Can Shortest-path Routing and TCP Maximize Utility *

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Abstract

TCP-AQM protocols can be interpreted as distributed primal-dual algorithms over the Internet to maximize aggregate utility over source rates. In this paper we study whether TCP–AQM together with shortest-path routing can maximize utility over both rates and routes. We show that this is generally impossible because the addition of route maximization makes the problem NP-hard. We exhibit an inevitable tradeoff between routing stability and utility maximization. For the special case of ring network, we prove rigorously that shortest-path routing based purely on congestion prices is unstable. Adding a sufficiently large static component to link cost stabilizes it, but the maximum utility achievable by shortest-path routing decreases with the weight on the static component. We present simulation results to illustrate that these conclusions extend to general network topology, and that rouging instability can reduce utility to less than that achievable by the necessarily stable static routing.

1 Introduction

Recent studies have shown that *any* TCP congestion control algorithm can be interpreted as carrying out a distributed primal-dual algorithm over the Internet to maximize aggregate utility, and a user's utility function is (often implicitly) defined by its TCP algorithm, see e.g. [8, 12, 15, 16, 13, 11, 9] for unicast and [7, 3] for multicast. All of these papers assume that routing is given and fixed at the time scale of interest, and TCP, together with active queue management (AQM), attempt to maximize aggregate utility over source rates. In this paper, we study utility maximization at the time scale of route changes.

One approach to joint routing and congestion control is to allow multi-path routing, i.e., a source can transmit its data on multiple paths to its destination in the unicast setting. In this formulation, a source's decision is decomposed into two – how much traffic to send (congestion control) and how

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to distributed it over the available paths (multi-path routing) – in order to maximize aggregate utility. This has been analyzed in, e.g., [4, 8, 6], assuming that both decisions operate on the same time scale. The general intuition is that, for each source-destination pair, only paths with the minimum, and hence equal, 'congestion price' will be used and this minimum price determines the total source rate as in the single-path case.

Routing (within Autonomous Systems) in the current Internet, however, does not utilize multiple paths. IP uses shortest-path routing[1] to select a single path for each source-destination pair and generally operates on a slower time scale than TCP–AQM. Within this context, we ask:

1. Can TCP–AQM/IP, with shortest path routing, jointly solve the utility maximization over both source rates and their routes?

The dual problem of utility maximization over both source rates and routing has an appealing structure that makes it solvable by shortest-path routing using congestion prices as link costs, together with TCP–AQM; see Section 2. This raises the tantalizing possibility that TCP–AQM/IP may indeed turn out to maximize utility with proper choice of link costs. We will show however that the primal problem is NP-hard, and hence cannot be solved by shortest-path routing unless P=NP.

This prompts the question:

2. How well can IP solve the utility maximization approximately? In particular, what is the effect of the choice of link cost on maximum utility and on routing stability?

We answer these questions rigorously in the special case of a ring network with a common destination (Section 3). For this special case, we show that the duality gap is trivial, due to integer constraint on routing, and is closed in the abstract convexified version of the model. This suggests that shortest-path routing based on prices may indeed maximize utility in this special case. We show however that there is an inevitable tradeoff between utility maximization and routing stability. Specifically, link costs and shortest-path routing form a feedback system. This system is unstable when link costs are pure prices. It can be stabilized by adding a static component to the link cost. The loss in utility however increases with the weight on the static component. Hence, while stability requires a small weight on prices, utility maximization favors a large weight.

This is not surprising as it is well-known that routing stability generally requires that the relative weight on the dynamic (traffic-sensitive) component of the link cost be small. Indeed, our conclusions are similar to those reached in [2, 10] that study the same ring network for routing stability using different link costs. Here, since the dynamic component is the dual-optimal price for the utility maximization problem computed by TCP–AQM, this implies a tradeoff between routing stability and utility maximization.

We present simulation results that suggest that these conclusions generalize qualitatively to general network topology (Section 4). Moreover these results indicate that routing instability can reduce

[1] By "shortest-path routing" we mean minimum-cost routing with some given link costs.

aggregate utility to less than that achievable by (the necessarily stable) purely static routing. Finally, we conclude in Section 5 with limitation of this work.

2 Model

A network is modeled as a set of L uni-directional links with finite capacities $c = (c_l, l = 1, \ldots, L)$, shared by a set of N source-destination pairs, indexed by i (we will also refer to the pair simply as 'source i'). Let $\mathcal{R}$ be a given finite set of possible paths connecting all source-destination pairs. A *routing* is an element of $\mathcal{R}$ and can be expressed as an $L \times N$ 0-1 matrix R defined by:

$$R_{li} = \begin{cases} 1 & \text{if } l \text{ is in path of } i \\ 0 & \text{otherwise} \end{cases}$$

All routes in $\mathcal{R}$ are single-path in that if source i transmits at rate x_i packets/sec, then the destination as well as all links in the path of source i receive at the same rate of x_i in equilibrium. Shortest-path routing has the further restriction that flows that diverge after a link cannot meet again at a downstream link, unless the divergent paths hat equal cost.

2.1 Duality model of TCP–AQM

Each source i has a utility function $U_i(x_i)$, as a function of its rate x_i. One can think of TCP–AQM as a distributed primal-dual algorithm to maximizing aggregate utility, given a routing matrix R, i.e., it solves the following constrained convex program (see e.g. [8, 12, 15, 16, 13, 11, 9]):

$$\max_{x_i} \quad \sum_i U_i(x_i) \tag{1}$$

$$\text{subject to} \quad Rx \leq c \tag{2}$$

and the associated dual problem [12, 13, 11]:

$$\min_{p_l \geq 0} \quad \sum_i \max_{x_i \geq 0} \left(U_i(x_i) - x_i \sum_l R_{li} p_l \right) + \sum_l p_l c_l \tag{3}$$

TCP algorithms adapt the primal variables $x = (x_i, i = 1, \ldots, N)$, and AQM algorithms adapt the dual variables $p = (p_l, l = 1, \ldots, L)$. These dual variables are measures of network congestion and we will call them 'prices'. To see the relation between the pair of problems, define the Lagrangian [1, 14]

$$L(x,p) = \sum_i U_i(x_i) + \sum_l p_l \left(c_l - \sum_i R_{li} x_i \right)$$

The primal problem (1)–(2) is $\max_x \min_p L(x,p)$ and the dual problem (3) is $\min_p \max_x L(x,p)$. The fact that there is no duality gap, i.e., $\max_x \min_p L(x,p) = \min_p \max_x L(x,p)$, means that TCP and AQM can carry out their individual optimization *asynchronously*, over x and p respectively, and the equilibrium (x^*, p^*) will be primal-dual optimal, i.e., solve both (1)–(2) and (3).

Conversely, given any TCP algorithm, the equilibrium rates x^* solve (1)–(2) with appropriate utility functions that are defined by the given TCP algorithm. For example, the utility function of TCP Reno (or its variants) is $\frac{\sqrt{2}}{D_i}\tan^{-1}(x_i D_i/\sqrt{2})$ where D_i is source i's round trip time, and the utility function of Vegas is $\alpha_i d_i \log x_i$ where α_i is protocol parameter and d_i is round trip propagation delay of source i; see [11, 13] and references therein for details and other variations. These utility functions are strictly concave increasing, and hence the problem (1)–(6) can be efficiently solved.

Note that we can go between utility maximization and the design of TCP-AQM algorithms in both directions. One can design utility functions that are tailored to one's applications and then derive TCP and/or AQM algorithms to maximize the chosen utility. Conversely, and historically, we designed TCP-AQM algorithms without explicit consideration of any utility functions. It turns out that these algorithms can be interpreted as solving a certain utility maximization problem. This simple convex program allows us to understand, and predict, the equilibrium properties of large-scale networks under TCP-AQM control (e.g., [11, 13]).

We now study whether the same methodology can be applied to the understanding of TCP-AQM and shortest-path routing.

2.2 TCP–AQM/IP

Consider the problem of maximizing utility over routes as well as rates

$$\max_{R\in\mathcal{R}} \max_{x_i\geq 0} \quad \sum_i U_i(x_i) \tag{4}$$

$$\text{subject to} \quad Rx \leq c \tag{5}$$

where $\mathcal{R}$ is the given finite set of available paths between all source-destination pairs. Clearly, an optimal routing R^* for (4)–(5) exists since the set $\mathcal{R}$ is finite, and, given R, the objective function is continuous and the feasible set is compact. Define the Lagrangian as

$$L(R,x,p) \;=\; \sum_i U(x_i) + \sum_l p_l \left(c_l - \sum_l R_{li} x_i \right)$$

and the dual problem as

$$\min_{p\geq 0} \max_{R\in\mathcal{R}, x\geq 0} L(R,x,p) \;=\; \min_{p\geq 0} \sum_i \max_{x_i\geq 0} \left(U(x_i) - x_i \min_{R_i\in\mathcal{R}_i} \sum_l R_{li} p_l \right) + \sum_l p_l c_l \tag{6}$$

where $\mathcal{R}_i$ denotes the set of available routes for source-destination pair i and R_i (column of routing matrix R) is an element of $\mathcal{R}_i$. The striking feature of the dual problem is that the maximization over R takes the form of shortest-path routing with prices p as link costs. This suggests that TCP–AQM/IP might turn out to be a distributed primal-dual algorithm that maximizes utility, with proper choice of link costs.

We show, however, that the primal problem is NP-hard and hence cannot be solved by shortest-path routing in general.

Theorem 1. *The problem (4)–(5) is NP-hard.*

Proof. We describe a polynomial time procedure that reduces an instance of integer partition problem [5, pp. 47] to a special case of the primal problem. Given a set of integers $c_1, \ldots, c_N$, the integer partition problem is to find a subset $A \subset \{1, \ldots, N\}$ such that

$$\sum_{i \in A} c_i \;=\; \sum_{i \notin A} c_i$$

Given an instance of the integer partition problem, consider the tree-like network in Figure 1, with N

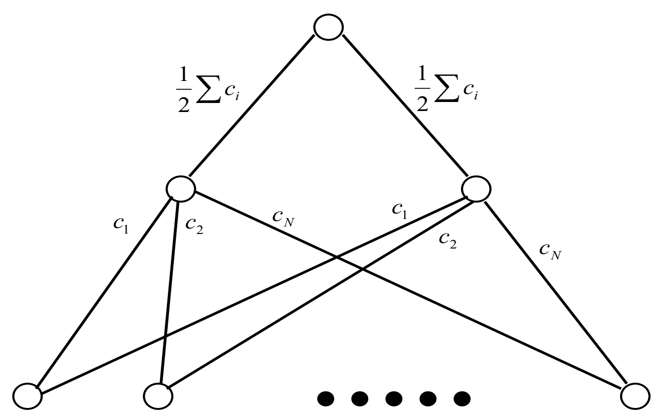

Figure 1: Network to which integer partition problem can be reduced.

sources at the root, two relay nodes, and N receivers, one at each of the N leaves. The two links from the root to the relay nodes have a capacity of $\sum_i c_i/2$ each, and the two links from each relay node to receiver i have a capacity of c_i. All receivers have the same utility function that is increasing. The routing decision for each source is to decide which relay node to traverse. Clearly, maximum utility of $\sum_i U_i(c_i)$ is attained when each receiver i receives at rate c_i, from exactly one of the relay nodes, and the links from the root to the two relay nodes are both saturated. Such a routing exists if and only if there is a solution to the integer partition problem. $\square$

How well does shortest-path routing solve it approximately? Specifically, suppose routing changes at a slower time-scale than TCP–AQM, so that in each discrete period t with routing $R(t)$, TCP–AQM converges instantly and source rates $x(t) = x(R(t))$ and prices $p(t) = p(R(t))$ are the primal and dual solutions of (1)–(3) with fixed routing $R = R(t)$. Clearly, if link costs are static, e.g., hop counts or fixed propagation delays, then routes remain unchanged at the time scale of interest, $R(t) = R(0)$ for all t. More generally, we will consider link cost $d_l(t)$ that has both a static and a dynamic component:

$$d_l(t) \quad = \quad \beta\tau_l + \alpha p_l(t) \tag{7}$$

where τ_l are the fixed propagation (and processing) delays and $p_l(t) = p_l(R(t))$ are the dual-optimal prices on links l in period t. The motivation of including prices in link cost is that they are precise measure of congestion. Indeed, these prices represent packet loss probability in TCP Reno and its variants and queueing delay in TCP Vegas [11, 13].

The protocol parameters α and β determine the responsiveness of routing to network traffic: $\alpha = 0$ corresponds to static routing, $\beta = 0$ corresponds to purely dynamic routing, and the larger the ratio of α/β, the more responsive routing is to network traffic. We are interested in the condition on α, β under which routing $R(t)$ is stable, i.e., $R(t)$ converges to some matrix R, and when it is stable, the maximum utility in equilibrium.

We next answer these questions in the special case of ring network with a common destination.

3 Ring network

Consider a ring network with $N + 1$ nodes, indexed by $i = 0, 1, \ldots, N$. Nodes $i \geq 1$ are sources and their common destination is node 0; see Figure 2. For notational convenience we will also refer to

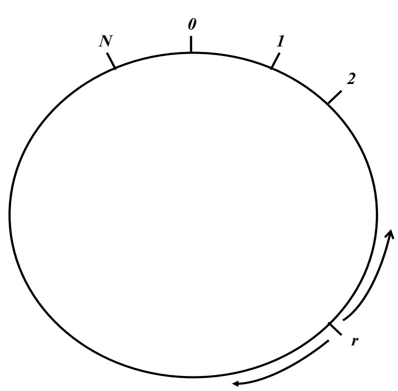

Figure 2: A ring network

node 0 as node $N + 1$. Each pair of nodes is connected by two links, one in each direction. We will refer to the two unidirectional links between node $i - 1$ and i as link i; the direction should be clear from the context. The fixed delay on link i is denoted as $\tau_i > 0$, $i = 1, \ldots, N + 1$, in each direction.

As mentioned above, the cost on link i in period t is $d_i(t) = \beta\tau_i + \alpha p_i(t)$ where $p_i(t)$ is the price on link i (see below). At time t, source i routes all its traffic in the direction, counterclockwise or clockwise, with the smaller cost. The ring network is particularly simple because the routing of the whole network can be represented by a single number r. Note that under shortest path routing, if node i sends in the counterclockwise direction, so must node $i-1$, and if node i sends in the clockwise direction, so must node $i+1$. Hence, we can represent routing on the network by $r \in \{0, \ldots, N\}$ with the interpretation that nodes $1, \ldots, r$ send in the counterclockwise direction and nodes $r+1, \ldots, N$ send in the clockwise direction.

For this special case, we now show that the duality gap is trivial, that shortest-path routing based just on prices ($\beta = 0$) indeed solves the primal and dual problems, but is unstable. Using a continuous model, we then show that routing can be stabilized if the weight β on the fixed delay is nonzero and the weight α on price is small enough. The maximum achievable utility however decreases with small α. There is thus an inevitable tradeoff between utility maximization and routing stability.

3.1 Utility maximization and shortest-path routing

Suppose all sources i have the same utility function $U(x_i)$, and all links have the same capacity of $c = 1$ unit. We assume that U is *strictly* concave increasing and differentiable. Then at any time, only link 1, in the counterclockwise direction, and link $N+1$, in the clockwise direction, can be saturated and have strictly positive price. The utility maximization problem (4)–(5) reduces to the following simple form:

$$\max_{r \in \{0,\ldots,N\}} \ \max_{x_i} \quad \sum_i U(x_i) \tag{8}$$

$$\text{subject to} \quad \sum_{i=1}^{r} x_i \leq 1 \quad \text{and} \quad \sum_{i=r+1}^{N} x_i \leq 1 \tag{9}$$

When routing is r, nodes $i = 1, \ldots, r$ see price $p_1(r)$ on their paths while nodes $i = r+1, \ldots, N$ see price $p_{N+1}(r)$ on their paths. Since these rates $x_i(r)$ and prices $p_i(r)$ are primal and dual optimal, they satisfy [12]

$$U'(x_i(r)) \ = \ p_1(r) \qquad \text{for } i = 1, \ldots, r \tag{10}$$

$$U'(x_i(r)) \ = \ p_{N+1}(r) \qquad \text{for } i = r+1, \ldots, N \tag{11}$$

This implies that $x_1(r) = \cdots = x_r(r)$ and $x_{r+1}(r) = \cdots = x_N(r)$. It is easy to see that the optimal routing $r^* \neq 0$ or N. Hence both constraints are active at optimality, implying that (from (9))

$$x_1(r) = \cdots = x_r(r) = \frac{1}{r} \quad \text{and} \quad x_{r+1}(r) = \cdots = x_N(r) = \frac{1}{N-r} \tag{12}$$

The problem (8)–(9) thus becomes

$$\max_{r \in \{1,\ldots,N-1\}} \quad r\, U\left(\frac{1}{r}\right) \ + \ (N-r)\, U\left(\frac{1}{N-r}\right)$$

Dividing the objective function by N and using the strict concavity of U, we have

$$\frac{r}{N}\,U\left(\frac{1}{r}\right) \;+\; \frac{N-r}{N}\,U\left(\frac{1}{N-r}\right) \;\;\leq\;\; U\left(\frac{2}{N}\right)$$

with equality if and only if $r = N/2$. This implies that the optimal routing is

$$r^* \;\;:=\;\; \lfloor N/2 \rfloor \tag{13}$$

and the maximum utility is

$$V^* \;\;:=\;\; \left\lfloor \frac{N}{2} \right\rfloor \, U\left(\frac{1}{\lfloor N/2 \rfloor}\right) \;+\; \left\lceil \frac{N}{2} \right\rceil \, U\left(\frac{1}{\lceil N/2 \rceil}\right) \tag{14}$$

where $\lfloor y \rfloor$ is the largest integer less or equal to y and $\lceil y \rceil$ is the smallest integer greater or equal to y.

It can be shown that there is no duality gap for the ring network considered here when N is even, by verifying that routing r^* in (13), rates $x_i(r^*)$ in (12), and prices $p_1(r^*), p_{N+1}(r^*)$ in (10)–(11) are indeed primal-dual optimal. When N is odd, there is generally a duality gap due to integral constraint on r; see Appendix 6.1 for a proof. This duality gap disappears in the convexified problem when routing is allowed to take real value in $[0, N]$, a model we consider in the next subsection. This suggests that TCP together with shortest-path routing based on prices can potentially maximize utility for this ring network. We next show, however, that shortest-path routing based only on prices is unstable.

Given routing r, we can combine (10)–(11) and (12) to obtain the prices $p_1(r)$ and $p_{N+1}(r)$ on links 1 and $N+1$:

$$p_1(r) \;=\; U'\left(\frac{1}{r}\right) \quad \text{and} \quad p_{N+1}(r) \;=\; U'\left(\frac{1}{N-r}\right) \tag{15}$$

The path cost for node i in the counterclockwise direction is

$$D^-(i;r) \;\;=\;\; \sum_{j=1}^{i} \beta\tau_j + \alpha p_1(r) \;\;=\;\; \beta \sum_{j=1}^{i} \tau_j + \alpha U'\left(\frac{1}{r}\right) \tag{16}$$

and the path cost in the clockwise direction is

$$D^+(i;r) \;\;=\;\; \sum_{j=i+1}^{N+1} \beta\tau_j + \alpha p_{N+1}(r) \;\;=\;\; \beta \sum_{j=i+1}^{N+1} \tau_j + \alpha U'\left(\frac{1}{N-r}\right) \tag{17}$$

In the next period, each node i will choose counterclockwise or clockwise direction according as $D^-(i;r)$ or $D^+(i;r)$ is smaller. Define $f(r)$ as

$$f(r) \;\;:=\;\; \max\,\{i \mid D^-(i;r) \leq D^+(i;r)\} \tag{18}$$

Then the resulting routing satisfies the recursive relation

$$r(t+1) = \begin{cases} 0 & \text{if } D^-(1;r(t)) > D^+(1;r(t)); \\ N & \text{if } D^-(N;r(t)) < D^+(N;r(t)); \\ f(r(t)) & \text{otherwise} \end{cases} \tag{19}$$

Theorem 2. *If $\beta = 0$ and $\alpha > 0$, then starting from any routing $r(0)$, except possibly the equilibrium $N/2$ when N is even, the subsequent routing oscillates between 0 and N.*

Proof. For any $r(0) \in \{0, \ldots, N\}$,

$$\begin{aligned} D^-(1; r(0)) - D^+(1; r(0)) &= D^-(N; r(0)) - D^+(N; r(0)) \\ &= \alpha \left(U' \left(\frac{1}{r(0)} \right) - U' \left(\frac{1}{N - r(0)} \right) \right) \end{aligned}$$

If N is even, then $N/2$ is the unique equilibrium routing that solves $D^-(i; N/2) = D^+(i; N/2)$. Suppose $r(0) \neq N/2$. If $r(0) > N/2$, then $1/r(0) < 2/N < 1/(N - r(0))$. Since U' is strictly decreasing, $U'(1/r(0)) > U'(1/(N - r(0))$ and hence $D^-(1; r(0)) > D^+(1; r(0))$ and $r(1) = 0$. Similarly, if $r(0) < N/2$, then $D^-(N; r(0)) < D^+(N; r(0))$ and $r(1) = N$. Hence r oscillates between 0 and N henceforth. □

Theorem 2 says that purely dynamic routing based on prices is unstable and hence we will not consider this strategy any further. For the rest of the paper, we will, without loss of generality, set $\beta = 1$ and consider the effect of α on utility maximization and stability.

3.2 Maximum utility of shortest-path routing

As mentioned above, the duality gap is of a trivial kind that disappears when integer constraint on routing is relaxed. For the rest of this section, we consider a continuous model where every point on the ring is a source. A point on the ring is labeled by $s \in [0, 1]$ and the common destination is the point 0 (or equivalently 1). The utility maximization problem becomes

$$\max_{r \in [0,1]} \max_{x(\cdot)} \quad \int_0^1 U(x(u))du \tag{20}$$

$$\text{subject to} \quad \int_0^r x(u)du \le 1 \tag{21}$$

$$\int_r^1 x(u)du \le 1 \tag{22}$$

As in the discrete case, both constraints are active at optimality, and hence the problem reduces to

$$\max_{r \in (0,1)} \quad rU\left(\frac{1}{r}\right) + (1 - r)U\left(\frac{1}{1 - r}\right)$$

which, by concavity, yields the optimal routing r^* and maximum utility V^*:

$$r^* = \frac{1}{2} \quad \text{and} \quad V^* = U(2) \tag{23}$$

To see that there is no duality gap, note that the problem (20)–(22) is equivalent to:

$$\begin{aligned} \max_{r \in [0,1]} \max_{x^-, x^+ \ge 0} \quad & rU(x^-) + (1 - r)U(x^+) \\ \text{subject to} \quad & rx^- \le 1, \quad (1 - r)x^+ \le 1 \end{aligned}$$

Define the Lagrangian as

$$L(r, x^-, x^+, p^-, p^+) \;\;=\;\; rU(x^-) + (1-r)U(x^+) + p^-(1 - rx^-) + p^+(1 - (1-r)x^+)$$

It is easy to verify that

$$r^* = \frac{1}{2}, \;\; x^{-*} = x^{+*} = 2, \;\; p^{-*} = p^{+*} = U'(2) \tag{24}$$

are primal-dual optimal and there is no duality gap; see Appendix 6.2.

We now look at the maximum utility achievable by the equilibrium of shortest-path routing.

Let the delay from s to the destination in the counterclockwise direction be

$$T(s) \;\;:=\;\; \int_0^s \tau(u)du$$

and the delay in the clockwise direction be

$$T(1) - T(s) \;\;=\;\; \int_s^1 \tau(u)du$$

where $\tau(u)$, $u \in [0,1]$, is given. Here, $\tau(u)$ corresponds to link cost in the discrete model. Given routing $r \in [0,1]$, the price in the counterclockwise direction is $U'(1/r)$ and the price in the clockwise direction is $U'(1/(1-r))$. Then the cost of source s in the counterclockwise direction is

$$D^-(s;r) \;\;=\;\; T(s) + \alpha U'\left(\frac{1}{r}\right) \tag{25}$$

and the cost in the clockwise direction is

$$D^+(s;r) \;\;=\;\; T(1) - T(s) + \alpha U'\left(\frac{1}{1-r}\right) \tag{26}$$

A routing r is in equilibrium if the costs of source r in both directions are the same.

Definition 3. *A routing r is called an* equilibrium routing *if $D^-(r;r) = D^+(r;r)$. It is denoted by r_α or $r(\alpha)$.*

By definition, r_α is the solution of

$$g(r) \;\;:=\;\; 2T(r) - T(1) + \alpha\left(U'\left(\frac{1}{r}\right) - U'\left(\frac{1}{1-r}\right)\right) \;=\; 0 \tag{27}$$

Since $g(0) < 0$, $g(1) > 0$ and $g'(r) > 0$, the equilibrium r_α is in $(0,1)$ and is unique.

Given a routing r, its utility is

$$V(r) \;\;:=\;\; rU\left(\frac{1}{r}\right) + (1-r)U\left(\frac{1}{1-r}\right)$$

The maximum utility achieved by shortest-path routing, with parameter α, is then $V(r_\alpha) \le V(r^*) = V^*$. The next result implies that r_α varies between r_0 and r^* and converges monotonically to r^* as $\alpha \to \infty$. As a result, the loss $V^* - V(r_\alpha) \ge 0$ in utility also approaches 0 as $\alpha \to \infty$. Denote the interval in which $1/r_\alpha$ and $1/(1-r_\alpha)$ vary as $I := [2, 1/\min\{r_0, 1-r_0\}]$.

Theorem 4. *Suppose U'' exists and is bounded on I. For all $\alpha \geq 0$, $|r_\alpha - r^*|$ is a strictly decreasing function of α. Moreover, as $\alpha \to \infty$, $|r_\alpha - r^*|$ and $V^* - V(r_\alpha)$ approach 0.*

Proof. The equation (27) defines the equilibrium routing $r(\alpha) := r_\alpha$ as an implicit function of α. By the implicit function theorem, $r'(\alpha)$ satisfies

$$r'(\alpha)\left[2\tau(r_\alpha) - \frac{\alpha}{r_\alpha^2}U''\left(\frac{1}{r_\alpha}\right) - \frac{\alpha}{(1-r_\alpha)^2}U''\left(\frac{1}{1-r_\alpha}\right)\right] \;=\; U'\left(\frac{1}{1-r_\alpha}\right) - U'\left(\frac{1}{r_\alpha}\right)$$

The term in the square bracket is positive since U is strictly concave. Hence $r'(\alpha)$ has the same sign as the right-hand side, i.e., since U' is decreasing,

$$r'(\alpha) \;=\; \begin{cases} > \;0 & \text{if } r_\alpha < r^* \\ < \;0 & \text{if } r_\alpha > r^* \\ = \;0 & \text{if } r_\alpha = r^* \end{cases} \tag{28}$$

This implies that $|r_\alpha - r^*|$ is a strictly decreasing function of α; see Figure 3.

Hence $|r_\alpha - r^*|$ converges to a limit as $\alpha \to \infty$. Since U'' is bounded on the closed interval I, so is U'. Hence, from (27), we must have $U'(1/r_\alpha) - U'(1/(1-r_\alpha)) \to 0$, or

$$U'(1/\lim_{\alpha\to\infty} r_\alpha) \;=\; U'(1/(1-\lim_{\alpha\to\infty} r_\alpha))$$

Since U' is strictly decreasing, this implies that $\lim_{\alpha\to\infty} r_\alpha = 1 - \lim_{\alpha\to\infty} r_\alpha = r^*$.

To show that $V^* - V(r_\alpha) \geq 0$ also converges to 0, note that $V'(r^*) = 0$ and hence we have, by Taylor expansion,

$$V(r_\alpha) - V^* \;=\; \frac{1}{2}V''(u)(r_\alpha - r^*)^2$$

for some u between r_α and r^*. Here

$$\begin{aligned} V''(u) &= \frac{1}{u^3}U''\left(\frac{1}{u}\right) + \frac{1}{(1-u)^3}U''\left(\frac{1}{1-u}\right) \\ &\geq -\frac{2\mu}{(\min\{r_0, 1-r_0\})^3} \end{aligned}$$

where μ is the upper bound of U'' on I. Hence

$$0 \;\leq\; V^* - V(r_\alpha) \;\leq\; \frac{\mu(r_\alpha - r^*)^2}{(\min\{r_0, 1-r_0\})^3}$$

Since $|r_\alpha - r^*| \to 0$, the proof is complete. ☐

The shape of $r'(\alpha)$ in (28) implies that, if $r(0) > r^*$ then $r(\alpha) \geq r^*$ for all α but $r(\alpha)$ decreases to r^* as $\alpha \to \infty$, and if $r(0) < r^*$ then $r(\alpha) \leq r^*$ for all α but $r(\alpha)$ increases to r^* monotonically, as illustrated in Figure 3. This is a consequence of the continuity of $r(\alpha)$.

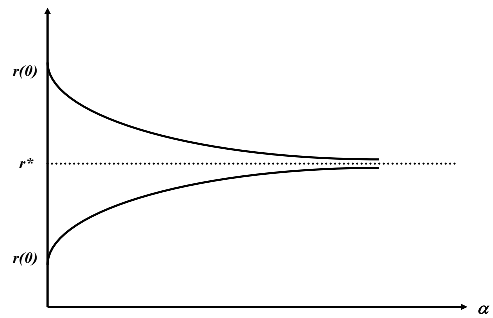

Figure 3: $r(\alpha)$

3.3 Stability of shortest-path routing

We now turn to the stability of r_α. For simplicity, we will take $U(x) = \log x$, the utility function of TCP Vegas [13]. With $\log$ utility function, $V'(r_\alpha) = \log(1-r)/r$ and hence Theorem 4 can be strengthened to show that $V^* - V(r_\alpha)$ is a strictly decreasing function of α, and hence converges monotonically to 0 as $\alpha \to \infty$.

Given r, let $f(r)$ denote the solution of

$$D^-(s;r) \quad = \quad D^+(s;r)$$

It is in the range $[0,1]$ if and only if $0 \le T(s) \le T(1)$, or if and only if

$$r^* - \frac{T(1)}{2\alpha} \ \le \ r \ \le \ r^* + \frac{T(1)}{2\alpha}$$

We will assume that $\min_{u \in [0,1]} \tau(u) > 0$. Then T^{-1} exists and

$$f(r) \quad = \quad T^{-1}\left(\frac{1}{2}(T(1)+\alpha) - \alpha r\right) \tag{29}$$

The routing iteration is

$$r(t+1) \quad = \quad [f(r(t))]_0^1 \tag{30}$$

where $[r]_0^1 = \max\{0, \min\{1, r\}\}$.

Definition 5. *The equilibrium routing r_α is* (globally) stable *if starting from any routing $r(0)$, $r(t)$ defined by (29)–(30) converges to r_α as $t \to \infty$.*

Example 6. *Suppose delay is uniform on the ring, $\tau(u) = \tau$ for all $u \in [0,1]$, so that $T(r) = r\tau$. From (27), the equilibrium routing is*

$$r_\alpha \quad = \quad \frac{1}{2} \quad = \quad r^*, \quad \forall \alpha \ge 0$$

coinciding with the utility-maximizing routing r^*. Suppose $\alpha < \tau$. Then the routing iteration becomes

$$r(t+1) \;=\; \frac{1}{2\tau}(\tau+\alpha) - \frac{\alpha}{\tau}r(t) \;=\; f(r(t))$$

Since $|f(s)-f(r)| = (\alpha/\tau)|s-r| < |s-r|$, $f(r)$ is a contraction mapping and hence r_α is globally stable for all $0 \le \alpha < \tau$.

Hence for the uniform delay case, adding a static component to link cost stabilizes routing provided the weight on prices is smaller than link delay. Moreover, the static component does not lead to any loss in utility ($r_\alpha = r^*$). The stability condition generalizes to the general delay case. The following theorem says that if α is smaller than the minimum 'link delay', then r_α is globally stable; if α is bigger than the maximum 'link delay', then it is globally unstable (diverge from any initial routing except r_α); otherwise, it may converge or diverge depending on initial routing.

Theorem 7. 1. *If $\alpha < \min_{u\in[0,1]} \tau(u)$ then r_α is globally stable.*

2. *Suppose $\alpha \ge T(1)$. Then there exists $\underline{r} < r_\alpha < \overline{r}$ such that*
   (a) *If $r(0) = \underline{r}$ or $r(0) = \overline{r}$ then subsequent routings oscillate between $\overline{r}$ and $\underline{r}$.*
   (b) *If $r(0) < \underline{r}$ or $r(0) > \overline{r}$ then subsequent routings after a finite number of iterations oscillate between 0 and 1.*
   (c) *If $\underline{r} < r(0) < \overline{r}$ then $r(t)$ converges to r_α provided $\alpha < \min_{u\in(\underline{r},\overline{r})} \tau(u)$.*
3. *If $\alpha > \max_{u\in[0,1]} \tau(u)$ then starting from any initial routing $r(0) \neq r_\alpha$, subsequent routings after a finite number of iterations oscillate between 0 and 1.*

Proof. 1. We show that the routing iteration (30) is a contraction mapping if $\alpha < \min_{u\in[0,1]} \tau(u)$. Now

$$\begin{aligned}
\left|[f(s)]_0^1 - [f(r)]_0^1\right| &\le |f(s)-f(r)| \\
&= \left|T^{-1}\left(\frac{1}{2}(T(1)+\alpha) - \alpha s\right) - T^{-1}\left(\frac{1}{2}(T(1)+\alpha) - \alpha r\right)\right| \\
&= \left|\frac{1}{T'(u)}\,(\alpha s - \alpha r)\right| \\
&\le \frac{\alpha}{\min_{u\in[0,1]} \tau(u)}\,|s-r|
\end{aligned}$$

for some u between r and s, by the mean value theorem. Hence $h(r)$ is a contraction mapping and starting from any $r(0) \in [0,1]$, $r(t)$ converges exponentially to r_α.

2. Define

$$h(r) \;=\; \frac{1}{2}(T(1)+\alpha) - \alpha r$$

Then the routing iteration can be written as

$$T(r(t+1)) \;=\; [h(r(t))]_0^1 \tag{31}$$

Define the following sequences:

$$\begin{aligned} a_0 &= 0, \qquad & b_0 = T(0) \\ a_{n+1} &= h^{-1}(b_n), \qquad & b_{n+1} = T(a_{n+1}) \end{aligned}$$

Note that $(a_n, n \geq 0)$ is a routing sequence going backward in time.

The following lemma is proved in the appendix, following [10].

Lemma 8. *Let* $b_\alpha = T(r_\alpha) = h(r_\alpha)$. *Then*

$$\begin{aligned} 0 = a_0 \;<\; a_2 \;<\; \ldots \;<\; r_\alpha \;<\; \ldots \;<\; a_3 \;<\; a_1 < 1 \\ T(0) = b_0 \;<\; b_2 \;<\; \ldots \;<\; b_\alpha \;<\; \ldots \;<\; b_3 \;<\; b_1 < T(1) \end{aligned}$$

Since the sequences are monotone, the lemma implies that there are $\underline{r}$ and $\overline{r}$ with $0 < \underline{r} < r_\alpha < \overline{r} < 1$ such that

$$\lim_{n\to\infty} a_{2n} = \underline{r} \quad \text{and} \quad \lim_{n\to\infty} a_{2n+1} = \overline{r}$$

By continuity of T and h, we have

$$T(\underline{r}) = h(\overline{r}) \quad \text{and} \quad T(\overline{r}) = h(\underline{r})$$

This implies that starting from $r(0) = \underline{r}$ or $r(0) = \overline{r}$, the subsequent routings oscillate between $\underline{r}$ and $\overline{r}$.

To show the second claim, suppose $r(0) < \underline{r}$. Specifically, suppose $a_{2n-2} < r(0) < a_{2n}$ for some n. If $h(r(0)) > T(1)$ (possible since $\alpha \geq T(1)$), then $r(1) = 1$ and subsequent routings oscillate between 0 and 1. Otherwise, from (31), $r(0) = h^{-1}(T(r(1)))$, and hence $a_{2n-2} < h^{-1}(T(r(1))) < a_{2n}$. Since h is strictly decreasing, we have $b_{2n-1} < T(r(1)) < b_{2n-3}$ by definition of b_n. Hence, since T is strictly increasing, $a_{2n-1} < r(1) < a_{2n-3}$. The same argument then shows that $a_{2n-4} < r(2) < a_{2n-2}$. Hence we have shown that $r(0) < a_{2n}$ implies $r(2) < a_{2n-2}$. This proves the second claim.

The proof of the third claim follows the same argument of part 1.

3. By the mean value theorem, we have

$$|h^{-1}(T(a)) - h^{-1}(T(a'))| = \frac{T'(u)}{\alpha}|a - a'|$$

for some u between a and a'. Hence the iteration map

$$a_{n+1} = h^{-1}(T(a_n))$$

is a contraction provided $\alpha > \max_{u\in[0,1]} \tau(u)$. This implies that the sequence $(a_n, n \geq 0)$ converges and, since r_α is the unique fixed point of $h^{-1}(T(\cdot))$, $\underline{r} = \overline{r} = r_\alpha$. The assertion then follows from part 2(b). □

4 General topology: simulations

It seems difficult to derive an analytical bound on α to guarantee routing stability or to compute optimal routing for general network. In this section, we present simulation results to illustrate that the intuition from the simple ring network analyzed in the last section extends to general topology.

We generate the random network based on Waxman's algorithm [18]. The nodes are uniformly distributed in a two dimensional plane. The probability that a pair of nodes u, v are connected is given by

$$\text{Prob}(u, v) = a \exp\left(\frac{d(u, v)}{bL}\right)$$

where the maximum probability $a > 0$ controls connectivity, $b \leq 1$ controls the length of the edges with a larger b favoring longer edges, $d(u, v)$ is the Euclidean distance between nodes u and v, and L is the maximum distance between any two nodes.

In our simulation, we set the number of nodes $N = 30$, $a = 0.8$, $b = 0.3$, which generated about 200 links; see Figure 4.

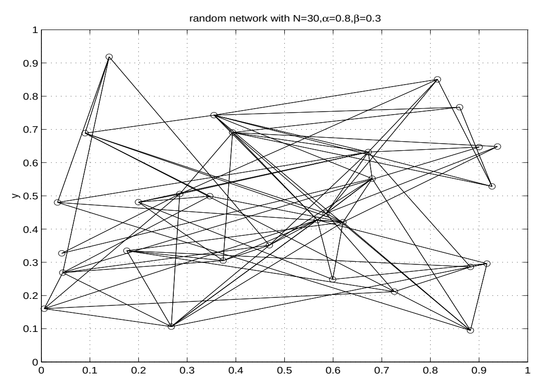

Figure 4: A random network

The fixed delay τ_l of each link l is randomly chosen according to a uniform distribution over [100, 400]ms. The link capacities are randomly chosen from the interval [1000, 4000] packets/sec, also with uniform distribution. There are 60 flows on the network with randomly chosen source and destination nodes.

Routing on this network is computed using Bellman-Ford shortest-path algorithm, with link cost $d_l(t) = \tau_l + ap_l(t)$ in each update period t, on a slower timescale than congestion control. In each routing period t, we first solve the link prices based on the current routing, using the gradient projection algorithm of [12]. We iterate the source algorithm to update rates and the link algorithm to update prices, until they converge. The link prices are then used to compute the shortest paths for the next period.

We measure the performance of the scheme at different α by the sum of all source's utilities. If the routing is stable (at small α), the aggregate utility is computed using the equilibrium routing.

Otherwise, the routing oscillates and the time-averaged aggregate utility is used. The result is shown in Figure 5.

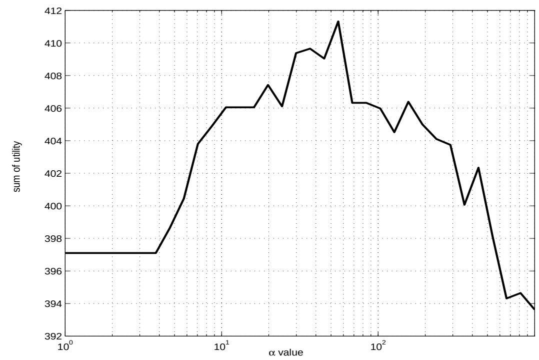

Figure 5: Aggregate utility as a function of α for random network

As expected, when α is small, routing is stable and the aggregate utility increases with α, as in the ring network analyzed in Section 3.2 (Theorem 4). When $\alpha < 4$, the static delay τ_l dominates the link cost and the routes computed with $d_l(t)$ remain the same as with static routing ($\alpha = 0$), and hence the aggregate utility is independent of α. Routing becomes unstable at around $\alpha = 10$. Even though the time-averaged utility continues to rise after routing instability sets in, eventually it peaks and drops to a level less than the utility achievable by the necessarily stable static routing.

5 Conclusion

Given a routing, TCP-AQM can be interpreted as a distributed primal-dual algorithm over the Internet to maximize aggregate utility over source rates. In this paper, we study whether TCP-AQM together with shortest-path routing can maximize utility over both source rates and routing, on a slower timescale. The answer is generally negative, because the problem of maximizing utility over both rates and routes is NP-hard and thus cannot always be solved by shortest-path routing. We exhibit an inevitable tradeoff between routing stability and utility maximization. For the special case of ring network with a common destination, we prove rigorously that shortest-path routing based purely on prices is unstable, and adding a sufficiently large static component to the link distance metric that is independent of congestion stabilizes it. The maximum utility achievable by shortest-path routing, however, decreases with the weight on the static component. Simulations suggest that these conclusions extend qualitatively in general network topology. Furthermore, they show that routing instability can reduce utility to less than that achievable by static routing which is necessarily stable.

The duality model of TCP-AQM has been useful to understand the equilibrium properties, including throughput loss, delay, and fairness, of large-scale networks under TCP-AQM control. This paper is a first attempt to apply the same methodology to understand the interaction of TCP-AQM and shortest-path routing. The analytical model, though provides clear insights on the tradeoff between utility maximization and routing stability, is simplistic. Although the simulation results provide hope that these insights may hold in a more realistic setting, we have not been able to find an analytical proof. One of the major difficulties is that, in a general network, shortest-path routing cannot be as conveniently represented as is done here, and a new approach will be needed to analyze the general case.

6 Appendix

6.1 Duality gap when N is odd

We prove that there is generally a duality gap between the primal problem (8)–(9) and its dual when N is odd.

It is easy to see that the primal optimal routing is

$$r^* = \frac{N-1}{2} \quad \text{or} \quad \frac{N+1}{2}$$

Suppose without loss of generality that $r^* = (N-1)/2$ (the other case is similar). Then, the source rates are

$$x_1 = \cdots = x_{r^*} = \frac{2}{N-1} \quad \text{and} \quad x_{r^*+1} = \cdots = x_N = \frac{2}{N+1}$$

yielding a primal objective value of

$$\begin{aligned} \frac{N-1}{2}U\left(\frac{2}{N-1}\right) + \frac{N+1}{2}U\left(\frac{2}{N+1}\right) &= N\left\{ \left(\frac{1}{2}-\frac{1}{N}\right)U\left(\frac{2}{N-1}\right) + \left(\frac{1}{2}+\frac{1}{N}\right)U\left(\frac{2}{N+1}\right) \right\} \\ &< NU\left(\frac{2}{N}\right) \end{aligned}$$

where the last inequality follows from the strict concavity of U. We now show that the right-hand side is the optimal dual objective value, and hence there is a duality gap.

The dual problem of (8)–(9) is (e.g., [12])

$$\min_{p_1, p_{N+1} \geq 0} \quad \sum_{i=1}^{N} \max_{x_i} \left(U(x_i) - x_i \, \min\{p_1, p_{N+1}\}\right) + (p_1 + p_{N+1})$$

First, note that the minimizing (p_1, p_{N+1}) must satisfy $p_1 = p_{N+1}$, for otherwise, if (say) $p_1 < p_{N+1}$, then the dual objective value is

$$\sum_{i=1}^{N} \max_{x_i} \left(U(x_i) - x_i p_1\right) + (p_1 + p_{N+1})$$

and can be reduced by decreasing p_{N+1} to p_1. Hence the dual problem is equivalent to

$$\min_{p \geq 0} \quad \sum_{i=1}^{N} \max_{x_i} \left(U(x_i) - x_i \, p\right) + 2p \tag{32}$$

Let p^* denote the minimizer and $x_i^* = x_i(p^*) = x(p^*) =: x^*$ denote the corresponding maximizers (they are equal for all i by symmetry). Then we have

$$U'(x^*) \quad = \quad p^* \tag{33}$$

Differentiating the objective function in (32) with respect to p and setting it to zero, we have

$$0 \quad = \quad N(U'(x^*)x'(p^*) - p^*x'(p^*) - x^*) + 2 \tag{34}$$

Using (33), we have

$$x^* \quad = \quad \frac{2}{N}$$

and hence the minimum dual objective value is

$$N(\max_{x^*} U(x^*) - x^* p^*) + 2p^* \quad = \quad NU\left(\frac{2}{N}\right)$$

as desired. □

6.2 Primal-dual optimality

We prove that the solution given by (24) is primal-dual optimal using the saddle-point theorem (e.g., [1, pp.427]). Clearly, (r^*, x^{-*}, x^{+*}) is primal feasible and (p^{-*}, p^{+*}) is dual feasible. We now show that $(r^*, x^{-*}, x^{+*}, p^{-*}, p^{+*})$ is a saddle point, i.e., for all (r, x^-, x^+, p^-, p^+),

$$L(r, x^-, x^+, p^{-*}, p^{+*}) \;\; \leq \;\; L(r^*, x^{-*}, x^{+*}, p^{-*}, p^{+*}) \;\; \leq \;\; L(r^*, x^{-*}, x^{+*}, p^-, p^+) \tag{35}$$

For the right inequality, substitute (r^*, x^{-*}, x^{+*}) from (24) into $L(r^*, x^{-*}, x^{+*}, p^-, p^+)$ to get, for all (p^-, p^+),

$$L(r^*, x^{-*}, x^{+*}, p^-, p^+) \quad = \quad U(2)$$

But $U(2) = L(r^*, x^{-*}, x^{+*}, p^{-*}, p^{+*})$, establishing the right inequality.

For the left inequality, we have from (24), denoting $p^* := p^{-*} = p^{+*}$,

$$\begin{aligned} L(r, x^-, x^+, p^{-*}, p^{+*}) \quad &= \quad rU(x^-) + (1-r)U(x^+) - (rx^- + (1-r)x^+)p^* + 2p^* \\ &\leq \quad U(y) - yp^* + 2p^* \qquad \text{(concavity of } U \end{aligned} \tag{36}$$

with $y := rx^- + (1-r)x^+$, where equality holds if and only if $x^- = x^+$ since U is *strictly* concave. Notice that the right-hand side is maximized over y if and only if y satisfies

$$U'(y) \quad = \quad p^*$$

This implies that $y = x^{-*} = x^{+*} = 2$ since U' is strictly monotonic. Substitute $y = 2$ into (36) yields, for all (r, x^-, x^+),

$$L(r, x^-, x^+, p^{-*}, p^{+*}) \quad \leq \; U(2)$$

as desired, since $U(2) = L(r^*, x^{-*}, x^{+*}, p^{-*}, p^{+*})$. □

6.3 Proof of Lemma 8

We will prove the lemma by induction. Note that $b_0 < b_\alpha$ implies that $a_1 = h^{-1}(b_0) > h^{-1}(b_\alpha) = r_\alpha$. Since $\alpha \geq T(1)$ and $h(1) < 0$, $a_1 = h^{-1}(b_0) < 1$ (see Figure 6). Hence

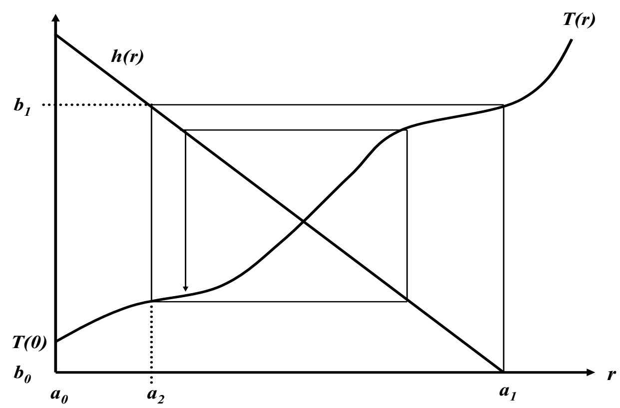

Figure 6: Lemma 8.

$$0 = a_0 < r_\alpha < a_1 < 1$$

This implies that $b_1 = T(a_1)$ satisfies

$$T(0) = b_0 < b_\alpha < b_1 < T(1)$$

Since $b_1 < T(1) < h(0)$, $a_2 = h^{-1}(b_1) > h^{-1}(h(0)) = 0$, we have

$$0 = a_0 < a_2 < a_\alpha < a_1 < 1$$

Let the induction hypothesis be

$$\begin{array}{ccccccccccc} a_0 & < & \ldots & < & a_{2n} & < & r_\alpha & < & a_{2n-1} & < & \ldots & < & a_1 \\ b_0 & < & \ldots & < & b_{2n-2} & < & b_\alpha & < & b_{2n-1} & < & \ldots & < & b_1 \end{array}$$

Then $b_{2n} = T(a_{2n}) > T(a_{2n-2}) = b_{2n-2}$ and that $b_{2n} = T(a_{2n}) < T(r_\alpha) = b_\alpha$. Hence,

$$b_{2n-2} \;<\; b_{2n} \;<\; b_\alpha$$

This implies that $r_\alpha < a_{2n+1} < a_{2n-1}$, which in turn implies that $b_\alpha < b_{2n+1} < b_{2n-1}$. This completes the induction. □

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