Stochastic Stability Under Fair Bandwidth Allocation: General File Size Distribution

Mung Chiang, Devavrat Shah, and Ao Tang

***Abstract*— We prove the stochastic stability of resource allocation under Network Utility Maximization (NUM) under general arrival process and file size distribution with bounded support, for α-fair utilities with α sufficiently small and possibly different for different sources' utility functions. In addition, our results imply that the system operating under α-fair utility is $1/(1+\alpha)$-approximate stable for any $\alpha \in (0, \infty)$ for any file size distribution with bounded support. Our results are in contrast to the recent stability result of Bramson (2005) for max-min fair (i.e. $\alpha = \infty$) under general arrival process and file size distribution, and that of Massoulie (2006) for proportional fair (i.e. $\alpha = 1$) under Poisson arrival process and phase-type distributions. We obtain our results by developing an appropriate Lyapunov function for the fluid model of Gromoll and Williams (2006)[1]**

I. Introduction

In 1998, Kelly, Maullo, and Tan [16] identified the current Internet congestion control protocol with an algorithm that allocates rates to flows according to certain 'fairness criteria' reflected through concave utility functions, which are maximized under linear capacity constraints. An extensive amount of research since then has shown many applications of this approach, from reverse-engineering of all major types of TCP congestion control protocols in use today to development of substantially improved new protocols. In [22], [28], an interested reader can find detailed surveys on the philosophy of viewing a resource allocation or congestion control algorithm as implicitly solving a global Network Utility Maximization (NUM) problem. In particular, such optimization problems have been studied as 'monotropic programming' [26] for a long time, and admit a simple, iterative, and distributed solution based on dual decomposition. Over the last several years, this line of work has further evolved to the following view of 'Layering as Optimization Decomposition': the entire protocol stack of network architecture can be thought of as optimizing a generalized network utility function over a constraint set of various types of variables, with different decomposition schemes corresponding to different layering architectural alternatives. Under a particular decomposition, the decomposed subproblems correspond to the functional modules (i.e., layers), and the interfaces among the layers are represented by some specific function of the primal or dual variables. As surveyed in [6], many researchers have contributed to this research area.

However, many results in the area adopt a deterministic NUM formulation. In reality, flows arrive to the network with finite workloads and depart after finishing the work. The service rates are determined by the solution to the NUM problem, which in turn takes in the number of flows as an argument. The key property of stochastic stability has been extensively studied since 1999. In [27], Robert and Massoulie introduced a stochastic dynamic model for Internet congestion control where flows with different service requirement (or file size when flow requests are 'file transfers') arrive, the rate allocation is done according to appropriate NUM, and flows depart on completion on their service. Subsequent to this work, de Veciana, Konstantopoulos and Lee [10], as well as Bonald and Massoulie [2], studied the stability property of the above introduced model under the assumption that arrival process has Poisson distribution while service requirement of flows have exponential distribution. In [10], stability of max-min fair and proportional fairness was established, while in [2], the stability of all weighted α-fair policies, $\alpha \in (0, \infty)$ was established. These results assumed that the rate allocation according the appropriate optimization is done instantaneously. This is called the 'time-scale separation', i.e., the time scale at which rate allocation algorithm operates is extremely fast compared to the time scale of the system dynamics. Lin and Shroff [20], as well as Srikant [29], established stability without time-scale separation assumption for α-fair policies for $\alpha \geq 1$ under the Poisson and exponential distributional assumptions. Natural generalizations of these results to other convex constraint sets were also obtained [32], [21].

While assuming a Markov traffic model (Poisson arrival with exponential file size distribution) leads to analytic tractability, it is widely recognized that file sizes in the Internet or wireless networks do *not* follow the exponential distribution. In this paper, we are interested in answering the question of whether the network is stable, under α-fair rate allocation, for general distributional assumption on arrival process and service requirement of the flows. Here, stability means that the departure rate is the same as the arrival rate, i.e., rate stability, or fluid stability. This question has been of great recent interest as a positive answer will provide justification for using NUM and its generalizations for network resource allocation and architecture design. A

Author names appear in the alphabetical order of their last names.
M. Chiang is with Department of Electrical Engineering at the Princeton University chiangm@princeton.edu
D. Shah is with Department of Electrical Engineering and Computer Science at MIT devavrat@mit.edu
A. Tang is with Division of Engineering and Applied Science at Caltech aotang@caltech.edu

[1]To be precise, our fluid model scaling is different so as to accommodate the case of heterogeneous utility functions of different sources. This scaling allows for possibility of larger class of utility functions as well. However, the justification is the same as that of Gromoll and Williams[14]. Hence, we call it the fluid model of [14].

stable system essentially means that the capacities that can be utilized in a deterministic NUM can also be utilized in the stochastic setting.

The following is a brief summary of what is currently known about this question (to the best of authors' knowledge based on the available preprints and personal communication). Bramson [5] has established stability for max-min fair (corresponding to $\alpha = \infty$) rate allocation under general arrival and file size distribution, and Massoulie [23] has established stability for proportional fair (corresponding to $\alpha = 1$) rate allocation for Poisson arrival and phase-type service distribution. The result of [23] is established by (a) justifying fluid model for system with exponential and Poisson assumption with routing, (b) establishing stability of this fluid model, and (c) using the known observation that network with phase type distribution for service requirement can be mapped to network with exponential-Poisson assumption and routing. We also make note of the following two results. Lakshmikantha, Beck and Srikant [19] established stability of Proportional fairness for a two resource linear network and 2×2 grid network for Poisson arrival and phase-type distribution of service requirement; Kelly and Williams [17] had formulated a proper fluid model for exponential service requirement to study the 'invariant states' as an intermediate step for obtaining diffusion approximation for all $\alpha \in (0, \infty)$.

Recently, Gromoll and Williams [14] have established fluid model for α-fair rate allocation, $\alpha \in (0, \infty)$, under general distributional condition on arrival process and service distribution. This is a very important step in the process of establishing stability via the means of fluid models. Using this fluid model, they have obtained a characterization of 'invariant states'. This led to stability of network under α-fair allocation, $\alpha \in (0, \infty)$, when the network topology is a tree.

We will establish the approximate stability of any α-fair rate allocation for any network topology under general distribution for $\alpha \in (0, \infty)$. We prove that any network with α-fair rate allocation is $1/(1+\alpha)$-approximate stable[2] under general distribution conditions. In a stronger characterization, we prove that the system is stable for a continuum of sufficiently small and strictly positive α_i, possibly different α_i for each source i. We will crucially use the fluid model established in [14] to obtain our results.

The paper is organized as follows. In Section II we present notations, technical preliminaries, system description, and stochastic model. In Section III we present the fluid model scaling and formal statement establishing relation between fluid model solutions and the stochastic system. The fluid model scaling presented in the paper is different from that used in [14] or [23] as it allows for heterogeneous utility functions for different sources, with possibly utilities coming from a larger class of utility functions compared to that in [14]. In Section IV, we present the main result of this paper (Corollary 7 and Theorem 6) establishing $1/(1+\alpha)$-approximate stability of network operating under α-fair rate allocation and general distributional conditions. This also implies the stability of network for a range of sufficiently small α. The stability is established by use of a new Lyapunov function, which is inspired by known Lyapunov functions in this research literature. However, as reader will notice, in contrast to the Markov arrival model, it is substantially more challenging to work with the fluid model for general distribution due to limited amount of information about fluid dynamics. We present some simple extensions and limitations of our results along with a discussion on future directions in Section V.

Even though our fluid model scaling is different, its justification is similar to that in [14]. An interested reader can find missing details in the longer version of this paper [7].

[2]Definition of $1/(1+\alpha)$-approximate stability will be made clear in Corollary 7. Roughly speaking, it means $100/(1+\alpha)$ % utilization of network's resource.

II. Setup

A. Notation and Technical Preliminaries

Let the natural number set be $\mathbb{N} = \{1, 2, \dots\}$, and the real number set be $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\} = [0, \infty)$. Let $\mathbb{R}^d$ be d-dimensional Euclidian space; similarly $\mathbb{N}^d$ and $\mathbb{R}^d_+$. Let $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. Let identity function be denoted by χ, i.e. $\chi(x) = x$ for all $x \in \mathbb{R}_+$. Let unit function be denoted by $\mathbf{1}$, i.e. $\mathbf{1}(x) = 1$ for all $x \in \mathbb{R}_+$. For vectors $\mathbf{u} = (u_1, \dots, u_\mathbf{I})$ and $\mathbf{v} = (v_1, \dots, v_\mathbf{I})$, let $\mathbf{u} \circ \mathbf{v}$ denote component-wise multiplication $(u_1 v_1, \dots, u_\mathbf{I} v_\mathbf{I})$.

For a real-valued function defined on $\mathbb{R}_+$, say $f : \mathbb{R}_+ \to \mathbb{R}$, its sup-norm is defined as $\|f\|_\infty = \sup_{x \in \mathbb{R}_+} |f(x)|$. Similarly, for $f : [0, T] \to \mathbb{R}$ define $\|f\|_T = \sup_{x \in [0,T]} |f(x)|$. Let f' denote the derivative of f, if exists. For any function f, let $f(\cdot - s), s > 0$, be its shifted copy by s, with the understanding that $f(x - s) = 0$ for all $x < s$.

Let $\mathbf{C}_b(\mathbb{R}_+)$ denote the set of bounded continuous functions defined on $\mathbb{R}_+$, $\mathbf{C}^1(\mathbb{R}_+)$ denote the set of once continuously differentiable functions and $\mathbf{C}^1_b(\mathbb{R}_+)$ denote the set of $f \in \mathbf{C}^1(\mathbb{R}_+)$ that have both f, f' bounded on $\mathbb{R}_+$. Define, $\mathcal{C} = \{f \in \mathbf{C}^1_b(\mathbb{R}_+) : f(0) = 0, f'(0) = 0\}$ and $\mathcal{C}_c = \{f \in \mathcal{C} : \ f \text{ has compact support}\}$.

Let $\mathbf{M}$ be set of finite non-negative measures (not necessarily probability measures) on $\mathbb{R}_+$. Let it be endowed with the topology induced by weak convergence: $\zeta^k \xrightarrow{\mathbf{w}} \zeta$ in $\mathbf{M}$ if and only if $\langle f, \zeta^k \rangle \to \langle f, \zeta \rangle$ for all $f \in \mathbf{C}_b(\mathbb{R}_+)$, where we have used notation[3] that, for $\zeta \in \mathbf{M}$,

$$\langle f, \zeta \rangle = \int_{\mathbb{R}_+} f d\zeta.$$

This topology is induced by the Prohorov's metric defined as follows: for $\zeta, \xi \in \mathbf{M}$, define

$$\begin{aligned} \mathbf{d}[\zeta, \xi] = \inf\{\varepsilon > 0 : \zeta(B) \leq \xi(B^\varepsilon) + \varepsilon, \text{ and} \\ \xi(B) \leq \zeta(B^\varepsilon) + \varepsilon, \text{for all closed } B \subset \mathbb{R}_+\}, \end{aligned} \tag{1}$$

[3]The notation $\langle f, \zeta \rangle$ for $\zeta = (\zeta_1, \dots, \zeta_d) \in \mathbf{M}^d$ will naturally mean $(\langle f, \zeta_1 \rangle, \dots, \langle f, \zeta_d \rangle)$.

where $B^\varepsilon = \{x \in \mathbb{R}_+ : \inf_{y\in B} |x-y| < \varepsilon\}$. For product space $\mathbf{M}^\mathbf{I}$ for any $\mathbf{I} \in \mathbb{N}$, define metric $\mathbf{d}_\mathbf{I}$ as follows: for $\zeta = (\zeta_1, \ldots, \zeta_\mathbf{I}), \xi = (\xi_1, \ldots, \xi_\mathbf{I}) \in \mathbf{M}^\mathbf{I}$,

$$\mathbf{d}_\mathbf{I}[\zeta, \xi] = \max_{1\le i\le \mathbf{I}} \mathbf{d}[\zeta_i, \xi_i].$$

It is well known that the metric space $\mathbf{M}^\mathbf{I}$ thus defined is a complete and separable, i.e. Polish space.

Let $\mathbf{D}([0,T], \mathbf{M}^\mathbf{I})$ denote the set of functions from $[0,T]$ to $\mathbf{M}^\mathbf{I}$ that are right continuous with left limits, also known as *cadlag* functions. In this paper, the domain $[0,T]$ will be time and hence use of 'time' should not confuse the reader. We will endow $\mathbf{D}([0,T], \mathbf{M}^\mathbf{I})$ with Skorohod's J_1-topology. Our interest will be in convergence of probability distributions on $\mathbf{D}([0,T], \mathbf{M}^\mathbf{I})$, for finite (time-interval) T. For this, we will be interested in an appropriate metric on $\mathbf{D}([0,T], \mathbf{M}^\mathbf{I})$ defined next.

Let Φ be set of nondecreasing function $\varphi : [0,T] \to [0,T]$ with $\varphi(0) = 0, \varphi(T) = T$. Define $\|\varphi\|^o = \sup_{0\le s<t\le T} \left|\log \frac{\varphi(t)-\varphi(s)}{t-s}\right|$. Let $\Phi_b = \{\varphi \in \Phi : \|\varphi\|^o < \infty\}$. Now, for any $\zeta, \xi \in \mathbf{D}([0,T], \mathbf{M}^\mathbf{I})$, the distance between them is defined as

$$d^o(\zeta, \xi) = \inf_{\varphi\in\Phi_b} \left\{ \|\varphi\|^o \vee \left(\sup_{0\le t\le T} \mathbf{d}_\mathbf{I}(\zeta(t), \xi(\varphi(t))) \right) \right\}.$$

The space $\mathbf{D}([0,T], \mathbf{M}^\mathbf{I})$ endowed with the above metric is complete and separable, i.e. Polish. Before we characterize the relatively compact sets in $\mathbf{D}([0,T], \mathbf{M}^\mathbf{I})$, we define the modulus of continuity for $\zeta \in \mathbf{D}([0,T], \mathbf{M}^\mathbf{I})$. Consider any $\delta \in (0,1)$ and any sequence $\{t_i\}$ of some $v \le 2T/\delta$ points, such that $0 = t_0 < t_1 < \cdots < t_v = T$ and $\min_i t_i - t_{i-1} > \delta$. Call the set of all such sequences as $\mathbf{T}_\delta$. Then, the modulus of continuity of ζ with δ precision is

$$\mathbf{w}'_T(\zeta, \delta) = \inf_{\{t_i\}\in\mathbf{T}_\delta} \max_i \sup_{s,t\in[t_{i-1},t_i]} \mathbf{d}_\mathbf{I}[\zeta(s), \zeta(t)].$$

In $\mathbf{D}([0,T], \mathbf{M}^\mathbf{I})$, a set A is relatively compact if the following holds: (1) there exists a compact set $\mathbf{K} \subset \mathbf{M}^\mathbf{I}$ such that for any $\zeta \in A$, $\zeta(t) \in \mathbf{K}$ for all $t \in [0,T]$, and (2) $\lim_{\delta\to 0} \sup_{\zeta\in A} \mathbf{w}'_T(\zeta, \delta) = 0$. This characterization of relatively compact set suggests the following criteria for proving tightness of a sequence of probability measures $\mathbb{P}_n, n \in \mathbb{N}$, on $D([0,T], \mathbf{M}^\mathbf{I})$ as follows: the sequence of probability measures $\mathbb{P}_n, n \in \mathbb{N}$ is tight if (1) for any $\varepsilon > 0$ there exists a compact set $\mathbf{K}_\varepsilon \subset \mathbf{M}^\mathbf{I}$ such that $\liminf_n \mathbb{P}_n(\zeta(t) \in \mathbf{K}_\varepsilon, \ \forall\, t \in [0,T]) \ge 1-\varepsilon$, and (2) for any $\varepsilon > 0$, $\lim_{\delta\to 0} \limsup_n \mathbb{P}_n(\{\zeta : \mathbf{w}'_T(\zeta, \delta) \ge \varepsilon\}) = 0$. This characterization of tightness of probability measures is used crucially in fluid model justification. We refer an interested reader to the book by Billingsley [1] to find details about the facts stated above.

B. Networks with Rate Allocation By NUM

We consider a connected network $G = (\mathcal{V}, \mathcal{J}, C, \mathcal{I})$, where $\mathcal{V}$ is the set of all vertices, $\mathcal{J}$ is the set of $\mathbf{J}$ links, $C = (C_j)_{1\le j\le \mathbf{J}}$ denote the capacity vector of the links, and $\mathcal{I}$ is the set of $\mathbf{I}$ routes. Let A be $\mathbf{J} \times \mathbf{I}$ routing incidence matrix, with $A_{ji} = 1$ if route i passes through link j and 0 otherwise.

In a network, multiple flows can be active on the same route. Further, flows of different routes (or types) can be sharing a link. The links have limited capacity. Hence, network need to assign the rates to the flows passing through it. In this paper, we are interested in bandwidth sharing policies in which each flow of the same type gets the same bandwidth allocated. Let λ_i be net bandwidth allocated to flows on route i. Since the links have limited capacity, we immediately have the following requirement:

$$A\lambda \le C.$$

The set of all $\lambda = (\lambda_1, \ldots, \lambda_\mathbf{I}) \in \mathbb{R}_+^\mathbf{I}$ satisfying the above inequality are called feasible bandwidth allocation.

In this paper, we are interested in the bandwidth allocation policies that maximizes certain network utility. Equivalently, bandwidth allocation corresponds to a solution of an appropriate Network Utility Maximization (NUM) problem. Let $\mathcal{U}_i(x)$ be utility of a flow of type i when it is allocated rate x. If there are z_i flows of type i and each one is allocated rate x_i, then the net bandwidth allocated to flows of type i is $\lambda_i = x_i z_i$. In this paper, we are primarily interested in the α-fair utility function, introduced by Mo and Walrand [24], which is commonly used in studying NUM type network resource allocation. For any $\alpha \in (0,\infty)$, define [4]

$$\varphi^\alpha(x) = \begin{cases} \frac{x^{1-\alpha}}{1-\alpha} & \text{for } \alpha \in (0,\infty)\backslash\{1\} \\ \log x & \text{for } \alpha = 1. \end{cases}$$

Then, under unweighted α-fair allocation, the utility of each flow i is $\mathcal{U}_i = \varphi^{\alpha_i}, \alpha_i \in (0,\infty)$. In the weighted α-fair allocation, $\mathcal{U}_i = \kappa_i \varphi^{\alpha_i}$, with κ_i some positive weights (constants). In this paper, for simplicity we will assume that $\kappa_i = 1$ for all i. However, as it will be clear to the reader that the results of this paper hold true for any choice of $\kappa_i > 0$.

Now the bandwidth or rate allocation happens according to an optimization problem which uses number of flows as argument. Let $z = (z_1, \ldots, z_\mathbf{I})$ be vector of number of flows. Then, each flow of type i is allocated rate $\mathbf{x}_i(z)$, $1 \le i \le \mathbf{I}$, where $\mathbf{x}(z) = (\mathbf{x}_1(z), \ldots, \mathbf{x}_\mathbf{I}(z))$ is a solution to the following optimization problem over $x \ge 0$:

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{\mathbf{I}} \mathcal{U}_i(x_i) z_i \\ \text{subject to} \quad & Ax \circ z \le C, \\ & x_i = 0 \ \text{ if } \ z_i = 0, \quad \text{for all } i \le \mathbf{I}, \end{aligned} \tag{2}$$

where $x \circ z = (x_1 z_1, \ldots, x_\mathbf{I} z_\mathbf{I})$ is the vector of net bandwidth allocated to flows. In this paper the utilities $\mathcal{U}_i(\cdot)$ are strictly concave on $(0,\infty)$. This will imply the uniqueness of the solution of the above optimization problem from standard arguments. Thus, $\mathbf{x}(z)$ can be viewed as a function from $\mathbb{R}_+^\mathbf{I}$ to $\mathbb{R}_+^\mathbf{I}$. We will assume that choice of utilities is such that $x(\cdot)$ satisfies the following assumptions.

[4]The general definition of α-fair utility allows for $\alpha = 0$, but such linear utility function leads to potential starvation and is not considered here.

Assumption 1: For each $i \leq \mathbf{I}$, $\mathbf{x}_i(z)$ is a continuous function on $\{z \in \mathbb{R}_+^{\mathbf{I}} : z_i > 0\}$. Further, if $z_i > 0$ then $\mathbf{x}_i(z) > 0$.

Kelly and Williams [17] showed that Assumption 1 holds when for all $i \leq \mathbf{I}$, the utility functions are the same and $\mathcal{U}_i = \varphi^\alpha$ for some $\alpha \in (0, \infty)$ for all $i \leq \mathbf{I}$. This assumption was verified by Ye, Qu and Yuan [33] as well. Next, we establish that Assumption 1 is satisfied even when the utilities of flow types are α fair with different α for different flow types.

Lemma 1: The Assumption 1 is satisfied when $\mathcal{U}_i = \varphi^{\alpha_i}$ with $\alpha_i \in (0, 1)$ for all $i \leq \mathbf{I}$.

Finally, define the vector of bandwidth allocated to flows, when vector of flows is z, as

$$\Lambda(z) = \mathbf{x}(z) \circ z = (\mathbf{x}_1(z)z_1, \ldots, \mathbf{x}_{\mathbf{I}}(z)z_{\mathbf{I}}).$$

We note the following obvious but crucial property of rate-allocation function $\mathbf{x}(\cdot) : \mathbb{R}_+^{\mathbf{I}} \to \mathbb{R}_+^{\mathbf{I}}$.

Lemma 2: For any $z \in \mathbb{R}_+^{\mathbf{I}}$ such that $z_i \geq \varepsilon$, $\mathbf{x}_i(z) \leq \|C\|/\varepsilon$.

C. Network Dynamics and Stochastic Model

Let $t \in \mathbb{R}_+$ denote the time index. Let $Z(t) = (Z_1(t), \ldots, Z_{\mathbf{I}}(t))$ denote the vector of the numbers of flows at time t. Let $E(t) = (E_1(t), \ldots, E_{\mathbf{I}}(t))$ be vector of cumulative number of arrivals of flows to the network in $[0, t]$ with $E(0) = \mathbf{0}$. Let U_{ik}, $k \geq 0$, denote the arrival time of k^{th} flow of type i with $U_{i0} = 0$. Each flow arrives with service requirement (or file-size). Let V_{ik} denote the service requirement of k^{th} flow of type i. Denote $V_i = (V_{ik}, k \geq 1)$ and $V = (V_1, \ldots, V_{\mathbf{I}})$. The system is assumed to start empty[5] at time $t = 0$.

Given the bandwidth allocation rule, the dynamics of the whole system can be obtained from the starting condition, arrival process, and service requirement process. We assume that the arrival process and service requirement process are defined on a common probability space, say $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{E}$ denoting the expectation. To this end, we assume that arrival process is such that inter-arrival times for flow $i \leq \mathbf{I}$, i.e. $U_{ik} - U_{i(k-1)}, k \geq 1$ are independent and identically distributed (i.i.d.) with $\nu_i^{-1} = \mathbb{E}[U_{i1} - U_{i0}] = \mathbb{E}[U_{i1}] \in (0, \infty)$. The service requirements for flow $i \leq \mathbf{I}$, $\{V_{ik}\}$ also form an i.i.d. sequence with density of distribution ϑ_i such that $\langle \mathbf{1}_{\{0\}}, \vartheta_i \rangle = 0$. Let the average service requirement be $\langle \chi, \vartheta_i \rangle = \mu_i^{-1} \in (0, \infty)$. The traffic intensity is defined as $\rho_i = \nu_i/\mu_i$. We assume that system is underloaded, that is,

$$A\rho \quad < \quad C. \tag{3}$$

The above condition is necessary for stability: the system can become unstable otherwise. We note that we have assumed the arrival and service processes to be i.i.d. just for simplicity. The only requirement is the existence of functional law of large numbers. As long as it is true, the fluid model (based on the proof in [14]) is justified and result of this paper holds true.

[5] Instead of empty, starting condition can be anything that is *not too bad*. Usually, such starting conditions are handled in a standard manner and we refer an interested reader to see [14].

Now, we describe system dynamics that will lead to the definition of a succinct system descriptor. Given the vector of number of flows in the system at time t, $Z(t)$, the rate allocation happens according to mapping $\mathbf{x}(Z(t))$. Define $S_i(t)$ to be the total amount of service allocated to a flows of type i in $[0, t]$. That is,

$$S_i(t) \quad = \quad \int_0^t \mathbf{x}_i(Z(\tau))d\tau. \tag{4}$$

Also define $S_i(t, t + \tau) = S_i(t + \tau) - S_i(t)$ for $\tau \in \mathbb{R}_+$. Finally, let $V_{ik}(t)$ be the remaining amount of service of k^{th} flow of type i at time t. Then,

$$V_{ik}(t) = (V_{ik} - S_i(U_{ik}, t)).$$

Let $W_i(t) = \sum_{k=1}^{E_i(t)} V_{ik}^+(t)$ be the total amount of unfinished work in the system at time t, where $x^+ = x\mathbf{1}_{\{x>0\}}$.

All of the above system information can be compactly represented via measure on $\mathbb{R}_+$ as follow: define $\mathcal{Z}(t) = (\mathcal{Z}_1(t), \ldots, \mathcal{Z}_{\mathbf{I}}(t)) \in \mathbf{M}^{\mathbf{I}}$ as

$$\mathcal{Z}_i(t) = \sum_{k=1}^{E_i(t)} \boldsymbol{\delta}^+_{V_{ik}(t)},$$

where $\boldsymbol{\delta}^+_x \in \mathbf{M}$ is a point mass measure at x if $x > 0$ and is $\mathbf{0}$ if $x \leq 0$. The $\mathcal{Z}_i(t)$ puts a unit amount of mass for each flow of type i in the system at time t at the positive value corresponding to the unfinished amount of work of the flow. For example, if the system has two flows of type 1 with remaining amount of work 2 and 4 at time t, then $\mathcal{Z}_1(t) = \boldsymbol{\delta}^+_2 + \boldsymbol{\delta}^+_4$. The $\mathcal{Z}(t)$ is sufficient to recover most of the relevant system information. For example, for $i \leq \mathbf{I}$

$$Z_i(t) \quad = \quad \langle \mathbf{1}, \mathcal{Z}_i(t) \rangle, \tag{5}$$

$$W_i(t) \quad = \quad \langle \chi, \mathcal{Z}_i(t) \rangle \;=\; \mathcal{L}_i(t) - T_i(t), \tag{6}$$

where $\mathcal{L}_i(t) = \sum_{k=1}^{E_i(t)} \boldsymbol{\delta}^+_{V_{ik}}$ and the process T_i is defined as follows: let $T(t) = (T_1(t), \ldots, T_{\mathbf{I}}(t))$ track the cumulative amount of work given to flows. That is,

$$T_i(t) \quad = \quad \int_0^t \mathbf{x}_i(Z(s))Z_i(s)ds \;=\; \int_0^t \Lambda_i(Z(s))ds. \tag{7}$$

Similarly, let process U track the cumulative amount of unused bandwidth in the network. That is,

$$U(t) \quad = \quad Ct - AT(t). \tag{8}$$

In summary, the system is determined by parameters $(A, C, \nu, \vartheta, \mathcal{U})$, where $\mathcal{U} = (\mathcal{U}_1, \ldots, \mathcal{U}_{\mathbf{I}})$. The processes describing system dynamics are (Z, W, T, U), which are induced by $(E, \mathcal{Z})$ and the NUM given that system starts empty, i.e., $\mathcal{Z}(0) = \mathbf{0}$.

III. Fluid Model Scaling

In this section, we describe fluid model scaling by considering a sequence of systems, indexed by scaling parameter $r \in \mathbb{N}$[6]. Specifically, the r^{th} system has corresponding

[6] We call parameter r instead of n, as scaling parameter is traditionally called r. This notation hopefully will not create much confusion.

parameters $(A, C^r, \nu^r, \vartheta, \mathcal{U})$ obeying the following relation: $C^r = rC = (rC_1, \ldots, rC_{\mathbf{J}})$ and $\nu^r = r\nu = (r\nu_1, \ldots, r\nu_{\mathbf{I}})$. That is, the capacity of each link and the arrival rate are scaled r times. However, the network routing matrix A, service requirement ϑ, and utility of the network remains the same. We make a quick remark that under this scaling the loading is $\rho^r = r\rho$, and, from (3),

$$rA\rho = A\rho^r < C^r = rC.$$

Now, we describe the arrival process and service requirement process of the r^{th} system. In this notation, the original system corresponds to the r^{th} system with $r = 1$. Recall that the original system's cumulative arrival process is E and its service requirement is given by V. The arrival process of the r^{th} system, denoted by E^r is $E^r(t) = E(rt)$. That is, requests arriving to the original system in time $[0, rt]$ arrive to the r^{th} system in time $[0, t]$. The requests retain their service requirement, that is, $V^r = V$. The stochastics of a system is completely in the arrival and service processes. Given the above described scaling, we have all the r systems, $r \in \mathbb{N}$, living on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Now, we define the scaled system variables. Let $(Z^r, W^r, T^r, U^r, \mathcal{Z}^r)$ be the variables corresponding to the r^{th} system. Now, define the scaled variables as follows. Given (5)-(8), it is sufficient to describe the scaled measure valued descriptor. Let it be defined as

$$\bar{\mathcal{Z}}^r(t) \;=\; \frac{1}{r}\mathcal{Z}^r(t). \tag{9}$$

Given the scaling of (9), we obtain that for other scaled variables

$$\begin{aligned} \bar{Z}^r(t) = \langle \mathbf{1}, \bar{\mathcal{Z}}^r(t)\rangle \;&=\; \frac{1}{r}\langle \mathbf{1}, \mathcal{Z}^r(t)\rangle \;=\; \frac{1}{r}Z^r(t), \\ \bar{W}^r(t) = \langle \chi, \bar{\mathcal{Z}}^r(t)\rangle \;&=\; \frac{1}{r}\langle \chi, \mathcal{Z}^r(t)\rangle \;=\; \frac{1}{r}W^r(t). \end{aligned} \tag{10}$$

Denote by $\mathbf{x}^r : \mathbb{R}^{\mathbf{I}} \to \mathbb{R}^{\mathbf{I}}$ the mapping from vector of number of flows to rates allocated to flows under NUM for r^{th} system with capacities C^r in place of C in (2).

Lemma 3: For any $r > 0$,

$$\mathbf{x}^r(rz) \;=\; \mathbf{x}(z). \tag{11}$$

Lemma 3 implies that for bandwidth allocation

$$\Lambda^r(rz) = \mathbf{x}^r(rz) \circ rz = \mathbf{x}(z) \circ rz = r\mathbf{x}(z) \circ z = r\Lambda(z).$$

Also, from Lemma 3, we have that, for $i \leq \mathbf{I}$ and $t, \tau > 0$,

$$\bar{S}_i^r(t, t+\tau) = \int_t^{t+\tau} \mathbf{x}_i(\bar{Z}^r(s))ds \;=\; S_i^r(t, t+\tau), \tag{12}$$

$$\bar{T}_i^r(t) = \int_0^t \Lambda_i(\bar{Z}^r(s))ds \;=\; \frac{1}{r}T_i^r(t), \tag{13}$$

$$\bar{U}^r(t) \;=\; Ct - A\bar{T}^r(t) \;=\; \frac{1}{r}U^r(t), \tag{14}$$

$$\bar{W}^r(t) \;=\; \frac{1}{r}\mathcal{L}^r(t) - \frac{1}{r}T^r(t) \;=\; \bar{\mathcal{L}}^r(t) - \bar{T}^r(t). \tag{15}$$

Here our interest is in studying the behavior of $(\bar{Z}^r, \bar{W}^r, \bar{T}^r, \bar{U}^r, \bar{\mathcal{Z}}^r)$ as $r \to \infty$. Under the stochastic assumptions on the arrival process and service requirement process, we will find that they will satisfy deterministic fluid model equations as defined below almost surely. Before proceeding further, we make the following remark about the scaling considered in this paper.

Remark. The scaling described above is different from the 'standard' fluid model scaling considered in [14], where the r^{th} system is obtained by scaling the variables of original system in time and space. For example, the $\bar{\mathcal{Z}}^r(t) = \mathcal{Z}(rt)/r$. For fluid model to be meaningfully defined, it is required that, for all $i \leq \mathbf{I}$, $\mathcal{U}_i(rz) = g(r)\mathcal{U}_i(z)$ for some function $g(r)$ (same for all $i \leq \mathbf{I}$) such that $g(r) > 0$ when $r > 0$. Instead, here we are scaling capacity, speeding up the arrival process, and scaling down the variables. A main advantage of such scaling is that it does not require the strictly concave utilities to have the above stated property. This allows for considering heterogeneous utility functions unlike in [14].

Definition 1 (Auxiliary variables): Given function $\zeta : \mathbb{R}_+ \to \mathbf{M}^{\mathbf{I}}$, define (z, w, τ, u) as follows:

$$\begin{aligned} z(t) &= \langle \mathbf{1}, \zeta(t)\rangle, \\ w(t) &= \langle \chi, \zeta(t)\rangle, \\ \tau(t) &= (\tau_i(t))_{i \leq \mathbf{I}}, \text{ where} \\ \tau_i(t) &= \int_0^t \left(\mathbf{x}_i(z(s))z_i(s)\mathbf{1}_{\{z_i(s)>0\}} + \rho_i \mathbf{1}_{\{z_i(s)=0\}}\right) ds, \\ u(t) &= Ct - A\tau(t). \end{aligned}$$

Definition 2 (Fluid model solution): Given system with parameters $(A, C, \nu, \vartheta, \mathcal{U})$, we call $\zeta : \mathbb{R}_+ \to \mathbf{M}^{\mathbf{I}}$ a solution to fluid model equation if ζ and corresponding auxiliary variables (z, w, τ, u) satisfy the following conditions:

(a) ζ is continuous.
(b) $\|\langle \mathbf{1}_{\{0\}}, \zeta(t)\rangle\| = 0$ for all $t \geq 0$.
(c) For any $f \in \mathcal{C}$ and $i \leq \mathbf{I}$,

$$\begin{aligned} \langle f, \zeta_i(t)\rangle = \nu_i \langle f, \vartheta_i\rangle &\left(\int_0^t \mathbf{1}_{\{z_i(s)>0\}} ds\right) \\ &- \int_0^t \langle f', \zeta_i(s)\rangle \mathbf{x}_i(z(s))ds. \end{aligned} \tag{16}$$

The following are useful properties of the auxiliary variables (z, w) associated with a fluid model solution. These properties are stated in [14] (specifically, Lemma 3.3 for property of w).

Lemma 4: Suppose ζ is a fluid model solution with $\zeta(0) = \mathbf{0}$. Then, for each $i \leq \mathbf{I}, t \geq 0$,

$$z_i(t) \;\leq\; \nu_i t, \tag{17}$$

$$\begin{aligned} w_i(t) \;&=\; \int_0^t \left(\rho_i - z_i(s)\mathbf{x}_i(z(s))\right) \mathbf{1}_{\{z_i(s)>0\}} ds \\ &=\; \rho_i t - \tau_i(t), \end{aligned} \tag{18}$$

$$\tau_i(t) \;\geq\; 0 \text{ and } \tau_i(t) \;\leq\; (\|C\| + \|\rho\|)\, t. \tag{19}$$

The following is a direct adaptation of Theorem 4.1 in [14] for the scaling described above. For this, let $\mathbb{P}_r^T$ denote the joint distribution of $(\bar{\mathcal{Z}}^r, E^r, \bar{Z}^r, \bar{W}^r, \bar{T}^r, \bar{U}^r)$ restricted to (compact) time interval $[0, T]$. Note that $\mathbb{P}_r^T$ has its support on the product space $\mathbf{D}([0,T],\mathbf{M}^{\mathbf{I}}) \times \mathbf{D}^5([0,T],\mathbb{R}_+^{\mathbf{I}})$ with the appropriately defined topology as described in Section II.

Theorem 5: Fix any $T > 0$. Then, the sequence of probability measures $\mathbb{P}_r^T, r \in \mathbb{N}$ is tight (component-wise and hence with respect to the product topology as well). Hence, any weak limit point is a probability measure on the same space. Under any such weak limit point, with probability 1 the tuple $(\mathcal{Z}, E, z, w, \tau, u)$ is such that $E(t) = \nu t$ and $(\mathcal{Z}(t), z(t), w(t), \tau(t), u(t))$ is a solution to fluid model for all $t \in [0,T]$.

IV. Main Result

We state and prove the main result of this paper regarding stability of the network operating under α fair utility based rate allocation for strictly positive and sufficiently small α_i, possibly a different α_i for each source i. The general approximate stability result, which is a Corollary of Theorem 6 is stated in the next subsection.

Theorem 6: Consider a sequence of networks $(A, C^r, \nu^r, \vartheta, \mathcal{U}), r \in \mathbb{N}$, as defined in Section III. Further,

(a) there exists $B > 0$ such that $\vartheta((B,\infty)) = 0$;
(b) there exists $\delta > 0$ such that $(1+\delta)A\rho < C$; and
(c) utility of a flow $i \le \mathbf{I}$ is $\mathcal{U}_i = \varphi^{\alpha_i}$ so that Assumption 1 is satisfied as well as $\alpha_i < \frac{\delta}{B\mu_i}$.

Then, for any finite $T > 0$ and $\theta > 0$,

$$\lim_{r\to\infty}\inf \mathbb{P}_r\left(\max_{i\le\mathbf{I}} \sup_{0\le t\le T} \bar{Z}_i^r(t) < \theta\right) = 1.$$

Before we dive into the proof of Theorem 6, we explain its consequences. The main claim of Theorem 6 implies that for large enough r, $\bar{Z}^r(\cdot)$ is uniformly "close" to 0 in the interval $[0,T]$ for any finite T with probability close to 1. Recall that $\bar{Z}^r(\cdot)$, the scaled vector of flows in the system, is equal to the difference between arrivals and departures at any time. Hence, we have that, for the limiting system as $r \to \infty$, the normalized cumulative arrivals is the same as normalized cumulative departures for any time $t \in [0,T]$. That is, the system is rate-stable. The main conditions required to prove the Theorem are uniform boundedness of file-size (or service requirement) by B and the fair utility parameter α_i being small enough (or close to, but strictly greater than, 0).

A. Another Implication of Theorem 6

Theorem 6 can be interpreted as an approximation stability result as well. Before we state a general implication, consider the following example.

Example 1: Suppose ϑ_i be uniform distribution on $[0,B]$ for $i \le \mathbf{I}$. Then $\mu_i = 2/B$. Then for $\delta = 2$, condition (c) is satisfied for any $\alpha_i \in (0,1)$. That is, the Theorem 6 proves stability of any ρ such that $3A\rho < C$. Thus, Theorem 6 implies 1/3-approximation of stability for the uniform distribution with bounded file-size.

Next, we state the implication of Theorem 6 that essentially shows that the system is $1/(1+\alpha)$-approximate stable when all $\alpha_i = \alpha \in (0,\infty)$ for *any* system with bounded file size distribution.

Corollary 7: Consider a sequence of networks $(A, C^r, \nu^r, \vartheta, \mathcal{U}), r \in \mathbb{N}$, as defined in Section III. Further,

(d) there exists $0 < b \le B < \infty$ such that $\vartheta_i([0,b) \cup (B,\infty)) = 0$ for all $i \le \mathbf{I}$;
(e) utility of flow $i \le \mathbf{I}$ is $\mathcal{U}_i = \varphi^\alpha$ for $\alpha \in (0,\infty)$; and
(f) $(1+\alpha)A\rho < C$.

Then, for any finite $T > 0$ and $\theta > 0$,

$$\lim_{r\to\infty}\inf \mathbb{P}_r\left(\max_{i\le\mathbf{I}} \sup_{0\le t\le T} \bar{Z}_i^r(t) < \theta\right) = 1.$$

Proof: The condition (f) implies that there exists an $\varepsilon > 0$ such that

$$(1+\varepsilon)(1+\alpha)A\rho \quad < \quad C. \tag{20}$$

That is

$$(1+\delta)A\rho \quad < \quad C, \tag{21}$$

where $\delta = \alpha(1+\varepsilon)+\varepsilon$. Now define interval $I_k = [b(1+\varepsilon)^k, b(1+\varepsilon)^{k+1})$. Let $K_\varepsilon = \lceil \log(B/b)/\log(1+\varepsilon)\rceil$. Then

$$[b,B) \subset \cup_{k=0}^{K_\varepsilon} I_k.$$

Now consider ϑ_i any $i \le \mathbf{I}$. Since the support of ϑ_i is contained in $[b,B)$ we can write ϑ_i as follows.

$$\vartheta_i = \sum_{k=0}^{K_\varepsilon} p_{ik}\vartheta_{ik},$$

where $p_{ik} = \int_{I_k} \vartheta_i(x)dx$ and $\vartheta_{ik} = p_{ik}^{-1}\vartheta_i \mathbf{1}_{I_k}$. Also, define $\nu_{ik} = \nu_i p_{ik}$, $\mu_{ik}^{-1} = \langle \chi, \vartheta_{ik}\rangle$ and $\hat{\rho}_{ik} = \nu_{ik}/\mu_{ik}$. The above suggests that flow of type i with parameters ν_i, ϑ_i is equivalent to K_ε different flows, $(i,k),\ 0 \le k \le K_\varepsilon$, with parameters $(\nu_{ik}, \vartheta_{ik})_{0\le k\le K_\varepsilon}$. That is the original system with $\mathbf{I}$ flows is equivalent to $K_\varepsilon \mathbf{I}$ flows. The $\mathbf{J}\times\mathbf{I}$ routing matrix A naturally extends to $\mathbf{J}\times K_\varepsilon\mathbf{I}$ matrix $\hat{A}$. Then, (21) implies that

$$(1+\delta)\hat{A}\hat{\rho} \quad < \quad C. \tag{22}$$

We want to remind reader that this newly created system with K_ε times more flows has all the stochastic properties of the original system - primarily the arrival process and service requirement process of each satisfy the functional law of large numbers. This can be checked easily given how the construction of new system is done from the original system. Now to complete the proof it is sufficient to show that this new system satisfies the conditions of Theorem 6.

To this end, consider conditions (a)-(c) of Theorem 6. Given (22) it is straightforward to check that conditions (a)-(b) are satisfied and $\mathcal{U}_i = \varphi^\alpha \in (0,\infty)$ for all $i \le \mathbf{I}$ satisfy Assumption 1 (i.e. all $K_\varepsilon\mathbf{I}$ flows satisfy it as well). Thus we are required to check the second part of condition (c). Now for flow $(i,k), 0 \le k \le K_\varepsilon, i \le \mathbf{I}$, the bound on service requirement is $B_k \triangleq b(1+\varepsilon)^{k+1}$ while support of ϑ_{ik} is on interval $[b(1+\varepsilon)^k, b(1+\varepsilon)^{k+1})$. Hence, $\mu_{ik}^{-1} \in$

$[b(1+\varepsilon)^k, b(1+\varepsilon)^{k+1})$. That is,

$$\frac{1}{1+\varepsilon} \quad \leq \quad \frac{1}{\mu_{ik} B_k}. \tag{23}$$

Now the definition of δ in (22) and (23) imply the following.

$$\begin{aligned} \alpha \quad &< \quad \alpha + \frac{\varepsilon}{1+\varepsilon} = \frac{\alpha(1+\varepsilon)+\varepsilon}{1+\varepsilon} \\ &= \quad \frac{\delta}{1+\varepsilon} \;\leq\; \frac{\delta}{\mu_{ik} B_k}. \end{aligned} \tag{24}$$

The (24) completes the verification of the condition (c) of Theorem 6. Given that the sum of the number of flows of $(i,k), 0 \leq k \leq K_\varepsilon$ is the same as the number of flows of type i, conclusion of Theorem 6 implies the desired conclusion of Corollary 7 and thus completes its proof. ∎

B. Proof of Theorem 6

We will use Theorem 5 crucially to obtain proof of Theorem 6. We state the following result about the fluid model solutions.

Lemma 8: Consider any system satisfying conditions (a)-(c) of Theorem 6. Let $\mathcal{Z}$ be corresponding fluid model solution with auxiliary variables (z, w, τ, u), and $\mathcal{Z}(0) = \mathbf{0}$ since system starts empty. Then, for any $t \in [0,T]$,

$$\max_{i \leq \mathbf{I}} z_i(t) = 0,$$

where recall that $z_i(t) = \langle \mathbf{1}, \mathcal{Z}_i(t) \rangle$.

The proof of the Lemma 8 will be presented in the next sub-section. First, we use it to complete the proof of Theorem 6. To this end, consider the following. Let

$$\mathbf{S}_B = \{\mathcal{Z} \in \mathbf{D}([0,T], \mathbf{M}^{\mathbf{I}}) : \langle \mathbf{1}_{(B,\infty)}, \mathcal{Z}_i(t) \rangle = 0,\ i \leq \mathbf{I}, \\ \forall t \in [0,T]\}.$$

The $\mathbf{S}_B$ is closed set of $\mathbf{D}([0,T], \mathbf{M}^{\mathbf{I}})$ justified as follows. Let $\{\mathcal{Z}^k\} \subset \mathbf{S}_B$ and $\mathcal{Z}^k \to \mathcal{Z}$. Then by definition of d^o, there exists $t_k \in [0,T]$ with $t_k \to t$ and $\mathcal{Z}_i^k(t_k) \xrightarrow{\mathbf{w}} \mathcal{Z}_i(t)$ for $t \in [0,T]$ and $i \leq \mathbf{I}$. But we know that $\langle \mathbf{1}_{(B,\infty)}, \mathcal{Z}_i^k(t_k) \rangle = 0$ for all $i \leq \mathbf{I}$. Hence, Portmantau's theorem implies that (for open set (B,∞))

$$0 = \liminf_k \langle \mathbf{1}_{(B,\infty)}, \mathcal{Z}_i^k(t_k) \rangle \geq \langle \mathbf{1}_{(B,\infty)}, \mathcal{Z}_i(t) \rangle.$$

That is, $\mathcal{Z} \in \mathbf{S}_B$. Thus, $\mathbf{S}_B$ is closed. Hence, the $\mathbf{S}_B$ is complete and separable as well under the topology induced by d^o as well. In what follows, we will be interested in this Polish space $\mathbf{S}_B$.

For $\theta > 0$, let $A_\theta = \{\mathcal{Z}(\cdot) \in \mathbf{S}_B : \max_{i \leq \mathbf{I}} \sup_{0 \leq t \leq T} \langle \mathbf{1}, \mathcal{Z}_i(t) \rangle < \theta\}$. Then, we claim that A_θ is open set with respect to topology induced by metric d^o on $\mathbf{S}_B$. It is justified as follows. Consider $B_\theta = A_\theta^c \subset \mathbf{S}_B$. It is sufficient to show that B_θ is closed (w.r.t. topology induced on $\mathbf{S}_B$ by d^o). Equivalently, it is sufficient to show that if $\zeta^k \to \zeta$ with $\{\zeta^k\} \subset B_\theta$ then $\zeta \in B_\theta$. Since the topology is induced by metric d^o, we have that $d^o(\zeta^k, \zeta) \to 0$.

Proposition 9: For any $\zeta, \xi \in \mathbf{S}_B$,

$$\left| \max_{i \leq \mathbf{I}} \sup_t \langle \mathbf{1}, \zeta_i(t) \rangle - \max_{i \leq \mathbf{I}} \sup_t \langle \mathbf{1}, \xi_i(t) \rangle \right| \leq d^o(\zeta, \xi).$$

Proof: Recall from Section II that

$$d^o(\zeta, \xi) = \inf_{\varphi \in \Phi_b} \left\{ \|\varphi\|^o \vee \left(\sup_{t \in [0,T]} \mathbf{d}_{\mathbf{I}}(\zeta(t), \xi(t)) \right) \right\}.$$

Note that by definition of Φ_b, all $\varphi \in \Phi_b$ must be continuous in addition to being nondecreasing and $\varphi(0) = 0, \varphi(T) = T$. Hence, every $\varphi \in \Phi_b$ map $[0,T]$ onto $[0,T]$. Given $\delta > 0$ there exists $\varphi \in \Phi_b$ such that

$$\max_{i \leq \mathbf{I}} \sup_{t \in [0,T]} \mathbf{d}[\zeta_i(\varphi(t)), \xi_i(t)] \leq d^o(\zeta, \xi) + \delta.$$

Let $\ell_\delta = d^o(\zeta, \xi) + \delta$. Then from above and definition of Prohov's metric $\mathbf{d}(\cdot,\cdot)$ imply that for any Borel set S

$$\zeta_i(\varphi(t))(S) \leq \xi_i(t)(S^{\ell_\delta}) + \ell_\delta;\ \ \xi_i(t)(S) \leq \zeta_i(\varphi(t))(S^{\ell_\delta}) + \ell_\delta.$$

Now, since $\zeta, \xi \in \mathbf{S}_B$ we have $\zeta_i((B,\infty)) = \xi_i((B,\infty)) = 0$. Hence, we have that for $S = [0, B + 2\ell_\delta]$

$$\zeta_i(\varphi(t))(S) = \zeta_i(\varphi(t))(S^{\ell_\delta});\ \ \ \xi_i(t)(S) = \xi_i(t)(S^{\ell_\delta}).$$

Further, for such choice of S

$$\zeta_i(\varphi(t))(S) = \langle \mathbf{1}, \zeta_i(\varphi(t)) \rangle;\ \ \ \xi_i(t)(S) = \langle \mathbf{1}, \xi_i(t) \rangle.$$

Putting above together, we have that

$$\langle \mathbf{1}, \zeta_i(\varphi(t)) \rangle \leq \langle \mathbf{1}, \xi_i(t) \rangle + \ell_\delta;\ \ \langle \mathbf{1}, \xi_i(t) \rangle \leq \langle \mathbf{1}, \zeta_i(\varphi(t)) \rangle + \ell_\delta.$$

Now since φ maps $[0,T]$ onto $[0,T]$, the above implies that

$$\left| \max_{i \leq \mathbf{I}} \sup_{t \in [0,T]} \langle \mathbf{1}, \zeta_i(t) \rangle - \max_{i \leq \mathbf{I}} \sup_{t \in [0,T]} \langle \mathbf{1}, \xi_i(t) \rangle \right| \leq \ell_\delta \; = \; d^o(\zeta, \xi) + \delta.$$

Since $\delta > 0$ is arbitrary, we conclude that

$$\left| \max_{i \leq \mathbf{I}} \sup_{t \in [0,T]} \langle \mathbf{1}, \zeta_i(t) \rangle - \max_{i \leq \mathbf{I}} \sup_{t \in [0,T]} \langle \mathbf{1}, \xi_i(t) \rangle \right| \leq d^o(\zeta, \xi).$$

∎

From Proposition 9, we obtain that if $\zeta^k \to \zeta$ with $\{\zeta^k\} \subset \mathbf{S}_B$, then

$$\max_{i \leq \mathbf{I}} \sup_{t \in [0,T]} \langle \mathbf{1}, \zeta_i^k(t) \rangle \to \max_{i \leq \mathbf{I}} \sup_{t \in [0,T]} \langle \mathbf{1}, \zeta_i(t) \rangle.$$

Hence, if $\{\zeta^k\} \subset B_\theta$ then $\zeta \in B_\theta$. That is, B_θ is closed and hence A_θ is open in $\mathbf{S}_B$.

Now suppose Theorem 6 is false for a given $\theta > 0$. Then, from above discussion it must be that there is a sequence $r_q, q \in \mathbb{N}$, $r_q \to \infty$, such that $\mathbb{P}^T_{r_q}(A_\theta) < 1 - \delta$ for all q and some $\delta > 0$. By Theorem 5, there exists a further subsequence $r_{q_m}, m \in \mathbb{N}$ of $r_q, q \in \mathbb{N}$, so that $\mathbb{P}^T_{r_{q_m}}$ converges to some $\mathbb{P}^T_\star$ under which the system satisfies fluid model solution with probability 1. By Lemma 8, we have that, for any $\theta > 0$,

$$\mathbb{P}^T_\star\left(A_\theta\right) = 1.$$

By Portmantau's characterization of weak-convergence and A_θ being open we have that

$$\liminf_{r_{q_m}} \mathbb{P}^T_{r_{q_m}}\left(A_\theta\right) \quad \geq \quad \mathbb{P}^T_\star\left(A_\theta\right) = 1. \tag{25}$$

This contradicts our assumption that Theorem 6 is false. This completes the proof of Theorem 6.

C. Proof of Lemma 8

Consider a system satisfying hypothesis (a)-(c) of the Theorem 6. Let $\mathcal{Z}$ be a fluid model solution with its auxiliary variables (z, w, τ, u), and $\mathcal{Z}(0) = \mathbf{0}$. Let $y_i(t) = (1+\delta)\rho_i/z_i(t)$ for $i \le \mathbf{I}$. Define the following Lyapunov function

$$L(t) = \sum_{i \le \mathbf{I}} L_i(t), \quad \text{where} \quad L_i(t) = w_i(t)\mathcal{U}_i'(y_i(t)).$$

Now

$$\mathcal{U}_i'(x) = x^{-\alpha_i} \quad \text{and} \quad \mathcal{U}_i''(x) = -\alpha_i x^{-1-\alpha_i}.$$

In what follows, we wish to upper bound $\limsup_{h\to 0^+} \frac{L(t+h)-L(t)}{h}$ for all t. By Fatou's Lemma,

$$\limsup_{h\to 0^+} \frac{L(t+h)-L(t)}{h} \le \sum_{i\le \mathbf{I}} \limsup_{h\to 0^+} \frac{L_i(t+h)-L_i(t)}{h}. \tag{26}$$

Next, we bound $\limsup_{h\to 0^+} \frac{L_i(t+h)-L_i(t)}{h}$. To this end, replacing the value of $\mathcal{U}_i'(\cdot)$ and using simple manipulation give us

$$\begin{aligned}\frac{L_i(t+h)-L_i(t)}{h} &= y_i^{-\alpha_i}(t)\frac{w_i(t+h)-w_i(t)}{h} \\ &\quad + w_i(t+h)\frac{y_i^{-\alpha_i}(t+h)-y^{-\alpha_i}(t)}{h}\end{aligned} \tag{27}$$

Next, we bound (27) as $h \to 0^+$ in many steps as follows.

Step 1. Bound on $y_i^{-\alpha_i}(t)$*:* We have $y_i^{-\alpha_i}(t) = z_i^{\alpha_i}(t)(1+\delta)^{-\alpha_i}\rho_i^{-\alpha_i}$. For any $t \in [0,T]$, (17) of Lemma 4 imply that $z_i(t) \le \nu_i t \le \nu_i T$. Putting this together, we have that, for any $t \in [0,T]$,

$$y_i^{-\alpha_i}(t) \le (1+\delta)^{-\alpha_i}\rho_i^{-\alpha_i}\nu_i^{\alpha_i}T^{\alpha_i} \triangleq K_1^T. \tag{28}$$

Step 2. Bound on $w_i(t)$*:* From (18) of Lemma 4, for any $t \in [0,T]$, we have

$$w_i(t) \le \rho_i t \le \rho_i T. \tag{29}$$

Step 3. Bound on $\limsup_{h\to 0^+} \frac{w_i(t+h)-w_i(t)}{h}$*:* The (18) and (19) of Lemma 4 imply that $w_i(\cdot)$ is a Lipschitz continuous function with constant $(\|C\| + \|\rho\|)$. It is well-known that Lipschitz continuous function are differentiable almost everywhere. Since we have finite $\mathbf{I}$, we have that all w_i, $i \le \mathbf{I}$, are differentiable almost everywhere. Such t are called regular points. At such t, the term $\limsup_{h\to 0^+} \frac{w_i(t+h)-w_i(t)}{h} = \frac{dw_i(t)}{dt}$. From Lemma 4, for such regular point t, we have

$$\begin{aligned}&\limsup_{h\to 0^+} \frac{w_i(t+h)-w_i(t)}{h} = \frac{dw_i(t)}{dt} \\ &= (\rho_i - x_i(z(t))z_i(t))\,\mathbf{1}_{z_i(t)>0} \le \rho_i - x_i(z(t))z_i(t).\end{aligned} \tag{30}$$

Here the last inequality follows from the fact that, for $z_i(t) = 0$, $x_i(z(t))z_i(t) = 0$. Note that Lipschitz continuity of $w_i(\cdot)$ implies that, for all t, we have $\limsup_{h\to 0^+} \frac{w_i(t+h)-w_i(t)}{h} \le (\|C\| + \|\rho\|)$ and $\lim_{h\to 0^+} w_i(t+h) = w_i(t)$.

Step 4. Bound on $\limsup_{h\to 0^+} \frac{y_i^{-\alpha_i}(t+h)-y^{-\alpha_i}(t)}{h}$*:* Consider the following.

$$\begin{aligned}y_i^{-\alpha_i}(t+h) - y^{-\alpha_i}(t) &= \frac{z_i^{\alpha_i}(t+h) - z_i^{\alpha_i}(t)}{(1+\delta)^{\alpha_i}\rho_i^{\alpha_i}} \\ &\le \frac{(z_i(t)+h\nu_i)^{\alpha_i} - z_i^{\alpha_i}(t)}{(1+\delta)^{\alpha_i}\rho_i^{\alpha_i}},\end{aligned} \tag{31}$$

where the last inequality follows from Lemma 4. Taking $h \to 0^+$ in (31), we obtain

$$\begin{aligned}\limsup_{h\to 0^+} \frac{y_i^{-\alpha_i}(t+h)-y^{-\alpha_i}(t)}{h} &\le \frac{\alpha_i\nu_i}{(1+\delta)^{\alpha_i}\rho_i^{\alpha_i}z_i^{1-\alpha_i}(t)} \\ &= \frac{\alpha_i\nu_i y_i^{1-\alpha_i}}{(1+\delta)\rho_i}.\end{aligned} \tag{32}$$

Using the bounds from Steps 1-4, we obtain the following: for almost every t,

$$\begin{aligned}&\limsup_{h\to 0^+} \frac{L(t+h)-L(t)}{h} \le \\ &\sum_{i\le\mathbf{I}}\left[(\rho_i - x_i(z(t))z_i(t))y_i^{-\alpha_i}(t) + \frac{w_i(t)\alpha_i\nu_i y_i^{1-\alpha_i}}{(1+\delta)\rho_i}\right].\end{aligned} \tag{33}$$

Further, for any $t \in [0,T]$, there exists a finite constant $K_2^T < \infty$ such that

$$\limsup_{h\to 0^+} \frac{L(t+h)-L(t)}{h} \le K_2^T. \tag{34}$$

Next, we study the bound on the right hand side (RHS) of (33) with goal of establishing it to be negative if $\max_{i\le\mathbf{I}} z_i(t)$ is positive. To this end let $z_i(t) > 0$. Consider term $\sum_{i\le\mathbf{I}}(\rho_i - x_i(z(t))z_i(t))y_i^{-\alpha_i}(t)$. For an $i \le \mathbf{I}$,

$$\begin{aligned}&(\rho_i - x_i(z(t))z_i(t))\,y_i^{-\alpha_i}(t) \\ &= -\delta\rho_i y_i^{-\alpha_i}(t) + z_i(t)\left(\frac{(1+\delta)\rho_i}{z_i(t)} - x_i(z(t))\right)y_i^{-\alpha_i}(t) \\ &= -\delta\rho_i y_i^{-\alpha_i}(t) + z_i(t)\,(y_i(t) - x_i(z(t)))\,y_i^{-\alpha_i}(t).\end{aligned} \tag{35}$$

From the hypothesis of Theorem 6, $(1+\delta)A\rho < C$. Hence, the vector $y(t) = (y_1(t), \ldots, y_{\mathbf{I}}(t))$ is a feasible rate allocation. Now, utility function of i^{th} flow corresponds to α_i fair utility. Hence, it is strictly concave as discussed earlier. That is, given $z(t)$, the rate allocation vector $x(z(t))$ is unique and satisfies the zero gradient condition. From this, using standard argument it follows that

$$\sum_{i\le\mathbf{I}} z_i(t)(y_i(t) - x_i(z(t)))y_i^{-\alpha_i}(t) \le 0. \tag{36}$$

Therefore,

$$\sum_{i\le\mathbf{I}}(\rho_i - x_i(z(t))z_i(t))y_i^{-\alpha_i}(t) \le -\delta\sum_{i\le\mathbf{I}}\rho_i y_i^{-\alpha_i} \tag{37}$$

Now, the term $\sum_{i\leq \mathbf{I}} \frac{w_i(t)\alpha_i\nu_i y_i^{1-\alpha_i}(t)}{(1+\delta)\rho_i}$. For this, note that we have $\vartheta_i((B,\infty)) = 0$ from the hypothesis of Theorem 6. Subsequently, $\langle \mathbf{1}_{(B,\infty)}, \mathcal{Z}_i(t)\rangle = 0$ for all $i \leq \mathbf{I}$ and all $t > 0$. Hence,

$$w_i(t) = \langle \chi, \mathcal{Z}_i(t)\rangle \leq B\langle \mathbf{1}, \mathcal{Z}_i(t)\rangle = Bz_i(t). \quad (38)$$

Using (38) and recalling the definition of $y_i(t)$, we obtain

$$\sum_{i\leq \mathbf{I}} \frac{w_i(t)\alpha_i\nu_i y_i^{1-\alpha_i}(t)}{(1+\delta)\rho_i} \leq \sum_{i\leq \mathbf{I}} B\alpha_i\nu_i y_i^{-\alpha_i}. \quad (39)$$

Combining (37) and (39) in (33), we obtain that, for almost all $t \in [0,T]$,

$$\begin{aligned}\limsup_{h\to 0^+} \frac{L(t+h)-L(t)}{h} &\leq -\sum_{i\leq \mathbf{I}} (\delta\rho_i - \alpha_i B\nu_i) y_i^{-\alpha_i}(t) \\ &= -\sum_{i\leq \mathbf{I}} \frac{(\delta\rho_i - \alpha_i B\nu_i) z_i^{\alpha_i}}{(1+\delta)^{\alpha_i}\rho_i^{\alpha_i}}. \quad (40)\end{aligned}$$

By property of fluid model solution, we have $\|\langle \mathbf{1}_{\{0\}}, \zeta(t)\rangle\| = 0$ for all $t \in [0,T]$. Also, $w_i(t) \leq Bz_i(t)$. Hence, we have

$$L(t) = 0 \Leftrightarrow z(t) = \mathbf{0}.$$

In summary, we have the following: (1) for almost all $t \in [0,T]$, $\limsup_{h\to 0^+} \frac{L(t+h)-L(t)}{h} < 0$ if $L(t) > 0$; (2) $\limsup_{h\to 0^+} \frac{L(t+h)-L(t)}{h} \leq K_2^T$ for all $t \in [0,T]$. Given (1) and (2), simple analysis arguments imply that $L(t) = 0$ for all $t \in [0,T]$ given that $L(0) = 0$. This immediately implies that $z(t) = \mathbf{0}$ for all $t \in [0,T]$. This completes the proof of Lemma 8.

V. Concluding Remarks and Future Work

Deterministic versions of NUM and its generalizations have been extensively used in many network designs recently. However, most results on stochastic stability of NUM rely on the assumption of exponentially distributed file sizes. In this paper, we have established the stability of network operating under α-fair rate allocation with general file size distributions, when the α corresponding to each flow is close to 0 and the service requirement has bounded size. In addition, our results imply $1/(1+\alpha)$-approximate stability of network with any α-fair utility under general file size distribution. Our method was based on Lyapunov function analysis for fluid model solution of the scaled system. Due to different scaling, we could establish fluid model (and subsequently stability) for heterogeneous α-fair utilities for different flows. Our Lyapunov function is naturally valid for the fluid model scaling of Gromoll and Williams [14] since the fluid model solutions are identical.

It is straightforward to extend Theorems 5 and 6 with same utility functions and convex constraints such that **0** is a feasible point under constraints. Extending these results for the case of general set of concave utilities beyond α-fair would also be an interesting task.

The special cases of $\alpha = \infty$ and $\alpha = 1$ have recently been tackled in the preprints of [5] and [23], respectively. In contrast, this paper provides guarantees for a continuum of $\alpha \in (0,\infty)$, including stability for heterogeneous and sufficiently small α_i. However, stability for all $\alpha \in (0,\infty)$ and general file size distribution is still open. Note that we have ignored any dynamical information about the fluid quantity $z(t)$ by upper bounding its rate of change by ν. Further progress can be made by studying the details of the dynamics of $z(t)$, which can be obtained based on the fluid model solution of Theorem 5.

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