30}$$

where i is given in (18).

[1] We note that a closed-form solution to the optimization problem in (28) was first obtained in [26], whose solution is however based on a nonnegative (increased dimension) matrix different from $\mathbf{B}$. Lemma 5 is obtained using the nonlinear Perron-Frobenius theory, similarly to the proof given in the Appendix [13].

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Then, η is an approximation ratio to (1) by solving (9).

Similarly, the approximation ratio of Algorithm SAPC in solving (1) can be stated as follows.

Theorem 8: Suppose $\tilde{\mathbf{B}}$ exists. Let

$$\eta = \frac{\sum_l w_l \log(1+\mathsf{SIR}(\mathbf{p}'))}{\|\mathbf{w}\|_\infty^{\mathbf{x}(\mathbf{B})\circ\mathbf{y}(\mathbf{B})}\log(1+1/\rho(\mathbf{B}))}, \tag{31}$$

where $\mathbf{p}'$ is the optimal solution to (6). Then, η is an approximation ratio to (1) by solving (6).

B. General approximability

The results in Section IV-A depend on the existence of a nonnegative $\tilde{\mathbf{B}}$. If this sufficient condition is not satisfied, we show how the results in Section IV-A can still be used to construct useful upper bounds to (1). First, we define a transmission configuration as a set of links $\mathcal{C} = \{l\,|\,l = 1,\ldots,L\}$ with $|\mathcal{C}| \leq L$. Users in $\mathcal{C}$ transmit with positive power. In general, a transmission configuration can be used to construct a scheduling policy in which users that belong to $\mathcal{C}$ solve (1) using power control, whereas users that belong to $\bar{\mathcal{C}}$ transmit one at a time. Clearly, $|\mathcal{C}| + |\bar{\mathcal{C}}| = L$. For example, when $\tilde{\mathbf{B}}$ exists and all users have positive optimal power, we need to only consider $\mathcal{C}$ such that $|\mathcal{C}| = L$. A scheduling policy determines how users in $\mathcal{C}$ and $\bar{\mathcal{C}}$ are time-shared in a time-division multiple access manner. When there are L users, there are

$$\sum_{l=1}^{L-2}\binom{L}{l} + 2 \tag{32}$$

possible transmission configurations. Now, for any transmission configuration $\mathcal{C}$ and any $\mathbf{p}$, we have

$$\sum_{l=1}^{L} w_l \log(1+\mathsf{SIR}_l(\mathbf{p})) \leq \sum_{l\in\mathcal{C}} w_l \log(1+\mathsf{SIR}_l(\mathbf{p})) + \textstyle\sum_{l\in\bar{\mathcal{C}}} w_l \log(1+p_l/v_l), \tag{33}$$

where $\mathsf{SIR}_l(\mathbf{p}), l\in\mathcal{C}$, in the first summand on the right-hand side of (33) contains only interference terms coming from users in $\mathcal{C}$. Therefore, it can be deduced that

$$\sum_{l=1}^{L} w_l \log(1+\mathsf{SIR}_l(\mathbf{p}^\star)) \leq \max_{\mathbf{0}\leq\mathbf{p}\leq\bar{\mathbf{p}}} \sum_{l\in\mathcal{C}} w_l \log(1+\mathsf{SIR}_l(\mathbf{p})) + \textstyle\sum_{l\in\bar{\mathcal{C}}} w_l \log(1+\bar{p}_l/v_l). \tag{34}$$

Let $\mathbf{B}_\mathcal{C}$ denote a submatrix obtained from $\mathbf{B}$ by deleting those rows and columns whose indices belong to $\bar{\mathcal{C}}$. If $\mathbf{B}_\mathcal{C}$ is a quasi-inverse of a nonnegative matrix, we denote that matrix as $\tilde{\mathbf{B}}_\mathcal{C}$. Hence, for any transmission configuration $\mathcal{C}$, if $\tilde{\mathbf{B}}_\mathcal{C}$ exists, we can use the previous results to deduce the following bound

$$\begin{aligned}&\textstyle\sum_{l=1}^{L} w_l \log(1+\mathsf{SIR}_l(\mathbf{p}^\star)) \\ &\leq \max_{l\in\mathcal{C}} \frac{w_l}{(\mathbf{x}(\mathbf{B}_\mathcal{C})\circ\mathbf{y}(\mathbf{B}_\mathcal{C}))_l} \log\left(1+\frac{1}{\rho(\mathbf{B}_\mathcal{C})}\right) + \\ &\textstyle\sum_{l\in\bar{\mathcal{C}}} w_l \log(1+G_{ll}\bar{p}_l/n_l).\end{aligned} \tag{35}$$

Subject to the existence of quasi-inverses, the tightest upper bounds to (1) can be found by searching all $\sum_{l=1}^{L-2}\binom{L}{l}+2$ transmission configurations. Note that if $\tilde{\mathbf{B}}$ exists, then $\tilde{\mathbf{B}}_\mathcal{C}$ exists for any $\mathcal{C}$. On the other hand, as $|\mathcal{C}|$ gets smaller, then it is more likely that $\tilde{\mathbf{B}}_\mathcal{C}$ exists, because the second summand in a submatrix of (23) tends to dominate the first summand.

Parameter	Avg. % of $\tilde{\mathbf{B}}$ exists	SAPC (η)	Max-min SIR (η)	On-off sched. (η)
$\bar{p}_l = 33\text{mW}\,\forall l$ SNR = 7dB	99	0.97 (0.93)	0.99 (0.96)	0.89 (0.84)
$\bar{p}_l = 1\text{W}\,\forall l$ SNR = 40dB	65	0.87 (0.82)	0.91 (0.83)	0.87 (0.82)

TABLE I
A TYPICAL NUMERICAL EXAMPLE IN A TEN-USER NETWORK WITH TWO MAXIMUM POWER CONSTRAINT SETTINGS: $\bar{p}_l = 33$MW OR $\bar{p}_l = 1$W FOR ALL l. THE PERCENTAGE OF INSTANCES WHERE $\tilde{\mathbf{B}}$ EXISTS IS RECORDED.

Example 3: We now examine the implications of our results for practical interference-limited networks such as the IEEE 802.11b ad hoc network using a numerical example. We also consider the on-off scheduling algorithm, which finds the power vector that maximizes sum rates in which users either transmit at maximum or zero power, as a baseline for comparison with SAPC and max-min SIR. Table I records a typical numerical example for a ten-user network, where the maximum power is set as 33mW and 1W (the largest possible value allowed in IEEE 802.11b). We set $\mathbf{w} = \mathbf{x}(\mathbf{B})\circ\mathbf{y}(\mathbf{B})$.

For each maximum power constraint, we average the percentage of instances where $\tilde{\mathbf{B}}$ exists, the sum rates, and the approximation ratios over 10,000 random instances. In the case where $\tilde{\mathbf{B}}$ does not exist, the tightest upper bound using the general approximability technique in Section IV-B is computed and the approximation ratios of SAPC, max-min weighted SIR (with $\boldsymbol{\beta} = \mathbf{w}$) and on-off scheduling are computed. Recorded in parentheses, the approximation ratios for max-min weighted SIR (with $\boldsymbol{\beta} = \mathbf{w}$) and SAPC are computed using Theorem 7 and 8 respectively, and the approximation ratio of on-off scheduling is computed using Theorem 6. Also shown, without parentheses in each cell, are the actual ratios of the achieved rates by the respective algorithms to the global optimal sum rates. As shown in Table I, the sufficient condition that $\tilde{\mathbf{B}}$ exists occurs over a large proportion of time, and the relatively large approximation ratios using SAPC and max-min SIR indicate the usefulness of using power control to maximize sum rates in a typical IEEE 802.11b network.

V. GEOMETRY OF POWER CONTROLLED RATE REGION

We now move from the nonconvexity of the sum rate maximization problem to the nonconvexity of the resulting rate region. We first present an equivalent formulation of the weighted sum rate maximization problem, which enables us to derive the exact achievable rate region of (1). This allows us to investigate the geometry of the achievable rate region in different SNR regimes. The results quantifies the intuition that if the interference channel gains are sufficiently small or if the maximum powers are sufficiently small, then the rate region is convex.

Theorem 9: Consider the following maximization problem:

$$\begin{array}{ll} \text{maximize} & \sum_l w_l \log(1+\gamma_l) \\ \text{subject to} & \rho(\text{diag}(\boldsymbol{\gamma})(\mathbf{F}+(1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top)) \le 1, \quad \forall\, l \\ \text{variables} & \gamma_l, \quad \forall\, l. \end{array} \tag{36}$$

The optimal SIR vector $\boldsymbol{\gamma}^\star$ in (36) is related to the optimal power vector $\mathbf{p}^\star$ in (1) as follows:

$$\mathbf{p}^\star = (\mathbf{I}-\text{diag}(\boldsymbol{\gamma}^\star)\mathbf{F})^{-1}\,\text{diag}(\boldsymbol{\gamma}^\star)\mathbf{v}. \tag{37}$$

Further, there exists a link i such that

$$\rho(\text{diag}(\boldsymbol{\gamma}^\star)(\mathbf{F}+(1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top)) \le \rho(\text{diag}(\boldsymbol{\gamma}^\star)(\mathbf{F}+(1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)) = 1 \tag{38}$$

for all l.

Remark 8: From (36), we deduce that $\mathbf{p}^\star$ can be interpreted as the Perron eigenvector (unique up to a scaling constant) of a nonnegative matrix $\text{diag}(\boldsymbol{\gamma})(\mathbf{F}+(1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)$ for some i with the Perron-Frobenius eigenvalue of 1. Clearly, $\boldsymbol{\gamma}^\star > \mathbf{0}$ if and only if $\text{diag}(\boldsymbol{\gamma})(\mathbf{F}+(1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)$ is irreducible.

One of the advantages gained through (36) is to characterize the power controlled rate region, which is defined as all possible points $r_l = \log(1+\mathsf{SIR}_l(\mathbf{p}))$ satisfying the constraints in (1) for all l obtained by varying $\mathbf{w}$. We next investigate the geometry of the rate region using power control and time-sharing as the SNR of each user varies.

Theorem 10: The achievable rate region $\mathcal{R}$ using power control only is given by

$$\{\mathbf{r} \in R_+^L : \rho(\text{diag}(\exp(\mathbf{r})-\mathbf{1})(\mathbf{F}+(1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top)) \le 1\}. \tag{39}$$

The achievable rate region $\mathcal{R}$ using power control and time-sharing is given by the convex hull of $\mathcal{R}$,

Remark 9: From Theorem 10, we deduce that $\text{Conv}\{\mathbf{r} \in \mathcal{R} : \rho(\text{diag}(\exp(\mathbf{r})-\mathbf{1})\mathbf{F}) \le 1\}$ is the achievable rate region in the high SNR regime (as $\bar{\mathbf{p}} \to \infty$).

In general, computing $\mathcal{R}$ explicitly is difficult. The following example illustrates the use of (36) in Theorem 9 when $L=2$.

Example 4: When $L=2$, (36) simplifies as

$$\begin{array}{ll} \text{maximize} & w_1\log(1+\gamma_1)+w_2\log(1+\gamma_2) \\ \text{subject to} & v_1\gamma_1+(F_{12}v_2+\bar{p}_1F_{12}F_{21})\gamma_1\gamma_2 \le \bar{p}_1, \\ & v_2\gamma_2+(F_{21}v_1+\bar{p}_2F_{12}F_{21})\gamma_1\gamma_2 \le \bar{p}_2 \\ \text{variables} & \gamma_1,\ \gamma_2. \end{array} \tag{40}$$

When $L=2$ and $w_1 = w_2$, one of the three vectors: $(\bar{p}_1,0)^\top$, $(0,\bar{p}_2)^\top$ or $(\bar{p}_1,\bar{p}_2)^\top$ solves (1) optimally [4], [5]. Using (40), we can further determine a priori the user that transmits at maximum power. In particular, if $v_1/\bar{p}_1 \le v_2/\bar{p}_2$ and $F_{12}v_2/\bar{p}_1 \le F_{21}v_1/\bar{p}_2$, then the first user transmits at maximum power.

When $L=2$, Theorem 10 simplifies as follows.

Corollary 3: When $L=2$, the achievable rate region $\mathcal{R}$ using power control only is given by

$$\{(r_1,r_2)\in R_+^2 : r_1 \le \min\{\log\left(1+\frac{\bar{p}_2/(e^{r_2}-1)-v_2}{F_{21}(F_{12}\bar{p}_2+v_1)}\right), \log\left(1+\frac{\bar{p}_1}{F_{12}(F_{21}\bar{p}_1+v_2)(e^{r_2}-1)+v_1}\right)\}\}. \tag{41}$$

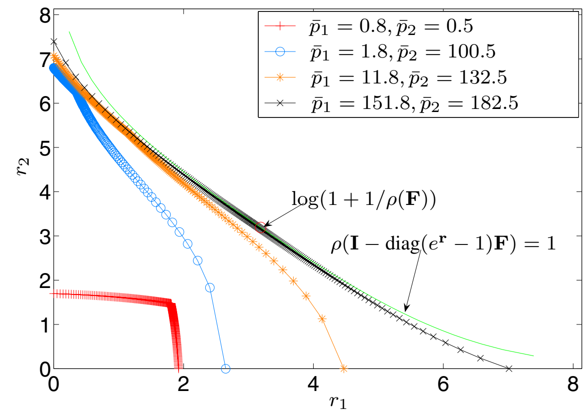

Fig. 4. Achievable rate region for a 2-user interference-limited channel. The channel gains are given by $G_{11} = 0.73, G_{12} = 0.04, G_{21} = 0.03, G_{22} = 0.89$ and the AWGN is 0.1. The maximum power for User 1 and User 2 are varied and are set as $\bar{\mathbf{p}} = [0.8,\ 0.5]^\top$ with $\mathsf{SNR} = [7.66,\ 6.48]^\top$ in dB, $\bar{\mathbf{p}} = [1.8,\ 100.5]^\top$ with $\mathsf{SNR} = [11.19,\ 29.52]^\top$ in dB, $\bar{\mathbf{p}} = [11.8,\ 132.5]^\top$ with $\mathsf{SNR} = [19.35,\ 30.72]^\top$ in dB and $\bar{\mathbf{p}} = [151.8,\ 182.5]^\top$ with $\mathsf{SNR} = [30.45,\ 32.11]^\top$ in dB.

Furthermore, as $\bar{p}_l \to \infty$ for all l, the asymptotic rate region $\bar{\mathcal{R}}$ is given by

$$\{(r_1,r_2)\in R_+^2 : r_1 \le \log\left(1+\frac{1}{F_{12}F_{21}(e^{r_2}-1)}\right)\}. \tag{42}$$

We note that (41) is independently and contemporaneously derived in [7] using a different approach.

Figure 4 illustrates how the rate region is shaped by interference when $L=2$ in different SNR regimes. As observed, the rate region becomes convex in the low SNR regime and nonconvex in the high SNR regime. The green curve illustrates the asymptotic bound in (42), and the point labelled $\log(1+1/\rho(\mathbf{F}))$ is the rate that Algorithms 1, 2 and 3 converge to, when $\bar{p}_l \to \infty$ for all l and $w_1 = w_2$ ($\beta_1 = \beta_2$).

VI. Conclusion

Sum rate maximization is a hard problem in power control and any cross-layer design involving transmit power. But large approximation ratios can be obtained through two related problems: SIR approximation power control (SAPC) and max-min weighted SIR optimization. These two problems have also been extensively studied before, and now we have faster algorithms for them with geometric convergence rate, often independent of stepsize. These results are derived based on new techniques from nonnegative matrix theory.

Even if the sum rate maximization problem is solved exactly, the resulting rate region may still need to be convexified by scheduling if it is a nonconvex set. Extending our methodology, we characterize the rate region and its convexity property. However, how to optimally control both the times to transmit and the power levels during transmission, possibly leveraging the characterization of power controlled rate region in Theorem 10, remains a challenging open problem.

ACKNOWLEDGEMENT

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APPENDIX: PROOF OF THEOREM 4

Our proof is based on the nonlinear Perron-Frobenius theory [27]. We let the optimal max-min weighted $\mathsf{SIR}(\mathbf{p}^*)$ in (12) be τ^*. This implies, at optimality of (12),

$$\frac{(p_l^*/\bar{p})}{\sum_{j\neq l} F_{lj}(p_l^*/\bar{p}) + (v_l/\bar{p})} = \tau^* \beta_l \tag{43}$$

for all l. Letting $\mathbf{s}^* = (1/\bar{p})\mathbf{p}^*$, (43) can be rewritten as

$$(1/\tau^*)\mathbf{s}^* = \text{diag}(\boldsymbol{\beta})\mathbf{F}\mathbf{s}^* + (1/\bar{p})\text{diag}(\boldsymbol{\beta})\mathbf{v}. \tag{44}$$

We first state the following lemma:

Lemma 6 (Conditional eigenvalue [27], Corollary 14): Let $\mathbf{A}$ be a nonnegative matrix and $\mathbf{b}$ be a nonnegative vector. If $\rho(\mathbf{A} + \mathbf{b}\mathbf{e}_i^\top) > \rho(\mathbf{A})$, where $i = \arg\min_l 1/\rho(\mathbf{A} + \mathbf{b}\mathbf{e}_l^\top)$, then the conditional eigenvalue problem

$$\lambda\mathbf{s} = \mathbf{A}\mathbf{s} + \mathbf{b}, \quad \lambda \in \mathbb{R}, \quad \mathbf{s} \geq \mathbf{0}, \quad \max_l s_l = 1,$$

has a unique solution given by $\lambda = \rho(\mathbf{A} + \mathbf{b}\mathbf{e}_i^\top)$ and $\mathbf{s}$ being the unique normalized Perron eigenvector of $\mathbf{A} + \mathbf{b}\mathbf{e}_i^\top$.

Letting $\lambda = 1/\tau^*, \mathbf{A} = \text{diag}(\boldsymbol{\beta})\mathbf{F}, \mathbf{b} = (1/\bar{p})\text{diag}(\boldsymbol{\beta})\mathbf{v}$ in Lemma 6 and noting that $\max_l s_l^* = 1$ shows that $\mathbf{p}^* = (\bar{s}_i/\bar{p})\mathbf{x}(\text{diag}(\boldsymbol{\beta})\mathbf{F} + (1/\bar{p})\text{diag}(\boldsymbol{\beta})\mathbf{v})$ is a fixed point of (44) as it should be (cf. Theorem 2). Now, the fixed point in Lemma 6 is also a unique fixed point of the following equation [27]:

$$\mathbf{s} = \frac{\mathbf{A}\mathbf{s} + \mathbf{b}}{\|\mathbf{A}\mathbf{s} + \mathbf{b}\|_\infty}. \tag{45}$$

Applying the power method in (45) to the system of equations in (44) yields the following iterative method:

1) Update auxiliary variable $\mathbf{s}(k+1)$:

$$\mathbf{s}(k+1) = \text{diag}(\boldsymbol{\beta})\mathbf{F}\mathbf{s}(k) + (1/\bar{p})\text{diag}(\boldsymbol{\beta})\mathbf{v}. \tag{46}$$

2) Normalize $\mathbf{s}(k+1)$:

$$\mathbf{s}(k+1) \leftarrow \mathbf{s}(k+1)/\max_l s_l(k+1). \tag{47}$$

3) Compute power $\mathbf{p}(k+1)$:

$$\mathbf{p}(k+1) = \mathbf{s}(k+1)\bar{p}. \tag{48}$$

Combining Step 1 and 3 in the above and after some rearrangement yields Algorithm 3. This completes the proof of Theorem 4.

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