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Hopf bifurcation in REM algorithm with communication delay

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Abstract

The purpose of this paper is to study bifurcation of an Internet congestion control algorithm, namely REM (Random Exponential Marking) algorithm, with communication delay. By choosing the delay constant as a bifurcation parameter, we prove that REM algorithm exhibits Hopf bifurcation. The formulas for determining the direction of the Hopf bifurcation and the stability of bifurcating periodic solutions are obtained by applying the center manifold theorem and the normal form theory. Finally, a numerical simulation is present to verify the theoretical results.

1. Introduction

From the point of view of the control theory, Internet is a dynamic nonlinear system whose internal state is decided by the TCP congestion avoidance algorithms at sources and the active queue management (AQM) algorithms at link nodes. Generically, the congestion control mechanism in the Internet may be considered as a feedback system where the input is the congestion information in the network, and the output is the adjusted rates of the end system. In this system, AQM algorithms, such as DropTail [1], RED [2], REM [4,5], update a local congestion measure by the queueing size in the buffer, then feed the congestion marks back to all sources using this link, while TCP algorithms, such as TCP Reno [1], TCP Vegas [6], adjust a source rate in response to congestion information in its path. Shortly speaking, the TCP/AQM can be interpreted as a distributed algorithm to control Internet congestion by adjusting the transmission rates of sources.

Recently, stability and bifurcation analysis of congestion control algorithms with communication delays has attracted much attention. In [3], a conjecture on the stability of the first-order TCP algorithm was proved and generalized. The Hopf bifurcation properties of the TCP algorithm was studied in [10]. In this paper we will focus on a kind of new AQM algorithm, namely REM (Random Exponential Marking) algorithm [5]. This algorithm has an attractive feature that it decouples the equilibrium value of congestion measure and that of performance measure. It achieves high utilization with low loss and delay at equilibrium. The convergence of the REM with a single link in absence of communication delays was illustrated in [8]. The global asymptotic stability of the REM without delays was proved in [7]. The

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local stability of the REM algorithm with one or two-step identical delay was considered in [9]. In this paper, we are interested in the Hopf bifurcation properties of the REM algorithm. Note that the dynamics of the REM algorithm are quite different from the dynamics of the TCP algorithm discussed in [10]. The REM is described by the second order delayed differential equation while the later is of the first order. Moreover, unlike [10] in which the control gain was chosen as the bifurcation parameter, we will consider the communication delay as the bifurcation parameter. The reason is that in nonlinear systems including congestion control systems complex dynamics are often related to input delays or state delays.

The rest of the paper is organized as follows. The existence of Hopf bifurcation of REM with communication delay is investigated in Section 2. In Section 3, based on the center manifold theorem and the normal form theory, the formulas for determining the stability of bifurcating solutions and the direction of the Hopf bifurcation are derived. A numerical example is presented in Section 4 to illustrate the theoretical results. Finally, concluding remarks are given in Section 5.

2. Existence of Hopf bifurcation in REM algorithm with communication delay

In this section, we discuss the existence of Hopf bifurcation of the delayed REM algorithm with a single link accessed by a single source. The model is described by

$$\begin{cases} \dot{b}(t) = F(p(t-D)) - c, \\ \dot{p}(t) = \gamma(\alpha(b(t) - b_0) + F(p(t-D)) - c). \end{cases} \tag{1}$$

where $b(t)$ is the queueing size of the buffer at the link node at time t, $p(t)$ is the congestion indication function or congestion control rate at the link node. The control gains γ and α are positive constants, c is the serving capacity of the link node, D is the sum of forward and backward delays, b_0 is a target value of the buffer of the link node. The function $F(p(t))$ is the adjusted rate of the source based on the congestion rate $p(t)$ from the link node, which is a decreasing, nonnegative derivative function. Let the equilibrium point of the system (1) be b^*, x^*, which should satisfy

$$\begin{cases} b^* = b_0, \\ F(p^*) = c. \end{cases} \tag{2}$$

Let $y_1(t) = p(t) - p^*$, $y_2(t) = b(t) - b^*$. Linearizing the system (1) about the equilibrium point, we get

$$\begin{cases} \dot{y}_1(t) = \gamma\alpha y_2(t) + \gamma F' y_1(t-D) + O(y_1^2), \\ \dot{y}_2(t) = F' y_1(t-D) + O(y_1^2), \end{cases} \tag{3}$$

where $F' = F'(p^*) < 0$ is the first derivative value at the equilibrium point. Then the characteristic equation of the linearized equation (3) is

$$\lambda^2 + a_2\lambda e^{-\lambda D} + a_1 e^{-\lambda D} = 0, \tag{4}$$

where $a_1 = -\gamma\alpha F' > 0$, $a_2 = -\gamma F' > 0$.

Define $Q(\lambda, D) = \lambda^2 + a_2\lambda e^{-\lambda D} + a_1 e^{-\lambda D}$. Then we have the following lemma.

Lemma 1 [11]. *For the characteristic equation* (4), *we define*

$$M(D) = \#\{\lambda : \mathrm{Re}(\lambda) \geqslant 0, Q(\lambda, D) = 0\},$$

which denotes the number of the roots with nonnegative real part. Let $0 \leqslant D_1 < D_2$. *Suppose that for* $D \in [D_1, D_2]$, *there are no roots of* (4) *on the imaginary axis. Then* $M(D_1) = M(D_2)$.

In (4), by setting $\lambda = \pm i\omega_0$, where $\omega_0 > 0$, we obtain

$$\begin{cases} -\omega_0^2 + a_2\omega_0 \sin(\omega_0 D) + a_1 \cos(\omega_0 D) = 0, \\ a_2\omega_0 \cos(\omega_0 D) = a_1 \sin(\omega_0 D) \end{cases} \tag{5}$$

or equivalently,

$$\omega_0^4 - a_2^2\omega_0^2 - a_1^2 = 0.$$

Therefore, we have

$$\omega_0 = \sqrt{\frac{a_2^2 + \sqrt{a_2^4 + 4a_1^4}}{2}} \tag{6}$$

and

$$D_0 = \frac{1}{\omega_0} \arctan\left(\frac{\omega_0}{\alpha}\right). \tag{7}$$

From Eq. (7), we know that $D_0\omega_0 < \frac{\pi}{2}$. To proceed with the discussion, we need to prove the following lemmas.

Lemma 2. *When $D < D_0$, all the roots of (4) have strictly negative real parts.*

Proof. When $D = 0$, the characteristic equation is $Q(\lambda, 0) = \lambda^2 + a_2\lambda + a_1 = 0$. Since $a_1 > 0$, $a_2 > 0$, there are no roots of $Q(\lambda, 0) = 0$ with nonnegative real parts. So we have $M(0) = 0$. From the above analysis we know that there are no roots of (4) on the imaginary axis when $D < D_0$. By Lemma 1, we have $M(D) = M(0) = 0$. Therefore, when $D < D_0$, there are no roots of (4) with nonnegative real parts. □

Lemma 3. *Suppose $D = D_0$. Then Eq. (4) has a pair of purely imaginary roots $\lambda = \pm i\omega_0$ which are simple and satisfy $D_0\omega_0 < \frac{\pi}{2}$, and all the other roots have strictly negative real parts.*

Proof. From the above analysis we know that, when $D = D_0$, Eq. (4) has a pair of purely imaginary roots $\lambda = \pm i\omega_0$ which satisfy $D_0\omega_0 < \frac{\pi}{2}$.

Next, we show that $\lambda = \pm i\omega_0$ are simple roots of Eq. (4) when $D = D_0$. Based on the function $Q(\lambda, D) = \lambda^2 + a_2\lambda e^{-\lambda D} + a_1 e^{-\lambda D}$, we have

$$\frac{dQ(\lambda)}{d\lambda} = 2\lambda + a_2 e^{-\lambda D} - a_2\lambda D e^{-\lambda D} - a_1 D e^{-\lambda D}.$$

Let $\lambda = i\omega_0$. Then we obtain

$$\begin{aligned}
\frac{dQ(i\omega_0)}{d\lambda} &= i2\omega_0 + a_2 e^{-i\omega_0 D_0} - ia_2\omega_0 D_0 e^{-i\omega_0 D_0} - a_1 D_0 e^{-i\omega_0 D_0} \\
&= (a_2\cos(\omega_0 D_0) - a_2\omega_0 D_0 \sin(\omega_0 D_0) - a_1 D_0 \cos(\omega_0 D_0)) + i(2\omega_0 - a_2\sin(\omega_0 D_0) - a_2\omega_0 D_0\cos(\omega_0 D_0) \\
&\quad + a_1 D_0 \sin(\omega_0 D_0)) = (a_2\cos(\omega_0 D_0) - D_0\omega_0^2) + i(2\omega_0 - a_2\sin(\omega_0 D_0)) \\
&= (a_2\cos(\omega_0 D_0) - D_0\omega_0^2) + i(\omega_0^2 + a_1\cos(\omega_0 D_0))/\omega_0 \\
&\neq 0.
\end{aligned}$$

Through the same process, we get

$$\frac{dQ(-i\omega_0)}{d\lambda} \neq 0.$$

Hence, $\lambda = \pm i\omega_0$ are simple roots of Eq. (4) when $D = D_0$.

Finally, we show that all the other roots of Eq. (4) have strictly negative real parts when $D = D_0$. Suppose to the contrary that there exist a pair of roots of Eq. (4) $\lambda_{1,2} = \beta \pm i\omega$, where $\beta > 0$. Since the roots are continuous in parameter D, for any sufficiently small positive number ϵ there exists a positive number δ, which depends on ϵ, such that $|\mathrm{Re}(\lambda_1) - \beta| < \epsilon$ holds when $D \in (D_0 - \delta, D_0 + \delta)$. Let $\epsilon = \frac{\beta}{2}$, we have $\mathrm{Re}(\lambda_1) > \frac{\beta}{2}$ when $D \in (D_0 - \delta, D_0)$. This contradicts the conclusion of Lemma 2. Thus, we complete the proof of Lemma 3. □

Lemma 4. *Suppose $D = D_0 + \mu$. Let $\lambda(\mu) = \alpha(\mu) + i\omega(\mu)$ be the root of Eq. (4) satisfying $\alpha(0) = 0$, $\omega(0) = \omega_0$. Then $\frac{d\mathrm{Re}(\lambda)}{d\mu}|_{\mu=0} > 0$.*

Proof. Since $\lambda(\mu) = \alpha(\mu) + i\omega(\mu)$ is a root of Eq. (4), it satisfies

$$\lambda^2 + a_2\lambda e^{-\lambda(D_0+\mu)} + a_1 e^{-\lambda(D_0+\mu)} = 0.$$

Applying the Implicit Function Theorem, we get

$$\left.\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}\right|_{\mu=0} = -\frac{\lambda^3}{2\lambda + a_2\mathrm{e}^{-\lambda D_0} + \lambda^2 D_0} = \frac{\mathrm{i}\omega_0^3}{(a_2\cos(\omega_0 D_0) + D_0\omega_0^2) + \mathrm{i}(2\omega_0 - a_2\sin(\omega_0 D_0))}$$
$$= \frac{\omega_0^2(a_1\cos(\omega_0 D_0) + \omega_0^2) + \mathrm{i}\omega_0^2(a_1\sin(\omega_0 D_0) - \omega_0^2 D_0)}{(a_2\cos(\omega_0 D_0) + D_0\omega_0^2)^2 + (2\omega_0 - a_2\sin(\omega_0 D_0))^2}.$$

So

$$\left.\frac{\mathrm{dRe}(\lambda)}{\mathrm{d}\mu}\right|_{\mu=0} = \frac{\omega_0^2(a_1\cos(\omega_0 D_0) + \omega_0^2)}{(a_2\cos(\omega_0 D_0) + D_0\omega_0^2)^2 + (2\omega_0 - a_2\sin(\omega_0 D_0))^2} > 0 \tag{8}$$

follows because $\cos(D_0\omega_0) > 0$ when $D_0\omega_0 < \frac{\pi}{2}$. Thus, the proof is completed. □

Based on Lemmas 2–4, we obtain the following bifurcation theorem for Eq. (4) by applying the Hopf bifurcation theorem for delayed differential equations [12].

Theorem 1. *When $D = D_0$, Eq. (4) exhibits a Hopf bifurcation.*

3. Stability and direction of Hopf bifurcating periodic solutions

In this section, we use the normal form theory and the center manifold theorem in [13] to study the stability and the direction of bifurcating periodic solutions. The expansion of Eq. (1) about the equilibrium point is

$$\begin{cases} \dot{y}_1(t) = \gamma\alpha y_2(t) + \gamma F' y_1(t-D) + \frac{\gamma F''}{2} y_1^2(t-D) + \frac{\gamma F^{(3)}}{3!} y_1^3(t-D) + \cdots, \\ \dot{y}_2(t) = F' y_1(t-D) + \frac{F''}{2} y_1^2(t-D) + \frac{F^{(3)}}{3!} y_1^3(t-D) + \cdots, \end{cases} \tag{9}$$

where $F'' = F''(p^*)$, $F^{(3)} = F^{(3)}(p^*)$. Let $D = D_0 + \mu$. Denote $C^k[-D,0] = \{\varphi|\varphi : [-D,0] \to R^2, \varphi$ has k-order continuous derivative$\}$. For $\varphi(\theta) = (\varphi_1(\theta), \varphi_2(\theta))^{\mathrm{T}} \in C[-D,0]$, we define

$$L_\mu\varphi = \begin{pmatrix} 0 & \gamma\alpha \\ 0 & 0 \end{pmatrix}\varphi(0) + \begin{pmatrix} \gamma F' & 0 \\ F' & 0 \end{pmatrix}\varphi(-D). \tag{10}$$

By the Reisz representation theorem, there exists a 2×2 matrix-valued function

$$\eta(\cdot,\mu) : [-D,0] \to R^{2\times 2}$$

for $\varphi \in C[-D,0]$ such that

$$L_\mu\varphi = \int_{-D}^{0} \mathrm{d}\eta(\theta,\mu)\varphi(\theta),$$

where

$$\eta(\theta,\mu) = \begin{pmatrix} 0 & \gamma\alpha \\ 0 & 0 \end{pmatrix}\delta(\theta) + \begin{pmatrix} \gamma F' & 0 \\ F' & 0 \end{pmatrix}\delta(\theta + D), \tag{11}$$

where $\delta(\theta)$ is the Dirac Delta function.

For $\varphi \in C[-D,0]$, we define

$$A(\mu)\varphi = \begin{cases} \frac{\mathrm{d}\varphi}{\mathrm{d}\theta}, & \theta \in [-D,0); \\ \int_{-D}^{0} \mathrm{d}\eta(s,\mu)\varphi(s), & \theta = 0 \end{cases}$$

and

$$R(\mu)\varphi = \begin{cases} 0, & \theta \in [-D,0); \\ f_0(\mu,\varphi), & \theta = 0, \end{cases}$$

where

$$f_0(\mu,\varphi) = \begin{pmatrix} \frac{\gamma F''}{2}\varphi_1^2(t-D) + \frac{\gamma F^{(3)}}{3!}\varphi_1^3(t-D) + \cdots \\ \frac{F''}{2}\varphi_1^2(t-D) + \frac{F^{(3)}}{3!}\varphi_1^3(t-D) + \cdots \end{pmatrix} = \left(\frac{F''}{2}\varphi_1^2(t-D) + \frac{F^{(3)}}{3!}\varphi_1^3(t-D) + \cdots\right)\begin{pmatrix} \gamma \\ 1 \end{pmatrix}.$$

Then Eq. (9) can be rewritten as

$$\dot{y}_t = A(\mu)y_t + R(\mu)y_t, \tag{12}$$

where $y_t = (y_{1t}, y_{2t})^{\mathrm{T}} = (y_1(t+\theta), y_2(t+\theta))^{\mathrm{T}}$, $\theta \in [-D, 0]$.

For $\psi \in C[0, D]$, the adjoint operator A^* of A is defined as

$$A^*\psi = \begin{cases} -\frac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & s \in (0, D]; \\ \int_{-D}^{0} \mathrm{d}\eta^{\mathrm{T}}(t, 0)\psi(-t), & s = 0. \end{cases}$$

For $\varphi \in C[-D, 0]$, $\psi \in C[0, D]$, we define a bilinear form by

$$\langle \psi, \varphi \rangle = \bar{\psi}^{\mathrm{T}}(0)\varphi(0) - \int_{\theta=-D}^{0} \int_{\xi=0}^{\theta} \bar{\psi}^{\mathrm{T}}(\xi - \theta)[\mathrm{d}\eta(\theta)]\varphi(\xi)\mathrm{d}\xi. \tag{13}$$

From the above analysis, we know that $\pm \mathrm{i}\omega_0$ are the eigenvalues of A and A^*. Let $q(\theta)$ be the eigenvector of A associated with the eigenvalue $\mathrm{i}\omega_0$ and $q^*(\theta)$ be the eigenvector of A^* associated with the eigenvalue $-\mathrm{i}\omega_0$. We can obtain the eigenvectors by the following Lemma.

Lemma 5. *Let $q(\theta)$ be the eigenvector of A associated with the eigenvalue $\mathrm{i}\omega_0$ and $q^*(\theta)$ be the eigenvector of A^* associated with the eigenvalue $-\mathrm{i}\omega_0$. Then*

$$q(\theta) = (1, \rho_1)^{\mathrm{T}} \mathrm{e}^{\mathrm{i}\omega_0\theta},$$
$$q^*(\theta) = B(\rho_2, 1)^{\mathrm{T}} \mathrm{e}^{\mathrm{i}\omega_0\theta},$$

where

$$\rho_1 = \frac{\mathrm{i}\omega_0}{\gamma\alpha + \mathrm{i}\gamma\omega_0},$$
$$\rho_2 = -\frac{\mathrm{i}\omega_0}{\gamma\alpha},$$
$$\bar{B} = [(\rho_1 + \bar{\rho}_2) + D_0 \mathrm{e}^{-\mathrm{i}\omega_0 D_0} F'(\gamma\bar{\rho}_2 + 1)]^{-1}.$$

And moreover, the following equalities hold:

$$\langle q^*, q \rangle = 1,$$
$$\langle q^*, \bar{q} \rangle = 0.$$

Proof. Suppose $q(\theta)$ is the eigenvector of A associated with the eigenvalue $\mathrm{i}\omega_0$. Then we have

$$Aq(\theta) = \mathrm{i}\omega_0 q(\theta).$$

It follows that $q(\theta) = q(0)\mathrm{e}^{\mathrm{i}\omega_0\theta}$. When $\theta = 0$, since

$$\begin{pmatrix} \mathrm{i}\omega_0 & 0 \\ 0 & \mathrm{i}\omega_0 \end{pmatrix} q(0) = \begin{pmatrix} 0 & \gamma\alpha \\ 0 & 0 \end{pmatrix} q(0) + \begin{pmatrix} \gamma F' & 0 \\ F' & 0 \end{pmatrix} q(0)\mathrm{e}^{-\mathrm{i}\omega_0 D_0},$$

we get

$$q(0) = \left(1, \frac{\mathrm{i}\omega_0}{\gamma\alpha + i\gamma\omega_0}\right)^{\mathrm{T}}.$$

Similarly, we can get

$$q^*(0) = \left(-\frac{\mathrm{i}\omega_0}{\gamma\alpha}, 1\right)^{\mathrm{T}}.$$

So

$$q(\theta) = (1, \rho_1)^{\mathrm{T}} \mathrm{e}^{\mathrm{i}\omega_0\theta},$$
$$q^*(\theta) = B(\rho_2, 1)^{\mathrm{T}} \mathrm{e}^{\mathrm{i}\omega_0\theta},$$

where

$$\rho_1 = \frac{i\omega_0}{\gamma\alpha + i\gamma\omega_0}, \quad \rho_2 = -\frac{i\omega_0}{\gamma\alpha}.$$

Next, we calculate the parameter B, and prove $\langle q^*, q\rangle = 1, \langle q^*, \bar{q}\rangle = 0$. From Eq. (13), we obtain

$$\begin{aligned}
\langle q^*, q\rangle &= \bar{q}^{*\mathrm{T}}(0)q(0) - \int_{\theta=-D_0}^{0}\int_{\xi=0}^{\theta}\bar{q}^{*\mathrm{T}}(\xi-\theta)[\mathrm{d}\eta(\theta)]q(\xi)\mathrm{d}\xi \\
&= \bar{B}(\rho_1 + \bar{\rho}_2) - \bar{B}\int_{\theta=-D_0}^{0}\int_{\xi=0}^{\theta}\bar{q}^{*\mathrm{T}}(0)\mathrm{e}^{-\mathrm{i}\omega_0(\xi-\theta)}[\mathrm{d}\eta(\theta)]q(0)\mathrm{e}^{\mathrm{i}\omega_0\xi}\mathrm{d}\xi \\
&= \bar{B}(\rho_1 + \bar{\rho}_2) - \bar{B}\int_{\theta=-D_0}^{0}\theta\mathrm{e}^{\mathrm{i}\omega_0\theta}\bar{q}^{*\mathrm{T}}(0)[\mathrm{d}\eta(\theta)]q(0) \\
&= \bar{B}(\rho_1 + \bar{\rho}_2) + \bar{B}D_0\mathrm{e}^{-\mathrm{i}\omega_0 D_0}(\,\bar{\rho}_2 \quad 1\,)\begin{pmatrix}\gamma F' & 0 \\ F' & 0\end{pmatrix}\begin{pmatrix}1 \\ \rho_1\end{pmatrix} \\
&= \bar{B}[(\rho_1 + \bar{\rho}_2) + D_0\mathrm{e}^{-\mathrm{i}\omega_0 D_0}F'(\gamma\bar{\rho}_2 + 1)].
\end{aligned}$$

Letting

$$\bar{B} = [(\rho_1 + \bar{\rho}_2) + D_0\mathrm{e}^{-\mathrm{i}\omega_0 D_0}F'(\gamma\bar{\rho}_2 + 1)]^{-1},$$

we have

$$\langle q^*, q\rangle = 1.$$

Since

$$\begin{aligned}
\langle q^*, \bar{q}\rangle &= \bar{q}^{*\mathrm{T}}(0)\bar{q}(0) - \int_{\theta=-D_0}^{0}\int_{\xi=0}^{\theta}\bar{q}^{*\mathrm{T}}(\xi-\theta)[\mathrm{d}\eta(\theta)]\bar{q}(\xi)\mathrm{d}\xi \\
&= \bar{B}(\bar{\rho}_1 + \bar{\rho}_2) - \bar{B}\int_{\theta=-D_0}^{0}\int_{\xi=0}^{\theta}\bar{q}^{*\mathrm{T}}(0)\mathrm{e}^{-\mathrm{i}\omega_0(\xi-\theta)}[\mathrm{d}\eta(\theta)]\bar{q}(0)\mathrm{e}^{-\mathrm{i}\omega_0\xi}\mathrm{d}\xi \\
&= \bar{B}(\bar{\rho}_1 + \bar{\rho}_2) + \frac{\bar{B}}{2\mathrm{i}\omega_0}\int_{\theta=-D_0}^{0}\mathrm{e}^{\mathrm{i}\omega_0\theta}(\mathrm{e}^{-2\mathrm{i}\omega_0\theta} - 1)\bar{q}^{*\mathrm{T}}(0)[\mathrm{d}\eta(\theta)]\bar{q}(0) \\
&= \bar{B}(\bar{\rho}_1 + \bar{\rho}_2) + \frac{\bar{B}}{2\mathrm{i}\omega_0}(\mathrm{e}^{\mathrm{i}\omega_0 D_0} - \mathrm{e}^{-\mathrm{i}\omega_0 D_0})(\bar{\rho}_2 \ 1)\begin{pmatrix}\gamma F' & 0 \\ F' & 0\end{pmatrix}\begin{pmatrix}1 \\ \bar{\rho}_1\end{pmatrix} \\
&= \bar{B}\left[(\bar{\rho}_1 + \bar{\rho}_2) + \frac{\sin(\omega_0 D_0)}{\omega_0}F'(\gamma\bar{\rho}_2 + 1)\right]
\end{aligned}$$

and

$$\bar{\rho}_1 + \bar{\rho}_2 = \frac{\mathrm{i}\omega_0}{-\gamma\alpha + \mathrm{i}\gamma\omega_0} + \frac{\mathrm{i}\omega_0}{\gamma\alpha} = \frac{1}{\gamma}\left(\frac{\omega_0^2 - \mathrm{i}\alpha\omega_0}{\omega_0^2 + \alpha^2} + \frac{\mathrm{i}\omega_0}{\alpha}\right) = \frac{\omega_0^2}{\gamma\alpha}\left(\frac{\alpha + \mathrm{i}\omega_0}{\omega_0^2 + \alpha^2}\right),$$

from Eq. (5), we obtain

$$\frac{\sin(\omega_0 D_0)}{\omega_0} = \frac{a_2\omega_0^2}{a_1^2 + a_2^2\omega_0^2}.$$

So we have

$$\frac{\sin(\omega_0 D_0)}{\omega_0}F'(\gamma\bar{\rho}_2 + 1) = -\frac{\gamma F'\omega_0^2}{(\gamma\alpha F')^2 + (\gamma F'\omega_0)^2}F'\left(\frac{\mathrm{i}\omega_0}{\alpha} + 1\right) = -\frac{\omega_0^2}{\gamma(\alpha^2 + \omega_0^2)}\left(\frac{\mathrm{i}\omega_0}{\alpha} + 1\right).$$

Therefore

$$\langle q^*, \bar{q}\rangle = 0.$$

To this end the lemma is proved. □

Using the same notation as in Hassard et al. [13], we first compute the coordinates to describe the center manifold S at $\mu = 0$. Let y_t be the solution of Eq. (12) when $\mu = 0$. Define

$$z(t) = \langle q^*, y_t \rangle,$$
$$w(t,\theta) = y_t - zq - \bar{z}\bar{q} = y_t - 2\mathrm{Re}\{z(t)q(\theta)\}.$$

On the center manifold S, we have

$$w(t,\theta) = w(z,\bar{z},\theta),$$

where

$$w(z,\bar{z},\theta) = w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\bar{z} + w_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots, \tag{14}$$

where z and $\bar{z}$ are local coordinates for the center manifold in the direction of q^* and $\bar{q}^*$. For a solution $y_t \in$ S of Eq. (12), since $\mu = 0$, it satisfies

$$y_t = w(t,\theta) + 2\mathrm{Re}\{z(t)q(\theta)\}.$$

So we have

$$\dot{z}(t) = \langle q^*, \dot{y}_t \rangle = \mathrm{i}\omega_0 z(t) + \bar{q}^*(0)f_0(0, w(z,\bar{z},0) + 2\mathrm{Re}\{zq(0)\}) = \mathrm{i}\omega_0 z(t) + \bar{q}^*(0)f_0(z,\bar{z}).$$

Rewrite $\dot{z}(t)$ as

$$\dot{z}(t) = \mathrm{i}\omega_0 z(t) + g(z,\bar{z}), \tag{15}$$

where

$$g(z,\bar{z}) = \bar{q}^*(0)f_0(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots \tag{16}$$

Since

$$\dot{w} = \dot{y}_t - \dot{z}q - \dot{\bar{z}}\bar{q},$$

by Eqs. (15) and (12), we have

$$\dot{w} = \begin{cases} Aw - 2\mathrm{Re}\{g(z,\bar{z})q(\theta)\}, & \theta \in [-D_0, 0); \\ Aw - 2\mathrm{Re}\{g(z,\bar{z})q(\theta)\} + f_0, & \theta = 0. \end{cases} \tag{17}$$

Rewrite $\dot{w}$ as

$$\dot{w} = Aw + H(z,\bar{z},\theta), \tag{18}$$

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots \tag{19}$$

Substituting (14) and (15) into $\dot{w} = w_z\dot{z} + w_{\bar{z}}\dot{\bar{z}}$, we obtain

$$\dot{w} = \mathrm{i}\omega_0 w_{20}(\theta)z^2 - \mathrm{i}\omega_0 w_{02}(\theta)\bar{z}^2 + \cdots$$

Comparing the coefficients of the above equation with those of Eq. (18), we get

$$\begin{aligned} &(2\mathrm{i}\omega_0 - A)w_{20}(\theta) = H_{20}(\theta), \\ &Aw_{11}(\theta) = -H_{11}(\theta), \\ &(2\mathrm{i}\omega_0 + A)w_{02}(\theta) = -H_{02}(\theta). \end{aligned} \tag{20}$$

Our objective is to obtain the values of μ_2, τ_2, and β_2. To this end, firstly we need to compute the values of g_{20}, g_{11}, g_{02} and g_{21}. Since $y_t = w(t,\theta) + 2\mathrm{Re}\{z(t)q(\theta)\}$, we have

$$y_t = \begin{pmatrix} w^{(1)}(z,\bar{z},\theta) \\ w^{(2)}(z,\bar{z},\theta) \end{pmatrix} + z\begin{pmatrix} 1 \\ \rho_1 \end{pmatrix}\mathrm{e}^{\mathrm{i}\omega_0\theta} + \bar{z}\begin{pmatrix} 1 \\ \bar{\rho}_1 \end{pmatrix}\mathrm{e}^{-\mathrm{i}\omega_0\theta}.$$

Straightforward calculation yields

$$y_{1t}(-D_0) = w^{(1)}(z,\bar{z},-D_0) + zq_1(-D_0) + \bar{z}\bar{q}_1(-D_0)$$
$$= w_{20}^{(1)}(-D_0)\frac{z^2}{2} + w_{11}^{(1)}(-D_0)z\bar{z} + w_{02}^{(1)}(-D_0)\frac{\bar{z}^2}{2} + \cdots + z\mathrm{e}^{-\mathrm{i}\omega_0 D_0} + \bar{z}\mathrm{e}^{\mathrm{i}\omega_0 D_0}.$$

So we have

$$\varphi_1^2(-D_0) = z^2\mathrm{e}^{-\mathrm{i}2\omega_0 D_0} + 2z\bar{z} + \bar{z}^2\mathrm{e}^{\mathrm{i}2\omega_0 D_0} + (w_{20}^{(1)}(-D_0)\mathrm{e}^{\mathrm{i}\omega_0 D_0} + 2w_{11}^{(1)}(-D_0)\mathrm{e}^{-\mathrm{i}\omega_0 D_0})z^2\bar{z} + \cdots,$$
$$\varphi_1^3(-D_0) = 3z^2\bar{z}\mathrm{e}^{-\mathrm{i}\omega_0 D_0} + 3z\bar{z}^2\mathrm{e}^{\mathrm{i}\omega_0 D_0} + \cdots$$

The function f_0 can be expressed as

$$f_0(z,\bar{z}) = \left(\frac{F''}{2}\varphi_1^2(-D_0) + \frac{F^{(3)}}{3!}\varphi_1^3(-D_0) + \cdots\right)\begin{pmatrix}\gamma\\1\end{pmatrix}$$
$$= \left(\frac{F''}{2}[z^2\mathrm{e}^{-\mathrm{i}2\omega_0 D_0} + 2z\bar{z} + \bar{z}^2\mathrm{e}^{\mathrm{i}2\omega_0 D_0} + (w_{20}^{(1)}(-D_0)\mathrm{e}^{\mathrm{i}\omega_0 D_0} + 2w_{11}^{(1)}(-D_0)\mathrm{e}^{-\mathrm{i}\omega_0 D_0})z^2\bar{z} + \cdots]\right.$$
$$\left. + \frac{F^{(3)}}{3!}[3z^2\bar{z}\mathrm{e}^{-\mathrm{i}\omega_0 D_0} + 3z\bar{z}^2\mathrm{e}^{\mathrm{i}\omega_0 D_0} + \cdots] + \cdots\right)\begin{pmatrix}\gamma\\1\end{pmatrix}$$
$$= \left(F''\mathrm{e}^{-\mathrm{i}2\omega_0 D_0}\frac{z^2}{2} + F''z\bar{z} + F''\mathrm{e}^{\mathrm{i}2\omega_0 D_0}\frac{\bar{z}^2}{2} + [F''w_{20}^{(1)}(-D_0)\mathrm{e}^{\mathrm{i}\omega_0 D_0}\right.$$
$$\left. + 2F''w_{11}^{(1)}(-D_0)\mathrm{e}^{-\mathrm{i}\omega_0 D_0} + F^{(3)}\mathrm{e}^{-\mathrm{i}\omega_0 D_0}]\frac{z^2\bar{z}}{2} + \cdots\right)\begin{pmatrix}\gamma\\1\end{pmatrix}.$$

Since $\bar{q}^*(0) = \bar{B}(\,\bar{\rho}_2 \quad 1\,)^{\mathrm{T}}$, we have

$$g(z,\bar{z}) = \bar{q}^{*\mathrm{T}}(0)f_0(z,\bar{z}) = \bar{B}(\,\bar{\rho}_2 \quad 1\,)\left(F''\mathrm{e}^{-\mathrm{i}2\omega_0 D_0}\frac{z^2}{2} + F''z\bar{z} + F''\mathrm{e}^{\mathrm{i}2\omega_0 D_0}\frac{\bar{z}^2}{2}\right.$$
$$\left. + [F''w_{20}^{(1)}(-D_0)\mathrm{e}^{\mathrm{i}\omega_0 D_0} + 2F''w_{11}^{(1)}(-D_0)\mathrm{e}^{-\mathrm{i}\omega_0 D_0} + F^{(3)}\mathrm{e}^{-\mathrm{i}\omega_0 D_0}]\frac{z^2\bar{z}}{2} + \cdots\right)\begin{pmatrix}\gamma\\1\end{pmatrix}$$
$$= \bar{B}(\gamma\bar{\rho}_2 + 1)\left(F''\mathrm{e}^{-\mathrm{i}2\omega_0 D_0}\frac{z^2}{2} + F''z\bar{z} + F''\mathrm{e}^{\mathrm{i}2\omega_0 D_0}\frac{\bar{z}^2}{2} + [F''w_{20}^{(1)}(-D_0)\mathrm{e}^{\mathrm{i}\omega_0 D_0} + 2F''w_{11}^{(1)}(-D_0)\mathrm{e}^{-\mathrm{i}\omega_0 D_0}\right.$$
$$\left. + F^{(3)}\mathrm{e}^{-\mathrm{i}\omega_0 D_0}]\frac{z^2\bar{z}}{2} + \cdots\right).$$

Comparing the coefficients of the above equation with those in (16), we obtain

$$\begin{aligned}
g_{20} &= \bar{B}(\gamma\bar{\rho}_2 + 1)F''\mathrm{e}^{-\mathrm{i}2\omega_0 D_0},\\
g_{11} &= \bar{B}(\gamma\bar{\rho}_2 + 1)F'',\\
g_{02} &= \bar{B}(\gamma\bar{\rho}_2 + 1)F''\mathrm{e}^{\mathrm{i}2\omega_0 D_0},\\
g_{21} &= \bar{B}(\gamma\bar{\rho}_2 + 1)[F''w_{20}^{(1)}(-D_0)\mathrm{e}^{\mathrm{i}\omega_0 D_0} + 2F''w_{11}^{(1)}(-D_0)\mathrm{e}^{-\mathrm{i}\omega_0 D_0} + F^{(3)}\mathrm{e}^{-\mathrm{i}\omega_0 D_0}].
\end{aligned} \tag{21}$$

We still need to compute the values of $w_{20}(-D_0)$ and $w_{11}(-D_0)$ for the expression of g_{21}. By Eqs. (17) and (18), for $\theta \in [-D_0, 0)$, we have

$$H(z,\bar{z},\theta) = -2\mathrm{Re}\{g(z,\bar{z})q(\theta)\}$$
$$= -\left(g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots\right)q(\theta) - \left(\bar{g}_{20}\frac{\bar{z}^2}{2} + \bar{g}_{11}\bar{z}z + \bar{g}_{02}\frac{z^2}{2} + \bar{g}_{21}\frac{\bar{z}^2 z}{2} + \cdots\right)\bar{q}(\theta).$$

Comparing the coefficients of the above equation with those of (19), we obtain

$$\begin{aligned}
H_{20}(\theta) &= -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta),\\
H_{11}(\theta) &= -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta).
\end{aligned} \tag{22}$$

It follows from (20) that

$$\dot{w}_{20}(\theta) = 2\mathrm{i}\omega_0 w_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta).$$

Solving this equation we obtain

$$w_{20}(\theta) = \frac{\mathrm{i}g_{20}}{\omega_0}q(0)\mathrm{e}^{\mathrm{i}\omega_0\theta} + \frac{\mathrm{i}\bar{g}_{02}}{3\omega_0}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\omega_0\theta} + E_1\mathrm{e}^{\mathrm{i}2\omega_0\theta}. \tag{23}$$

By a similarly way, we get

$$w_{11}(\theta) = -\frac{\mathrm{i}g_{11}}{\omega_0}q(0)\mathrm{e}^{\mathrm{i}\omega_0\theta} + \frac{\mathrm{i}\bar{g}_{11}}{\omega_0}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\omega_0\theta} + E_2, \tag{24}$$

where E_1 and E_2 can be determined by setting $\theta = 0$ in H. It is evident that

$$H(z,\bar{z},0) = -2\mathrm{Re}\{g(z,\bar{z})q(0)\} + f_0.$$

Comparing the coefficients of the above equation with those of Eq. (19) yields

$$\begin{aligned} H_{20}(0) &= -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + F''\mathrm{e}^{-\mathrm{i}2\omega_0D_0}\begin{pmatrix}\gamma\\1\end{pmatrix},\\ H_{11}(0) &= -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + F''\begin{pmatrix}\gamma\\1\end{pmatrix}. \end{aligned} \tag{25}$$

By the definition of A and Eq. (20), we can obtain

$$\begin{aligned} \int_{-D_0}^{0}\mathrm{d}\eta(\theta)w_{20}(\theta) &= \mathrm{i}2\omega_0w_{20}(\theta) - H_{20}(0),\\ \int_{-D_0}^{0}\mathrm{d}\eta(\theta)w_{11}(\theta) &= -H_{11}(0). \end{aligned} \tag{26}$$

Since $q(\theta)$ is the eigenvector of A corresponding to $\mathrm{i}\omega_0$, it satisfies

$$\int_{-D_0}^{0}\mathrm{d}\eta(\theta)q(\theta) = \mathrm{i}\omega_0q(\theta). \tag{27}$$

Substituting (23) and (25) into the first equation of (26), and applying (27), we obtain

$$\left(\mathrm{i}2\omega_0 - \int_{-D_0}^{0}\mathrm{e}^{\mathrm{i}2\omega_0\theta}\mathrm{d}\eta(\theta)\right)E_1 = F''\mathrm{e}^{-\mathrm{i}2\omega_0D_0}\begin{pmatrix}\gamma\\1\end{pmatrix},$$

that is

$$\begin{pmatrix}\mathrm{i}2\omega_0 - \gamma F'\mathrm{e}^{-\mathrm{i}2\omega_0D_0} & -\gamma\alpha\\ -F'\mathrm{e}^{-\mathrm{i}2\omega_0D_0} & \mathrm{i}2\omega_0\end{pmatrix}\begin{pmatrix}E_1^{(1)}\\E_1^{(2)}\end{pmatrix} = F''\mathrm{e}^{-\mathrm{i}2\omega_0D_0}\begin{pmatrix}\gamma\\1\end{pmatrix}.$$

Finally, we get

$$\begin{aligned} E_1^{(1)} &= -\frac{\gamma\alpha + \mathrm{i}2\gamma\omega_0}{\mathrm{i}2\gamma\omega_0F'\mathrm{e}^{-\mathrm{i}2\omega_0D_0} + \gamma\alpha F'\mathrm{e}^{-\mathrm{i}2\omega_0D_0} + 4\omega_0^2}F''\mathrm{e}^{-\mathrm{i}2\omega_0D_0},\\ E_1^{(2)} &= -\frac{\mathrm{i}2\omega_0}{\mathrm{i}2\gamma\omega_0F'\mathrm{e}^{-\mathrm{i}2\omega_0D_0} + \gamma\alpha F'\mathrm{e}^{-\mathrm{i}2\omega_0D_0} + 4\omega_0^2}F''\mathrm{e}^{-\mathrm{i}2\omega_0D_0}. \end{aligned} \tag{28}$$

Similarly, we can get

$$\begin{aligned} E_2^{(1)} &= -\frac{F''}{F'},\\ E_2^{(2)} &= 0. \end{aligned} \tag{29}$$

Since $q(0) = (1 \quad \rho_1)^{\mathrm{T}}$, we have

$$\begin{aligned} w_{20}^{(1)}(-D_0) &= \frac{\mathrm{i}g_{20}}{\omega_0}\mathrm{e}^{-\mathrm{i}\omega_0D_0} + \frac{\mathrm{i}\bar{g}_{02}}{3\omega_0}\mathrm{e}^{\mathrm{i}\omega_0D_0} + E_1^{(1)}\mathrm{e}^{-\mathrm{i}2\omega_0D_0},\\ w_{11}^{(1)}(-D_0) &= -\frac{\mathrm{i}g_{11}}{\omega_0}\mathrm{e}^{-\mathrm{i}\omega_0D_0} + \frac{\mathrm{i}\bar{g}_{11}}{\omega_0}\mathrm{e}^{\mathrm{i}\omega_0D_0} + E_2^{(1)}. \end{aligned} \tag{30}$$

Based on the foregoing analysis, we can see that each g_{ij} is determined by the parameters and delays in Eq. (1). Thus, we have the formulas to compute the following quantities:

$$\begin{aligned}
C_1(0) &= \frac{\mathrm{i}}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, \\
\mu_2 &= -\frac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\{\lambda'(0)\}}, \\
\tau_2 &= -\frac{\mathrm{Im}\{C_1(0)\} + \mu_2 \mathrm{Im}\{\lambda'(0)\}}{\omega_0}, \\
\beta_2 &= 2\mathrm{Re}\{C_1(0)\}.
\end{aligned} \tag{31}$$

Now, we can give the main results of this section.

Theorem 2. *When $D_0 = \frac{1}{\omega_0}\arctan(\frac{\omega_0}{\alpha})$, the system (1) exhibits the Hopf bifurcation. The stability of the periodic solution of the Hopf bifurcation is determined by the formulas (31). Moreover, the following conclusions are true:*

(1) *μ_2 determines the direction of the Hopf bifurcation. If $\mu_2 > 0 (< 0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $D > D_0 (< D_0)$.*
(2) *β_2 determines the stability of the bifurcating periodic solutions. If $\beta_2 < 0 (>0)$, then the bifurcating periodic solutions are orbitally stable (unstable).*
(3) *τ_2 determines the period T of the bifurcating periodic solutions.*

$$T = \frac{2\pi}{\omega_0}(1 + \tau_2\varepsilon^2 + O(\varepsilon^4)),$$

where $\varepsilon^2 = \mu/\mu_2 + O(\mu^2)$.

4. An example

Supposing that the utility function of the source is set to logarithm function, Low et al. have shown that REM can achieve both high utilization and negligible loss and delay by the simulator ns-2 [4,5]. Let $U(x)$ be the utility function. Since

$$p(t) = U'(x(t)),$$

we can obtain

$$x(t) = v/p(x(t)),$$

where v is the target value of the marked packets received by the source. We assume that the capacity $c = 1$, the target of the buffer $b_0 = 10$, the control gains $\gamma = 0.05$, $\alpha = 0.005$, and $v = 0.1$. By Eq. (2), we have

$$b^* = 10, \quad x^* = 1, \quad p^* = 0.1.$$

Based on Eqs. (6) and (7), we obtain

$$\omega_0 = 0.5000, \quad D_0 = 3.1214.$$

It follows from Eq. (31) in last section that

$$C_1(0) = -25.7205 + \mathrm{i}9.5182,$$

$$\mu_2 = 356.6686,$$

$$\tau_2 = -99.8248,$$

$$\beta_2 = -51.4411.$$

Applying Theorem 2, we know that the system exhibits the Hopf bifurcation when $D_0 = 3.1214$. The Hopf bifurcation is supercritical and the bifurcating periodic solutions exist when $D > D_0$. The periodic solutions are orbitally stable and the period of the bifurcating periodic solutions is

$$T = 12.5655(1 - 0.2799\mu + \mathrm{O}(\mu^2)).$$

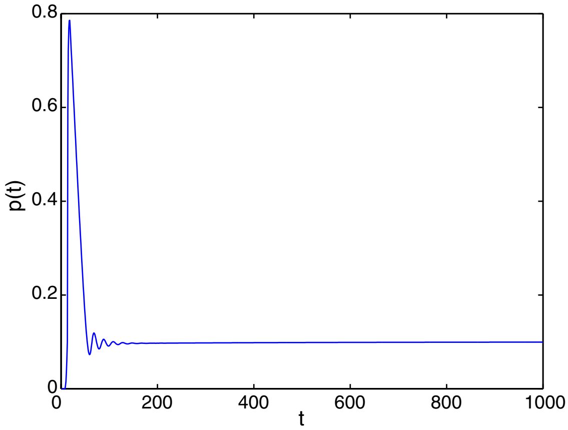

Fig. 1. State plot of $p(t)$ with $D = 2$.

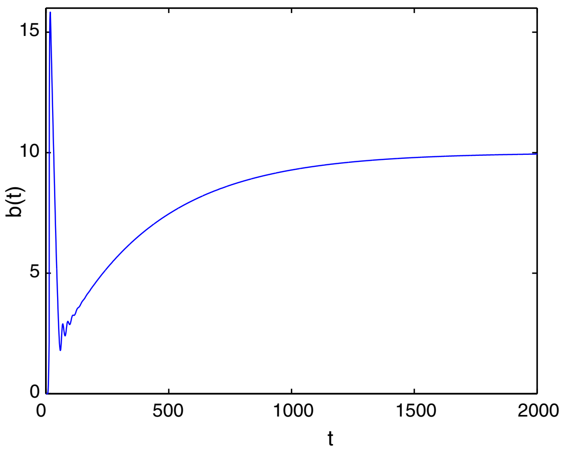

Fig. 2. State plot of $b(t)$ with $D = 2$.

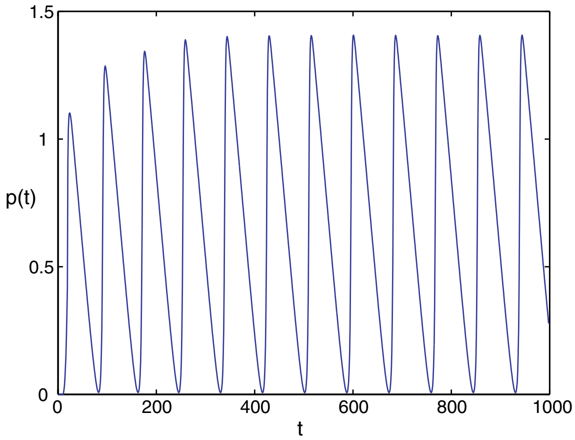

Fig. 3. State plot of $p(t)$ with $D = 4$.

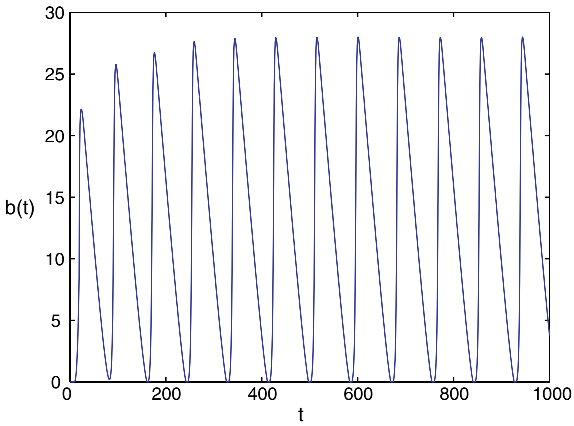

Fig. 4. State plot of $b(t)$ with $D = 4$.

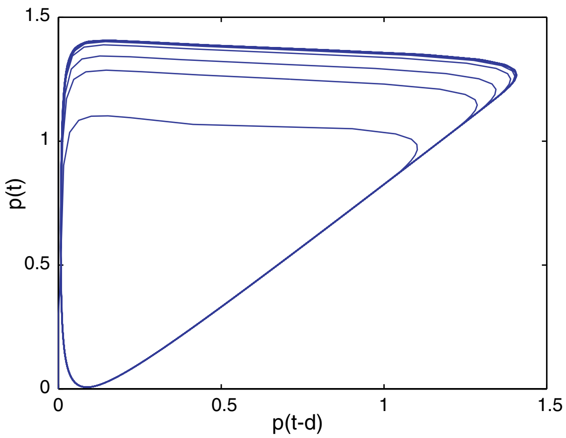

Fig. 5. Phase plot of $p(t)$ with $D = 4$.

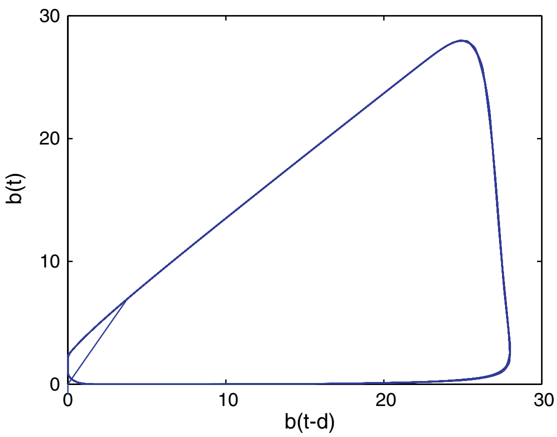

Fig. 6. Phase plot of $p(t)$ with $D = 4$.

Supposing the communication delay $D = 2$ (less than the critical value D_0), we know that the system is asymptotically stable (the computer simulations are shown in Figs. 1 and 2).

If we modify the delay D passing through the critical value D_0, a Hopf bifurcation occurs, i.e. there are periodic solutions bifurcating out from D_0. These results are illustrated by computer simulations as shown in Figs. 3 and 4. The phase plots are shown in Figs. 5 and 6.

5. Conclusion

In this paper, we have studied a kind of AQM algorithm, the REM algorithm, for the Internet congestion control. Sufficient conditions on the stability of the system have been obtained by analyzing the corresponding characteristic equation. By choosing the communication delay as a bifurcation parameter, we have shown that a Hopf bifurcation occurs in the delayed REM scheme. The direction of the Hopf bifurcation and the stability of bifurcating periodic orbits have been studied by using the center manifold theorem and the normal form theory. The correctness of the theoretical results have been verified by computer simulations.

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