

# MaxNet: An Efficient Max-Min Fair Allocation Scheme

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**Abstract.** It has been analytically proved that both MaxNet and SumNet can achieve max-min fair allocation in static networks. In SumNet networks, such as the TCP algorithm of the current Internet, the source rate is controlled by congestion signal which is the sum of signals from all of the congested links along the path from the source to the destination. In MaxNet networks, only the most congested link generates the control signal to dictate the source rate. This paper investigates the practical aspects of both schemes, specifically, the convergence speed and fairness tracking capability under transient network conditions. We have shown that the stability of SumNet's max-min fairness heavily depends on the network load. Within the stable ranges of operating points, SumNet's max-min fair allocation is obtained at the cost of response speed. An enhanced approach is proposed to improve the stability of SumNet to meet the max-min fairness criterion. We have shown both analytically and by simulation that under stable conditions, Maxnet has faster convergence speed and better fairness tracking capability than SumNet in a highly volatile environment.

**Keywords:** MaxNet, SumNet, max-min fairness, transient networks, performance

## 1. INTRODUCTION

Over more than two decades, the Internet has grown from a small scale network to a huge widely deployed system. As the Internet continues to expand in size, complexity and service variety, it is becoming increasingly difficult to design a fair and efficient resource management scheme. Congestion control aims to achieve fairness and maintain a high quality of service (QoS). It is an important component in the overall QoS infrastructure in IP networks. Generally speaking, fair resource allocation means that no user should be overly penalized to favor other users who share the same resources. With this loose definition, the

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purpose of congestion control is to guarantee the resources are fairly allocated and highly utilized under network capacity constraints.

Congestion control based on utility maximization has been developed in recent studies [1] [2], in which fairness is considered as a primary objective in shaping end-users' cooperative behavior in a distributed manner. In this approach, a network controls the rate of a source by sending it a scalar feedback congestion signal, which is generated by summing the "congestion prices", of all links traversed by the source's packets. The links adapt their congestion prices to achieve full utilization, without overload. Such networks are referred to as SumNet as in [3]. The term is used to distinguish it from another congestion control scheme, MaxNet, where the sources are not told the sum of prices, but are instead told the maximum price, such that only the most congested link controls the source rate.

The classical bandwidth sharing principle, max-min fairness, is the general guide in designing end-to-end congestion protocols [4] [5]. Recently, SumNet was shown to achieve max-min fairness with special selection of utility functions defined for sources [2, 6, 7]. With the selected utility functions, maximize the aggregated utility achieves max-min fairness. Low *et al.* suggested a centralized technique for selecting collaborative utility functions such that the output is max-min fair [2]. However, this requires the target max-min fair allocation to be known in advance. Other approaches proposed by Kelly [6] and Mo and Walrand [7], show that max-min fair allocation can be obtained in the limit of certain utility functions defined homogeneously by users. Adopting a different approach to communicating the link prices by routers, MaxNet can obtain max-min fairness for homogeneous sources with a wide range of well behaved demand functions [3].

Fairness is usually considered in static networks with a fixed number of source-destination pairs and shared resources. In volatile networks, in which resources and service request may vary unpredictably, the convergence becomes an important issue. The convergence speed could have big impacts on the QoS. Slow response results in long traffic transients and it is responsible for packet delay, delay-jitter, underutilization and buffer-overflow. Reducing the duration of transients can partially solve these problems and makes smaller buffer sizes possible. Therefore, to evaluate performance and feasibility of congestion control algorithms, both fairness and dynamic behavior have to be taken into consideration. Inevitably, a reasonable compromise has to be made.

In this paper, we study the convergence characteristics of both SumNet and MaxNet in obtaining max-min fair allocation in transient networks. The transient environment adopted in our study reflects a common Internet environment where sources come and go and data is transferred via fixed network links. We have defined two transient response properties: (1) *speed of response*: The time taken for the system to converge to a new equilibrium state after a transient starts; (2) *fairness tracking capability*: robustness of the fairness tracking in volatile networks where sources come and go with high frequency.

The paper is organized as follows: In Sections 2 and 3, SumNet and MaxNet are reviewed. In Section 4, we use control models introduced in [8] to analyse dynamic properties of the well-known max-min fairness approach for SumNet proposed by Mo and Walrand in [7]. We identify the stability problem in the approach of [7] that makes it impossible to obtain accurate max-min fair allocation. To compare the two algorithms, we have extended the work of [7] and proposed an enhanced approach to obtain scalable stable max-min fairness for SumNet networks, presented in Section 5. In Section 6, the transient performance of the enhanced SumNet and MaxNet protocols are examined and compared. The analytical results are verified by a series of experiments with fluid-flow approximated simulation in Section 7.

## **2. SUMNET: AN OPTIMIZATION BASED NETWORK CONGESTION CONTROL**

The SumNet congestion control architecture is formulated as a concave optimization problem over linear constraints, reported in remarkable studies [1] [2] [7]. Let  $L$  be the a set of links shared by set of

source destination pairs, indexed by  $i \in I$ . Each link  $l \in L$  has finite capacity  $C_l$ . The routing matrix  $R$  is defined as:

$$R_{li} = \begin{cases} 1 & \text{if source } i \text{ uses link } l \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Each source  $i$  has a transfer rate  $x_i(t)$  at time  $t$ ; the set of flows traversing link  $l$  generate an aggregate transmission rate  $y_l(t)$  at link  $l$  such that:

$$y_l(t) = \sum_i R_{li} x_i(t) \quad (2)$$

Suppose that each source's benefit can be characterized with a utility function of allocated transfer rate  $U(x_i)$ . Throughout this paper, we assume that the utility function is a strictly concave, increasing and continuously differentiable function of  $x_i$ , over  $x_i \geq 0$ . Such a utility function results in elastic traffic networks where each user tries to maximize its own net utility, defined as the utility minus the total price it pays. Since the resources are finite, this behavior should be controlled properly and fairly to avoid the resource overload.

Using the notation below, the price per unit bandwidth flowing through link  $l$  is adapted as:

$$\frac{d}{dt} p_l(t) = \begin{cases} \gamma_l (y_l(t) - C_{ol}) & \text{if } p_l(t) > 0 \\ \max\{0, \gamma_l (y_l(t) - C_{ol})\} & \text{if } p_l(t) = 0 \end{cases} \quad (3)$$

and

$$x_i(t) = D(q_i(t)) = U_i'^{-1} \left( \sum_l R_{li} p_l(t) \right) \quad (4)$$

The link integrator (3) tries to match the aggregate input rate  $y_l(t)$  with the virtual target capacity  $C_{ol}$  which is set slightly less than the real link's capacity  $C_l$  to eliminate the link's queue at equilibrium. Hence, the integral of  $(y_l - C_{ol})$  can be regarded as a "virtual" queue. The link gain,  $\gamma_l > 0$ , is to be chosen to determine the tradeoff between the stability and convergence speed, depending on the desired dynamic behavior. In [8], Paganini *et al.* suggested to define the step size  $\gamma_l = 1/C_{ol}$  for a scalable system; in this case, the congestion prices  $p_l(t)$  can be regarded as the virtual queuing delays.

In (4),  $U_r'^{-1}(\cdot)$  is the inverse of  $U_r'(\cdot)$ , and sources adjust their transfer rates through the decreasing demand function  $D$  in (4) in response to congestion price accumulated from all links on its path as

$$q_i(t) = \sum_l R_{li} p_l(t) \quad (5)$$

Given source's utility functions  $U_r(x_r)$ , the equations (3) and (4) have a unique stable point to which all trajectories of the algorithms converge and at equilibrium, the sources' rates seek to maximize the global profit [1]

$$\text{Maximize } \sum_i U_i(x_i) \quad (6)$$

$$\text{subject to } Rx \leq C \quad (7)$$

The demand function in (4) illustrates how users' demand is encoded by a concave increasing utility function. These demands are accommodated by (3) and (4) at an equilibrium whose fairness is dictated by the appropriate choices of source utility function [6] [7] [2]. This strategy provides a "soft" way of imposing fairness (weaker than, e.g., "max-min" fairness), or alternatively service differentiation. Classical max-min fairness was investigated as a limiting case of utility maximization with logarithmic and non-logarithmic utility functions [6] [7]. Adopting a logarithmic utility function proposed by Kelly's work results in so-called "proportional fairness"[1]. An alternative utility function with decreasing gradient,

$U(x_i) = (-1/x_i)$ , leads to the resource-sharing objective of minimizing the sum of the reciprocal of rates.

Mo and Walrand defined a class of utility functions

$$U_i(x_i) = w_i \frac{x_i^{1-\alpha}}{1-\alpha} \quad (8)$$

(with  $U(x_i) = w_i \ln(x_i)$  if  $\alpha = 1$ ) and the corresponding demand functions

$$D(q_i) = \left( \frac{w_i}{q_i} \right)^{1/\alpha} \quad (9)$$

At equilibrium, the resulting allocation obtained by the algorithm (3) and (4) with demand function (9) can be termed weighted  $\alpha$ -fair such that if  $w_i = 1, \forall i \in I$ , choosing values of  $\alpha = 0$ ,  $\alpha = 1$  and  $\alpha = \infty$  result in allocation which achieves maximum throughput, proportional fairness and max-min fairness, respectively [7]. The weighing factor  $w_i$  in all three criteria is regarded as an administrative means for differentiating policy such that an increase in this weight leads to an increase in the allocated resource for that source. Since we are primarily concerned about max-min fairness objective, throughout this paper, by default, the demand function (9) is referred with the weighing factor  $w_i = 1$ .

### 3. MAXNET: A MAX-MIN FAIR BASED CONGESTION CONTROL MECHANISM

The MaxNet network architecture and its resource allocation principle is presented in this section. As its name suggests, MaxNet uses the *Max* function in computation of congestion price at network links. Unlike SumNet network where all traversed links' prices are added up to provide the control signal as given by (5), MaxNet links select the *Maximum* value as the feedback signal. It implies that in MaxNet network, source  $i$  is controlled by feedback price  $\hat{q}_i$  generated by the tightest bottleneck link on its path:

$$\hat{q}_i = \max \{ \hat{p}_l, l \in L_i \} \quad (10)$$

Here, the hat ( $\hat{\cdot}$ ) identifies MaxNet variables which have related variables in SumNet. MaxNet is known to achieve max-min fair allocation for sources with wide range of homogeneous demand functions [3]. At steady state,  $p_l(t+1) = p_l(t)$ , resource allocated to source  $i$  depends on the magnitude of its demand function relative to those of other sources sharing its controlling link  $l$ . If  $T_l$  is the set of sources traversing link  $l$  and link  $l$  has the highest price on path of flow  $i$ , then the rate allocation to source  $i$  is

$$\hat{x}_i = C_{0l} \frac{D_i(\hat{q}_i)}{\sum_{k \in T_l} D_k(\hat{q}_k)} \quad (11)$$

Dynamic properties of MaxNet, such as scalability and convergence, were also studied by using control theoretic analysis in [9, 10]. By adopting similar scalable laws as in [8], MaxNet is proved to be stable for arbitrary capacities, delays and routing with appropriate choices of parameters [9]. More importantly, MaxNet was shown to achieve this robust stability with less requirements on availability of information. Moreover, under stable conditions, MaxNet was proved to outperform SumNet with faster convergence speed very near a network equilibrium [10]. The proof was based on analyzing the root loci of control loop models of MaxNet and SumNet networks where sources having the same value of round trip time.

### 4. MAX-MIN FAIRNESS AND INSTABILITY OF SUMNET

SumNet has been proven to obtain max-min fairness at equilibrium in a static network [6] [7]. The next question is how it guarantees system stability over arbitrary range of operating points. In this section, we investigate the stability of SumNet's max-min fair allocation from control point of view. We adopt the

scalable laws proposed in [8] which proposed the guideline to choose demand functions  $x_i = D_i(q_i)$  so that system stability remains robust with arbitrary network parameters such as capacities, delays and network topologies. This requires that

$$0 \leq -\frac{D_i'(q_i)}{D_i(q_i)} < \frac{\pi}{2M_i\tau_i} \quad (12)$$

Here  $M_i$  is the number of traversed links with non-zero price along source  $i$ 's path and  $\tau_i$  is source  $i$ 's round-trip-time, assumed to be fixed and monitored at the source. It is also assumed that the link gain is scaled down with own capacity at each link as follows

$$\gamma_l = \frac{1}{C_l}, \forall l \quad (13)$$

The stability law (12) provides a framework for analyzing scalable stability of system employed with specific demand functions. Following that guideline, we study the stability properties of SumNet with sources adopting the demand function in (9) and  $\alpha \rightarrow \infty$  for max-min fairness as proposed in [7].

*Proposition 1:* The framework (3) and (9) [7], with large  $\alpha$  for max-min fairness, results in system instability when network experiences small load such that the congestion price  $q_i$  is less than unity for some  $i$ .

*Proof:* For the demand function (9), the source gain is

$$\frac{\partial D_i}{\partial q_i} = -\frac{1}{\alpha q_i^{(\alpha-1)/\alpha}} \quad (14)$$

The negative sign indicates decreasing demand function. From (4) the relationship between the feedback price and source transfer rate can be expressed as

$$q_i = U_i'(x_i) = \left(\frac{1}{x_i}\right)^\alpha \quad (15)$$

Next, we apply the stability condition (12) to the max-min approach in (3) and (9). By (13), the link gain is

$$-\frac{D_i'(q_i)}{D_i(q_i)} = \frac{1}{\alpha q_i} = \frac{x_i^\alpha}{\alpha} < \frac{\pi}{2M_i\tau_i} \quad (16)$$

Since our major concern is the effect of the increasing  $\alpha$  on SumNet's stability for the max-min fairness approach with the demand function (9), we assume that link gain is fixed as specified in (13), as are the  $M_i$  and  $\tau_i$  values. Then the left hand side of (16) is determined only by values of  $\alpha$  and the operating point  $x_i$ . If the max-min fair operating point has  $x_i > 1$  then (16) is violated for sufficiently large  $\alpha$ . This leaves the question of what the units of  $x_i$  are. Is the threshold 1bit/s or 1Gb/s? The answer is that the units are implicit in (9). Thus, the system is unstable if  $q_i < w_i = 1$ , as was to be proved.

Note that if  $q_i \geq 1$  for all  $i$ , then the control gains all tend to 0, and the system becomes "frozen" with suboptimal rates.

## 5. STABLE APPROACH FOR SUMNET'S MAX-MIN FAIRNESS

In order to analyze and compare the dynamic properties of SumNet and MaxNet max-min fair allocation, we propose an enhanced framework for SumNet to obtain stable max-min fair allocation with arbitrary network parameters. The key challenge is how to solve the stability problem caused by large  $\alpha$  in the source's demand function mentioned in the earlier analysis, so that equilibrium allocation can converge more closely to the target max-min fairness with arbitrarily large  $\alpha$  values.

This can be achieved by a demand function that satisfies the bound (12) for all  $\alpha$  and still drives the system equilibrium to max-min fairness. Let  $\tau$  be an upper bound on the round trip time of any flow in the network. In general, the maximum RTT will not be known by the sources, but it is reasonable to assume that an upper bound is known. Consider now the demand function

$$x_i(t) = \left( \frac{q_i \beta \alpha}{M \tau} \right)^{\frac{1}{\alpha}} \bar{C} \quad (17)$$

again with

$$\frac{d}{dt} p_i(t) = \begin{cases} \gamma_i(y_i(t) - C_{oi}) & \text{if } p_i(t) > 0 \\ \max\{0, \gamma_i(y_i(t) - C_{oi})\} & \text{if } p_i(t) = 0 \end{cases} \quad (18)$$

where,  $\bar{M}$  is an upper bound of number of traversed links, also assumed to be known following [8]. We also define  $\bar{C}$  as a global bound of links' capacities and assume that it is known at sources.  $\beta$  is a positive constant to be chosen to adjust the source gain and will be detailed later. The corresponding source's utility function (for which  $D_i = U_i'^{-1}$ ) is

$$U_i(x_i) = \frac{\bar{M} \tau \bar{C} \left( \frac{x_i}{\bar{C}} \right)^{1-\alpha}}{\beta \alpha (1-\alpha)} \quad (19)$$

Now, we will prove that, with  $\alpha \rightarrow \infty$ , the system (3) and (4) maximizes our new utility function (19) and results in max-min fairness at equilibrium and the system is stable at arbitrary operating points. The new proposed utility function (19), with  $\alpha > 1$ , is a continuous, twice differentiable, strictly concave, increasing and negative function. Therefore, according to [7], the rates which maximize the sum of the utility functions (19) with  $\alpha \rightarrow \infty$  converges to max-min fairness.

Firstly, we analyze the stability of our system, by studying the inequality (12) with the demand function given by (17), yielding

$$0 \leq -\frac{D_i'(q_i)}{D_i(q_i)} = \frac{1}{\alpha q_i} < \frac{\pi}{2M_i \tau} \quad (20)$$

From the demand function (17) and from (4), we generalize the relationship between congestion price and transfer rate

$$q_i = U_i'(x_i) = \frac{\bar{M} \tau}{\beta \alpha} \left( \frac{x_i}{\bar{C}} \right)^{(-\alpha)} \quad (21)$$

Substituting (21) into (20), we obtain the requirement

$$-\frac{D_i'(q_i)}{D_i(q_i)} = \frac{1}{\alpha q_i} = \frac{\beta}{M \tau} \left( \frac{x_i}{\bar{C}} \right)^{\alpha} < \frac{\pi}{2M_i \tau} \quad (22)$$

If  $\beta$  is defined such that  $0 < \beta < \pi/2$ , the inequality (22) is satisfied with arbitrary large  $\alpha$  and other network parameter. Thus, as  $\alpha \rightarrow \infty$ , the obtained allocation approaches to max-min fairness without destabilizing the system.

Since the above system approaches max-min fairness only as  $\alpha \rightarrow \infty$ , we now study other important dynamic properties in this regime. In particular, we examine the system's responsiveness, as characterized by the source gain.

*Proposition 2:* The source gain implied by (17) and (18) tends to zero as  $\alpha \rightarrow \infty$ .

*Proof:* Differentiating the source  $i$ 's demand function (17) with respect to  $q_i$ , we yield the source  $i$ 's gain

$$\kappa_i = \frac{\partial D_i}{\partial q_i} = -\frac{\bar{C}}{\alpha q_i} \left( \frac{q_i \beta \alpha}{M \tau} \right)^{-1/\alpha} \quad (23)$$

Substituting (21) into (23) gives the expression of source gain in terms of  $x_i$  and  $\alpha$

$$\kappa_i = -\frac{\beta \bar{C}}{M \tau} \left( \frac{x_i}{\bar{C}} \right)^{1+\alpha} \quad (24)$$

Since  $\bar{C}$  is the upper bound of links capacities, the ratio of  $x_i / \bar{C} \leq 1, \forall i$  which implies the result.  $\square$

*Proposition 3:* As  $\alpha \rightarrow \infty$  in (17) and (18) the source gain for a given sources becoming a vanishing fraction of the source gain of a source with a larger bandwidth.

*Proof:* By (24), the ratio of source  $i$ 's gain to source  $j$ 's gain is  $(x_i / x_j)^{1+\alpha}$ , which tends to zero as  $\alpha \rightarrow \infty$  if  $x_i < x_j$ .  $\square$

Sources with lower source gains respond more slowly than sources with larger source gains. Although the rate of adaptation of all source goes to zero by Proposition 2, it goes to zero faster for sources with lower rates. If two sources are both below their max-min fair rates, the unfairness will be exacerbated by the fact that the source with the smaller rate will respond more slowly. However, if two sources are both above their max-min fair rates, this effect will improve the fairness (for a given overall rate of adaptation).

*Proposition 4:* Using a SumNet to obtain max-min fairness by using (17) and (18) with  $\alpha \rightarrow \infty$  results in exponential deceleration in responsiveness of sources when network congestion is increased.

*Proof:* When a network becomes more congested, the feedback price  $q_i$  increases. Since the source gain (23) is a decreasing function of  $q_i$ , source gain is consequently decreased. Moreover, when the demand function (17) is defined with great  $\alpha$  for max-min fair allocation, this dramatically decelerates the response speed of SumNet sources in a heavily congested network.

## 6. MAX-MIN FAIRNESS AND DYNAMIC PROPERTIES OF MAXNET

In this section, we will describe the version of MaxNet's which will be compared with SumNet in the following section. Since MaxNet was proven to obtain max-min with homogeneous well-behaved demand functions [3], we can again use the demand function of (17) for MaxNet sources, however with  $\alpha = 1$ . This gives the control laws at MaxNet's sources and links as follows:

$$\hat{x}_i(t) = \frac{\bar{C} M \cdot \tau}{\hat{q}_i \beta} \quad (25)$$

$$\frac{d}{dt} \hat{p}_l(t) = \begin{cases} \frac{1}{C_{ol}} (\hat{y}_l(t) - C_{ol}) & \text{if } \hat{p}_l(t) > 0 \\ \max \left\{ 0, \frac{1}{C_{ol}} (\hat{y}_l(t) - C_{ol}) \right\} & \text{if } \hat{p}_l(t) = 0 \end{cases} \quad (26)$$

This again assumes that sources know upper bounds on their round trip times ( $\tau$ ), the link capacities ( $\bar{C}$ ) and the number of congested links on any path ( $\bar{M}$ ). Following [3] and [9], we can conclude that for all  $0 < \beta < \pi/2$ , identified in Section 5, MaxNet's framework (25) and (26) is also globally stable with arbitrary routings, link capacities and delays and results in max-min fair allocation at equilibrium. Moreover, since the demand functions, and hence control gains, do not require  $\alpha \rightarrow \infty$ , MaxNet can avoid the drawbacks of SumNet's approach described in the previous section and thus can outperform SumNet to obtain accurate max-min fair allocation whilst maintaining a stable, faster and fairer system response.

## 7. NUMERICAL RESULTS

We have adopted the fluid flow approximated simulation [11] to simulate the linear models for SumNet and MaxNet networks. Various transient scenarios are generated and transient responses of the two algorithms are compared.

We assume that our test system is a physically realizable system. Even it is possible to devise a control strategy in which each source collects network parameters for tuning its own setting in demand function (equation (17) for SumNet or (25) for MaxNet) to optimize transient speed, a real time algorithm to achieve this is not easy. As our study mainly focuses on analyzing the dynamic properties of both algorithms designed for max-min fairness, an experimental environment setup is more practical and it can provide a fair comparison. For SumNet, stable maximization framework (17) and (18) with increasing  $\alpha \rightarrow \infty$  is simulated and the system (25) and (26) is adopted for the MaxNet network. Both algorithms are tested on the same linear network model, shown in Fig. 2 with two links of fixed capacities 8.0 and 10.0 shared by sources with homogeneous round trip times  $\tau_i = \tau \forall i$ . The other parameters are same for both SumNet and MaxNet: ie,  $\beta = 1.7$ ,  $\bar{M} = 2$ ,  $\bar{C} = 10$ . Also, for both algorithms, link gains are set to  $\gamma_l = 1/C_{0l}, \forall l$  and capacity utilization is set to 95% such that  $C_{0l} = 0.95 \times C_l, \forall l$ .

Taking max-min fairness as the target, we define the max-min unfairness index of rate allocation  $x$ , relative to the target max-min fair allocation  $\bar{x}$ , as:

$$UI(x) = \sum_i \frac{|x_i - \bar{x}_i|}{\bar{x}_i} \quad (27)$$

Essentially,  $UI(x)$  is a measure of the deviation of an allocation  $x$  from the target max-min fair allocation  $\bar{x}$ . It implies that the smaller the value of  $UI(x)$ , the closer the allocation  $x$  to the target max-min fair allocation  $\bar{x}$ .

Three experiments have been conducted to verify our propositions and to demonstrate the advantages of MaxNet over the SumNet's framework for obtaining max-min fairness. The first experiment verifies that in a static network model with a fixed number of sources and links, SumNet's convergence speed depends on the value of  $\alpha$  as stated in *Proposition 2*. Our second experiment illustrates the SumNet sources' responsiveness and aggressiveness when network congestion varies as shown in *Proposition 3* and *Proposition 4*. The third experiment investigates the change-tracking capabilities of the systems equipped with the two algorithms in a volatile network where sources turn ON and OFF periodically.

### A. Experiment 1: Effect of Large $\alpha$ on SumNet's Transient Performance

Our network model has 3 sources as shown in Fig. 2. We performed a series of tests on SumNet's maximization as in (17) and (18) with different  $\alpha$ . At starting step 0, 3 sources are ON as illustrated in Fig. 2. We generate the same transient scenarios for all experiments by activating 3 new sources on link 2 at the 3000<sup>th</sup> time step. After the first 3 sources reach equilibrium, the transient time, measured in time-steps from the time transient occurs until the time when the last source's rate is within  $\pm 1\%$  of its final value, is computed. Fig. 3 plots the system's convergence time against the  $UI$  values. The decreasing curve illustrates that, as  $\alpha$  increases, the equilibrium allocation approaches max-min along with a gradual increase in system's convergence time. It validates *Proposition 1*, ie SumNet obtains max-min fairness at the cost of decrease in system convergence speed. In contrast, MaxNet is shown to achieve absolute max-min fair allocation as  $UI_{MaxNet}$  is close to 0. Its convergence time is much shorter.

Fig. 7 provides a closer look at SumNet's and MaxNet's source's transient behavior. With  $\alpha = 2.75$ , SumNet source1 converges quickly to the new equilibrium when new sources are turned ON. However, with such a small  $\alpha$  value, source1's rate cannot converge to the target max-min fair allocation. With  $\alpha = 3.75$ , SumNet source1 converges closer to max-min fairness after transience, however, its response is much slower. Fig.7 also shows that MaxNet's source1 quickly tracks to the new equilibrium and converges to the target max-min fairness.

### B. Experiment 2: Fairness Tracking Capability in Volatile Networks

In this experiment, fairness tracking capabilities of both SumNet and MaxNet are tested under unstable network conditions in which sources turn ON and OFF with high frequency. We have simulated periodic deterministic transience with different frequencies to observe how both algorithms track to the new max-min fairness under different levels of volatility. In our simulation, the network with greater transient intervals is regarded as less volatile and vice versa. For two transient scenarios corresponding to two intervals, 10000 timestep and 20000 timestep, the former represents a faster (more volatile) transient network and the latter a slower one (less volatile) .

We used the same patterns in all experiments with 3 long-lived flows, indexed S1, S2, and S3 which remain ON throughout the experiment. On link 2, five “noise” sources indexed from S4 to S8 are activated and deactivated periodically with different intervals. We investigate how fast the system tracks to new expected equilibrium by defining tracking area (TA) metrics computed by the mean-square of the difference between sources’ rates and the target max-min fair rates during the period of transience:

$$TA = \int_{startstep}^{stopstep} \sum_i |x_i(t) - x_{i\_target}(t)|^2 dt \quad (28)$$

For SumNet, two series of experiments corresponding to two intervals, 10000 and 20000 time steps, have been conducted. The experiments vary  $\alpha$  over a wide range and both use the following procedure:

1. SumNet’s equations (17) and (18) are simulated for a duration of 300,000 time steps. In the period from the 60,000<sup>th</sup> to the 240,000<sup>th</sup> time step, transience is generated by five “noise” sources periodically turning ON and OFF on link 2 with the given interval.
2. Our strategy to choose the duration to measure the TA (28) is described in the following. Since the transience is periodic, the system’s response will also reach a “periodic steady state” as shown in Fig. 5 (a) and (b). It is the behaviour in this state that we measure. We prepared that by generating a warm-up transient period which is long enough for the system to be in such a “steady” state. In our case, as the transient starts at 60,000<sup>th</sup> time step, we measure TA from  $startstep = 200,000^{th}$  to  $stopstep = 240,000^{th}$  step for 3 long-lived sources 1,2 and 3, ignoring the “noise” sources’ flows.

For the MaxNet network, the sources and links are adjusted as in (25) and (26) and the experiments follow the same procedure as for SumNet. However, without necessary involvement of  $\alpha$  we just need to carry out two experiments corresponding to two intervals of transience. Fig. 4 gives the fairness tracking area  $TA$  against  $\alpha$  in transient scenarios of 10,000 and 20,000 time step intervals. Overall, SumNet exposes its weak tracking capability in high volatile networks as the values of the 20,000 timestep interval transient scenario,  $TA_{20,000}^{SumNet}$ , is smaller than that of the 10,000 timestep interval transient,  $TA_{10,000}^{SumNet}$ .

For SumNet, Fig. 4. clearly shows that the well-studied the large  $\alpha$  based max-min approach does not always yield max-min fairness in volatile networks. In fact, there exists a value  $\alpha^{optimal}$  which optimizes SumNet’s fairness tracking performance, i.e., TA is minimum if  $\alpha = \alpha^{optimal}$ . If  $\alpha < \alpha^{optimal}$ , TA is decreased when  $\alpha$  is increased. However, if  $\alpha > \alpha^{optimal}$ , then increasing  $\alpha$  would slow the response rate enough to increase the TA values as shown in Fig. 4 for both transient scenarios. The existence of such  $\alpha^{optimal}$  is consequence of the deceleration of SumNet source’s convergence speed caused by large  $\alpha$ . This can be explained by Fig. 6, which illustrates the distance between SumNet source1 rates, obtained with different ranges of  $\alpha$  such as  $\alpha < \alpha_{20,000}^{optimal}$  and  $\alpha > \alpha_{20,000}^{optimal}$ , and the target max-min fair rate under transient scenario of interval of 20,000 time steps.

Interestingly, the  $\alpha^{optimal}$  values are different in different transient scenarios. Fig. 4 shows that the optimal value of  $TA_{10,000}^{SumNet}$  is obtained as  $\alpha_{10,000}^{optimal} = 2$ , whilst  $TA_{20,000}^{SumNet}$  approaches optimal value at greater  $\alpha_{20,000}^{optimal} = 2.25$ . As greater  $\alpha$  value implies SumNet to obtain more accurate max-min fairness approximation, it can be forecasted that in highly volatile network, it’s hard to achieve good approximated max-min fair allocation while maintaining an adequate fairness tracking capability.

In contrast with SumNet, MaxNet, whose max-min fairness criteria does not rely on  $\alpha$ , is shown to have fast convergence and therefore a better fairness tracking capability in both high and low volatile networks. Both values of  $TA_{10,000}^{MaxNet}$  and  $TA_{20,000}^{MaxNet}$  are shown in Fig. 4 to be significantly smaller than those of

SumNet. This is also confirmed in Fig. 5(a) and Fig. 5(b), in which source1's rate under MaxNet converges quickly to its new max-min fair equilibrium in all transient scenarios.

## 8. CONCLUSION

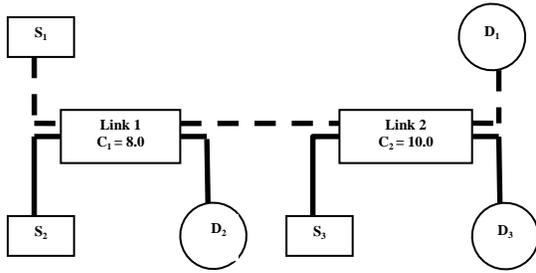
In this paper, we have discussed the stability and convergence speed of MaxNet and of SumNet configured to obtain max-min fairness. We also introduced an enhanced approach for SumNet to obtain max-min fairness without destabilizing the system. Our analysis has shown that MaxNet is a more efficient max-min fairness-based congestion control with better convergence properties. In volatile networks, MaxNet is shown to outperform SumNet in achieving approximated max-min fair allocation, convergence speed and fairness tracking capability. Numerical results for the complex transient network environments confirm the conclusions drawn from our analysis of simple systems.

## 9. ACKNOWLEDGMENT

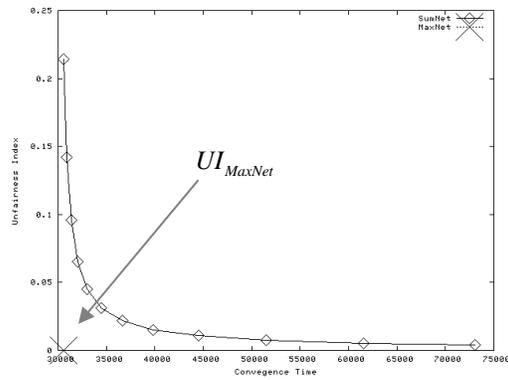
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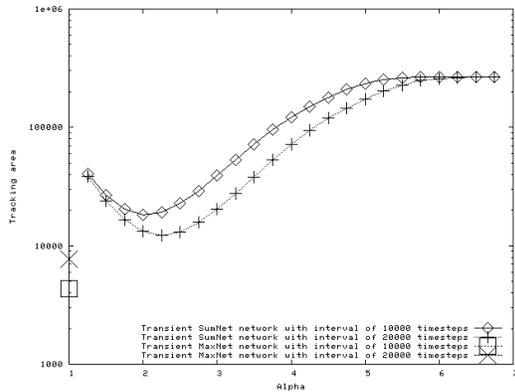
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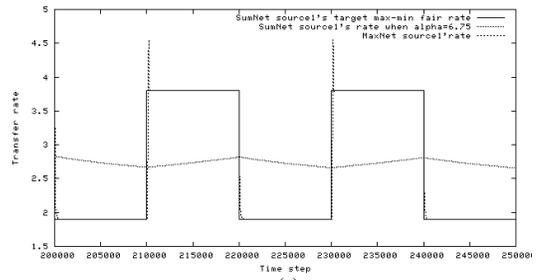
**Fig. 2.** : Network topology: S1, S2 and S3 are source1,source2 and source3 respectively.



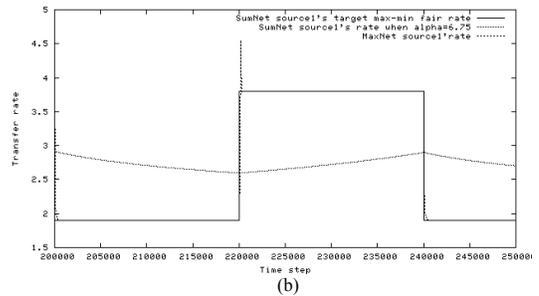
**Fig. 3.** : Convergence time vs. unfairness index ( $UI$ ) of SumNet and MaxNet under single transient occurs at 30,000<sup>th</sup> time step



**Fig. 4.** TA of SumNet with  $\alpha \in [1.25, 6.75]$  and of MaxNet in transient scenarios with two intervals: 10,000 and 20,000 time steps, respectively

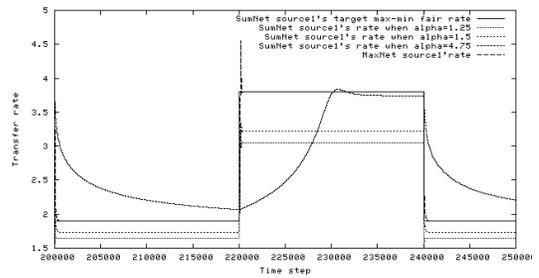


(a)

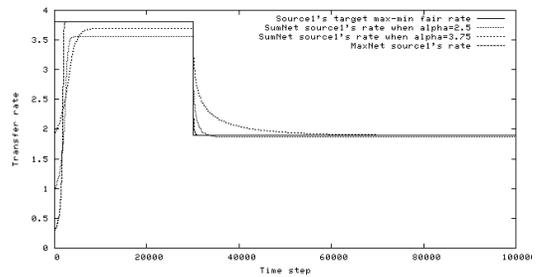


(b)

**Fig. 5.** Convergence properties of SumNet source1 rate with great  $\alpha = 6.75$  and MaxNet source1 rate in different transient scenarios: (a) 10,000 timestep interval transient, (b) 20,000 timestep interval transient



**Fig. 6.** Convergence properties of SumNet source1 with different value of  $\alpha$  and MaxNet source1 under transient with interval of 20,000 timestep



**Fig.7.** Source1's response in MaxNet and SumNet networks with single transient at 30,000<sup>th</sup> time step