# Global Finite-Time Convergence of TCP Vegas without Feedback Information Delay

**Joon-Young Choi***, **Kyungmo Koo, Jin S. Lee, and Steven H. Low**

**Abstract:** We prove that TCP Vegas globally converges to its equilibrium point in finite time assuming no feedback information delay. We analyze a continuous-time TCP Vegas model with discontinuity and high nonlinearity. Using the upper right-hand derivative and applying the comparison lemma, we cope with the discontinuous signum function in the TCP Vegas model; using a change of state variables, we deal with the high nonlinearity. Although we ignore feedback information delay in analyzing the model of TCP Vegas, the simulation results illustrate that TCP Vegas in the presence of feedback information delay shows very similar dynamic trends to TCP Vegas without feedback information delay. Consequently, dynamic properties of TCP Vegas without feedback information delay can be used to estimate those of TCP Vegas in the presence of feedback information delay.

**Keywords:** Congestion control; TCP Vegas; Nonlinear systems; Comparison lemma; Finite-time convergence

## 1. INTRODUCTION

TCP Vegas was introduced as a TCP implementation modified from TCP Reno in [1], and extensive experiments have been conducted to compare the performance of Vegas with that of Reno [1–4]. TCP Vegas was systematically analyzed and presented as a model of discrete-time dynamic system without feedback information delay in [5]. The TCP Vegas model is discontinuous and highly nonlinear, which has caused a theoretical difficulty in analyzing the dynamic properties of TCP Vegas. A sufficient condition for local asymptotic stability of TCP Vegas was presented in the presence of feedback information delay in [6], and a modified version of TCP Vegas was proposed to improve the local stability condition by adding a derivative term of queuing delay to the original TCP Vegas. The work in [6], however, was based on an approximated model assuming the time varying RTT(round trip time) as a constant value and ignoring the nonlinear and discontinuous properties of TCP Vegas dynamics, and showed only local stability results.

Global stability issues have been studied in other congestion control schemes and some initial results have been reported recently in [7–12] under various assumptions and limitations. The global asymptotic stability of the multi-link multi-source network with TCP Vegas sources was proved without feedback information delay in [13], but they used approximated models to avoid the theoretical difficulties associated with the original discontinuous TCP Vegas.

In this paper, we consider the multi-link multi-source network with TCP Vegas sources without feedback information delay. We rigorously analyze a continuous-time model of TCP Vegas with dynamics both in sources and in links without any approximation of the original TCP Vegas model. The continuous-time model has high nonlinearity and discontinuity due to a signum function in the right-hand side of the model equation. To deal with the discontinuous signum function in TCP Vegas model equations, we adopt the upper-right hand derivative and apply the comparison lemma [14]. In addition, to deal with the high nonlinearity in the TCP Vegas model, we make a change of state variables. We prove that assuming no feedback information delay, TCP Vegas globally converges to its equilibrium point in finite time without any other conditions.

Although we ignore feedback information delay in analyzing the model of TCP Vegas, the simulation results illustrate that TCP Vegas in the presence of feedback information delay shows very similar dynamic trends to TCP Vegas without feedback information delay. Consequently, the analysis results on TCP Vegas without considering feedback information delay are of significance in practice, and can be used to analyze and estimate the stability and performance of TCP Vegas in the presence of feedback information delay.

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This paper is organized as follows. Section 2 describes the multi-link multi-source network model of TCP Vegas. Section 3 analyzes the boundedness property of TCP Vegas. Section 4 proves the global finite-time convergence of TCP Vegas. Section 5 provides simulation results and discussions. Section 6 makes conclusions.

## 2. NETWORK MODEL

We consider a network of $L$ communication links shared by $N$ sources, where $L \leqslant N$. Each link indexed by $l$ has a finite transmission capacity $c_l$ $(1 \leqslant l \leqslant L)$ and is assumed to have infinite buffering storage. Each source is indexed by $i$ $(1 \leqslant i \leqslant N)$. The $L \times N$ routing matrix $R$ is defined by its $(l,i)$ elements:

$$R_{li} = \begin{cases} 1 & \text{if source } i \text{ uses link } l \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $R$ is assumed to be fixed and of full row rank to prevent the network from degenerating. Associated with the link $l$ is the queuing delay $p_l(t)$ and with the source $i$ is the source rate $x_i(t)$. Ignoring the feedback information delays in the interconnection, we assume at time $t$ that the source $i$ observes as a feedback signal the aggregated queuing delay in its path

$$q_i(t) := \sum_l R_{li} p_l(t) = R_i^T p(t) \qquad \forall\, 1 \leqslant i \leqslant N, \quad (1)$$

where $R_i$ is the $i$-th column of $R$ and $p^T = [p_1, p_2, \cdots, p_L]$, and link $l$ observes the aggregated source rate

$$y_l(t) := \sum_i R_{li} x_i(t) = R^l x(t) \qquad \forall\, 1 \leqslant l \leqslant L, \quad (2)$$

where $R^l$ is the $l$-th row of $R$ and $x^T = [x_1, x_2, \cdots, x_N]$. The RTT $T_i(t)$ is defined for each source $i$ as $T_i(t) = d_i + q_i(t)$, where $d_i$ is the constant round trip propagation time.

A model of TCP Vegas with its associated queue management was presented as a discrete-time dynamic system in [5]:

$$\begin{aligned} p_l(t+1) &= \left[ p_l(t) + \frac{1}{c_l}\left( \sum_i R_{li} x_i(t) - c_l \right) \right]^+ && (3) \\ w_i(t+1) &= \left[ w_i(t) + \frac{1}{T_i(t)} \operatorname{sgn}(\alpha_i d_i - x_i(t) q_i(t)) \right]^+ && (4) \end{aligned}$$

where $x_i(t) := \frac{w_i(t)}{T_i(t)}$, $w_i(t)$ is the window size of source $i$, $\alpha_i$ is the congestion control parameter of Vegas algorithm, $[z]^+ = \max\{0, z\}$, and $\operatorname{sgn}(z) = -1$ if $z < 0$, 0 if $z = 0$, 1 if $z > 0$. As in [5], the congestion control parameters $\alpha_i$ and $\beta_i$ of Vegas algorithm are assumed to be $\alpha_i = \beta_i$ for simplicity. Using the Euler's method, the discrete-time model of TCP Vegas (3) and (4) is transformed to a continuous-time dynamic system:

$$\begin{aligned} \dot{p}_l &= \frac{1}{c_l} \left( \sum_i R_{li} x_i(t) - c_l \right)^+_{p_l} && (5) \\ \dot{w}_i &= \frac{1}{T_i(t)} \operatorname{sgn}(\alpha_i d_i - x_i(t) q_i(t)) && (6) \end{aligned}$$

where

$$(h)^+_z := \begin{cases} h & \text{if } z > 0 \\ \max(0, h) & \text{if } z = 0. \end{cases}$$

The corresponding equilibrium point $(x_i^*, p_l^*)$ of (5) and (6) satisfies

$$\begin{cases} x_i^* q_i^* = \alpha_i d_i & \forall\, 1 \leqslant i \leqslant N \\ \left(\sum_i R_{li} x_i^* - c_l\right)^+_{p_l} = 0 & \forall\, 1 \leqslant l \leqslant L \end{cases}, \quad (7)$$

and the equilibrium point turns out to be unique as shown in the following lemma.

**Lemma 1:** The equilibrium point of TCP Vegas described by (5) and (6) is unique.

**Proof:** See Appendix 7.1. □

To facilitate the analysis, making the change of variables in (5) and (6) as

$$\begin{aligned} \tilde{x}_i &= x_i - x_i^* \\ s_i &= \frac{1}{w_i} - \frac{q_i}{\alpha_i d_i T_i} && (8) \end{aligned}$$

and using the fact that $\operatorname{sgn}(\alpha_i d_i - x_i q_i) = \operatorname{sgn}(s_i)$, we obtain the resulting system with the new variables $\tilde{x}_i$ and $s_i$:

$$\begin{aligned} \dot{\tilde{x}}_i &= -\frac{\dot{q}_i(t)}{T_i(t)} x_i(t) + \frac{1}{T_i^2(t)} \operatorname{sgn}(s_i(t)) && (9) \\ \dot{s}_i &= -\frac{1}{w_i^2(t) T_i(t)} \operatorname{sgn}(s_i(t)) - \frac{\dot{q}_i(t)}{\alpha_i T_i^2(t)} && (10) \end{aligned}$$

Before analyzing the stability of (9) and (10), it is necessary to consider the existence of solution of the initial-value problem (9) and (10). Since the right-hand sides of (9) and (10) don't satisfy the Lipschitz condition with respect to state variables because of the discontinuous signum function, the existence of solution is not guaranteed in the standard sense [14]. Fortunately, however, the discontinuity comes with only finite jump at $s_i = 0$, and it holds that the solution of (9) and (10) exists for any initial condition in the sense of Filippov [15]. As another property of the solution of TCP Vegas model, the following lemma shows the absolute continuity of the solution and this property will be used in the subsequent analysis.

**Lemma 2:** $p_l(t)$, $w_i(t)$, $\tilde{x}_i(t)$ and $s_i(t)$ developed by (5), (6), (9) and (10) respectively are absolutely continuous with respect to time $t$.

**Proof:** This lemma holds from Theorem 1 of Chapter 0 in [16]. □

## 3. BOUNDEDNESS

In this section, we investigate the boundedness properties of $x_i(t)$, $w_i(t)$, $q_i(t)$, $s_i(t)$, and $p_l(t)$ with respect to time $t$. First, we assume that the initial window size $w_i(0) > 0$ for all $1 \leqslant i \leqslant N$, which is justified by the slow-start phase [1], during which $w_i$ keeps increasing until either the packet is lost somewhere in the network or when some congestion feedback signal arrives from the network. When that happens, the congestion control phase takes over, at which instant the current $w_i > 0$ is set to $w_i(0)$ for the TCP Vegas. We now show in the following lemma that $w_i(t)$ for all $1 \leqslant i \leqslant N$ and $p_l(t)$ for all $1 \leqslant l \leqslant L$ are bounded below.

**Lemma 3:**
*(i)* $w_i(t)$ is bounded below for all $t \geqslant 0$ and for all $1 \leqslant i \leqslant N$ as

$$w_i(t) \geqslant w_i^{LB} := \min(w_i(0), \alpha_i d_i) > 0. \tag{11}$$

*(ii)* $p_l(t) \geqslant 0$ for all $t \geqslant 0$ and for all $1 \leqslant l \leqslant L$.

**Proof:** $(i)$ Whenever $0 < w_i < \alpha_i d_i$, it holds that

$$\begin{aligned} s_i &= \frac{1}{w_i}\left(1 - \frac{q_i w_i}{\alpha_i d_i T_i}\right) > 0 \quad \text{and} \\ \dot{w}_i &= \frac{1}{T_i}\operatorname{sgn}(s_i) > 0, \end{aligned}$$

from which we have $w_i \geqslant \min(w_i(0),\ \alpha_i d_i) > 0$ for all $t \geqslant 0$ and for all $1 \leqslant i \leqslant N$.

$(ii)$Since $\dot{p}_l = \left[\frac{1}{c_l}(y_l - c_l)\right]^+_{p_l}$ and $p_l(0) \geqslant 0$ for all $1 \leqslant l \leqslant L$, it is obvious that $p_l(t) \geqslant 0$ for all $t \geqslant 0$ and for all $1 \leqslant l \leqslant L$. □

We now show in the following that $x_i(t)$, $w_i(t)$, $q_i(t)$, and $s_i(t)$ are bounded above for all $1 \leqslant i \leqslant N$, and $p_l(t)$ is bounded above for all $1 \leqslant l \leqslant L$.

**Lemma 4:**
*(i)* $x_i(t)$ is bounded above for all $t \geqslant 0$ and for all $1 \leqslant i \leqslant N$ as

$$x_i(t) \leqslant x_i^{UB} := \max\left(x_i(0),\ 2\max_{l \in R_i^T} c_l,\ \frac{1}{d_i}\right).$$

*(ii)* $w_i(t)$ is bounded above for all $t \geqslant 0$ and for all $1 \leqslant i \leqslant N$ as

$$\begin{aligned} &w_i(t) \leqslant w_i^{UB} := \\ &\max\left(w_i(0),\ x_i^{UB}(d_i+1),\ \alpha_i d_i(d_i+1)\right). \end{aligned}$$

*(iii)* $q_i(t)$ is bounded above for all $t \geqslant 0$ and for all $1 \leqslant i \leqslant N$ as

$$q_i(t) \leqslant q_i^{UB} := \max\left(q_i(0), \frac{\max_{l \in R_i^T} |R^l|}{\min_{l \in R_i^T} c_l} \max_{i \in R^l} w_i^{UB}\right).$$

*(iv)* $s_i(t)$ is bounded above for all $t \geqslant 0$ and for all $1 \leqslant i \leqslant N$ as

$$s_i(t) \leqslant s_i^{UB} := \frac{1}{w_i^{LB}} + \frac{1}{\alpha_i d_i}.$$

*(v)* $p_l(t)$ is bounded above for all $t \geqslant 0$ and for all $1 \leqslant l \leqslant L$ as

$$p_l(t) \leqslant p_l^{UB} := \max\left(p_l(0),\ \frac{|R^l|}{c_l} \max_{i \in R^l} w_i^{UB}\right),$$

where $l \in R_i^T$ means that link $l$ belongs to the links that the source $i$ passes through, and $i \in R^l$ means that the source $i$ belongs to the sources that passes through the link $l$.

**Proof:** See Appendix 7.2. □

## 4. CONVERGENCE

In this section, we investigate the global finite-time convergence of TCP Vegas described by (9) and (10). To deal with the discontinuity in (9) and (10), we introduce the upper right-hand derivative $D^+V(t)$ defined by

$$D^+V(t) := \limsup_{h \to 0+} \frac{V(t+h) - V(t)}{h},$$

and the comparison lemma [14] stated as follows.

**Lemma 5:** Consider the scalar differential equation

$$\dot{u} = f(t,u),\ \ u(t_0) = u_0$$

where $f(t,u)$ is continuous in $t$ and locally Lipschitz in $u$, for all $t \geqslant 0$ and all $u \in J \subset \mathbb{R}$. Let $[t_0, T)$ ($T$ could be infinity) be the maximal interval of existence of the solution $u(t)$, and suppose $u(t) \in J$ for all $t \in [t_0, T)$. Let $v(t)$ be a continuous function whose upper right-hand derivative $D^+v(t)$ satisfies the differential inequality

$$D^+v(t) \leqslant f(t, v(t)),\ \ v(t_0) \leqslant u_0$$

with $v(t) \in J$ for all $t \in [t_0, T)$. Then, $v(t) \leqslant u(t)$ for all $t \in [t_0, T)$.

The following theorem shows that TCP Vegas described by (9) and (10) is globally finite-time convergent and $x_i(t)$ and $p_l(t)$ converge to their equilibrium points in finite time with any initial condition.

**Theorem 1:** TCP Vegas (9) and (10) is globally finite-time convergent, that is, $x_i(t)$ for all $1 \leqslant i \leqslant N$ and $p_l(t)$ for all $1 \leqslant l \leqslant L$ converge to their equilibrium points in finite time.

**Proof:** We consider a Lyapunov function candidate

$$V(s,\tilde{x}) = \sum_{i=1}^{N} \alpha_i |s_i| + \frac{1}{2}\sum_{l=1}^{L}\frac{1}{c_l}(R^l\tilde{x})^2, \tag{12}$$

which is absolutely continuous from Lemma 2 and bounded by Lemma 4 for all $t \geqslant 0$. Note that the continuity of (12) with respect to time $t$ is a necessary condition to apply the comparison lemma and we can choose the set $J \subset \mathbb{R}$ in the comparison lemma such that $V(t) \in J$ for all $t \geqslant 0$ because of the boundedness of $V(t)$. Equation (12) is not differentiable with the standard derivative when $s_i = 0$, but even when $s_i = 0$, we can calculate the upper right-hand derivative of (12) along the trajectories of (9) and (10):

$$\begin{aligned}
& D^+V(s,\tilde{x}) \\
&= \sum_{i=1}^{N}\left(\alpha_i \frac{s_i \dot{s}_i}{|s_i|}\right) + \sum_{l=1}^{L}\frac{1}{c_l}\left(R^l\tilde{x}\right)\left(R^l\dot{\tilde{x}}\right) \\
&= -\sum_{i=1}^{N}\left(\frac{\alpha_i}{w_i^2 T_i} + \frac{\dot{q}_i}{T_i^2}\mathrm{sgn}(s_i)\right) + \sum_{l=1}^{L}\dot{p}_l\left(R^l\dot{\tilde{x}}\right) \\
&= -\sum_{i=1}^{N}\left(\frac{\alpha_i}{w_i^2 T_i} + \frac{\dot{q}_i}{T_i^2}\mathrm{sgn}(s_i)\right) + \sum_{i=1}^{N}\left(\sum_{l=1}^{L} R_{li}\dot{p}_l\right)\dot{\tilde{x}}_i \\
&= -\sum_{i=1}^{N}\left(\frac{\alpha_i}{w_i^2 T_i} + \frac{\dot{q}_i}{T_i^2}\mathrm{sgn}(s_i)\right) + \sum_{i=1}^{N}\dot{q}_i\dot{\tilde{x}}_i \\
&= -\sum_{i=1}^{N}\left(\frac{\alpha_i}{w_i^2 T_i} + \frac{\dot{q}_i}{T_i^2}\mathrm{sgn}(s_i)\right) \\
&\quad + \sum_{i=1}^{N}\dot{q}_i\left(-\frac{\dot{q}_i}{T_i}x_i + \frac{1}{T_i^2}\mathrm{sgn}(s_i)\right) \\
&= -\sum_{i=1}^{N}\left(\frac{\alpha_i}{w_i^2 T_i}\right) - \sum_{i=1}^{N}\left(\frac{x_i}{T_i}\dot{q}_i^2\right) \\
&= -\sum_{i=1}^{N}\left(\frac{\alpha_i}{w_i^2 T_i}\right) - \sum_{i=1}^{N}\frac{x_i}{T_i}\left(R_i^T\dot{p}\right)^2 \\
&= -\sum_{i=1}^{N}\left(\frac{\alpha_i}{w_i^2 T_i}\right) - \dot{p}^T\left(\sum_{i=1}^{N}\frac{x_i}{T_i}R_iR_i^T\right)\dot{p} \\
&\leq -\sum_{i=1}^{N}\left(\frac{\alpha_i}{w_i^2 T_i}\right),
\end{aligned} \tag{13}$$

where the last inequality is derived by using $\dot{p}^T\left(\sum_{i=1}^{N}\frac{x_i}{T_i}R_iR_i^T\right)\dot{p} \geq 0$. On the other hand, since $w_i(t) \leq w_i^{UB}$ and $q_i(t) \leq q_i^{UB}$ by Lemma 4, we have the inequality

$$\sum_{i=1}^{N}\left(\frac{\alpha_i}{w_i^2 T_i}\right) \geq \sum_{i=1}^{N}\left(\frac{\alpha_i}{w_i^{UB^2}(q_i^{UB}+d_i)}\right) := \varepsilon > 0, \tag{14}$$

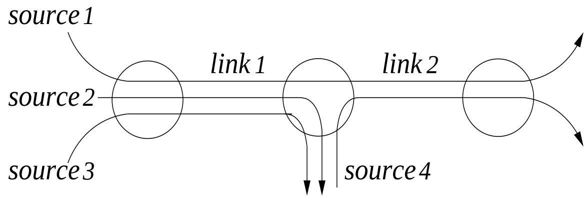



Fig 1: Network topology

from which we obtain the differential inequality

$$D^+V(s,\tilde{x}) \leqslant -\varepsilon.$$

Applying the comparison lemma, we achieve that

$$V(s(t),\tilde{x}(t)) \leqslant V(s(0),\tilde{x}(0)) - \varepsilon t, \tag{15}$$

which implies that $V(s,\tilde{x})$ reaches zero before or at time $t = \frac{V(s(0),\tilde{x}(0))}{\varepsilon}$ [14, p. 553]. $V(s,\tilde{x}) = 0$ implies a set of equations described with the variables $x_i$ and $q_i$: $s_i = 0$, that is, $x_iq_i = \alpha_i d_i$ for $1 \leqslant i \leqslant N$ and $R^l\tilde{x} = \sum_i R_{li}x_i - c_l = 0$ for $1 \leqslant l \leqslant L$, which are identical to (7). Consequently, it is proved by Lemma 1 that $x_i(t)$ for $1 \leqslant i \leqslant N$ and $p_l(t)$ for $1 \leqslant l \leqslant L$ converge to their equilibrium points in finite time. □

It is worth while to take a close look at how the congestion control parameter $\alpha_i$ affects the performance of TCP Vegas. It is shown from (7) that the equilibrium points of $w_i$ and $T_i(= d_i + q_i)$ in (13) increase as $\alpha_i$ increases, which implies that the terms $\frac{\alpha_i}{w_i^2 T_i}$ and $\frac{x_i}{T_i}(= \frac{w_i}{T_i^2})$ in (13) decrease in the steady state as $\alpha_i$ increases. Therefore, the sizes of negative terms in (13) decrease as $\alpha_i$ increases, and it is concluded that the convergent time of TCP Vegas lengthens as $\alpha_i$ increases, which will be illustrated in the simulation section. Note, however, that an extremely small $\alpha_i$ may cause sensitivity to the measurement noise because the queuing delay, the feedback information to be measured at each source, has the equilibrium point proportional to $\alpha_i$.

## 5. SIMULATION

In this section, we present the simulation results to illustrate the global finite-time convergence of TCP Vegas by using both MATLAB and *ns-2* network simulator. We simulate TCP Vegas described by (9) and (10) without feedback information delay using MATLAB, and simulate TCP Vegas in the presence of feedback information delay with *ns-2* simulator.

We conduct the simulation with two sets of scenarios, denoted by Scenario 1 and 2. Scenario 1 and 2 simulate a multi-link multi-source network that consists of two links and four heterogeneous sources as shown in Fig. 1. Each source transfers FTP packets to its counterpart with a packet size of 1KB. Link 1 is shared by source 1, 2, and

Table 1: Scenario 1: Equilibrium points

| $c_1 = 50$ $c_2 = 30$ $\alpha = 3$ | $d$ (ms) | Window size (pkts) | Sending rate (pkts/ms) | Queuing delay (ms) |
|---|---|---|---|---|
| Source 1 | 50 | 1076 | 18.53 | 8.09 |
| Source 2 | 10 | 135 | 10.49 | 2.86 |
| Source 3 | 20 | 480 | 20.98 | 2.86 |
| Source 4 | 20 | 289 | 11.47 | 5.23 |

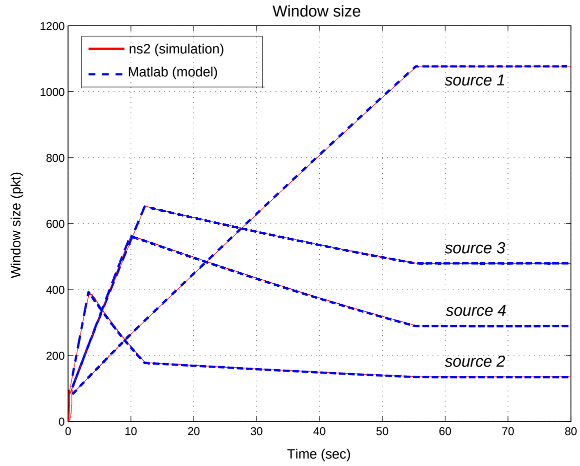
Table 2: Scenario 2: Equilibrium points

| $c_1 = 50$ $c_2 = 30$ $\alpha = 6$ | $d$ (ms) | Window size (pkts) | Sending rate (pkts/ms) | Queuing delay (ms) |
|---|---|---|---|---|
| Source 1 | 50 | 1226 | 18.53 | 16.18 |
| Source 2 | 10 | 165 | 10.49 | 5.72 |
| Source 3 | 20 | 540 | 20.98 | 5.72 |
| Source 4 | 20 | 349 | 11.47 | 10.46 |

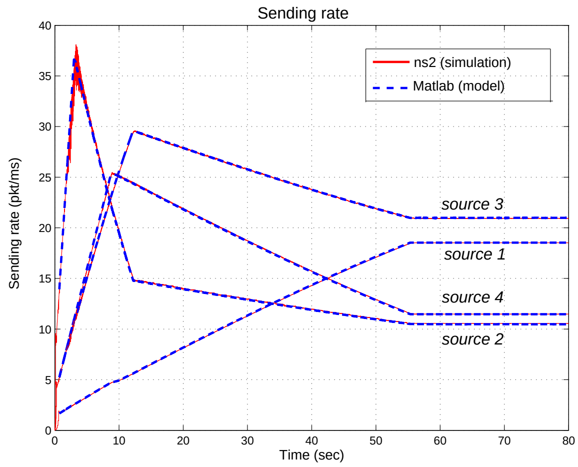
3 and has a capacity of 50pkts/ms; link 2 is shared by source 1 and 4 and has a capacity of 30pkts/ms. Source 1 sends packets through both link 1 and 2, and has a round trip latency of 50ms. The round trip latencies of source 2, 3, and 4 are 10ms, 20ms, and 20ms, respectively. The routers maintain a FIFO queue with a capacity of 20000 pkts. We set the congestion control parameter as follows: $\alpha = 3$ pkts/RTT in Scenario 1, $\alpha = 6$ pkts/RTT in Scenario 2, and $\alpha = \beta$ in *ns-2*.

Table 1 and 2 summarize the calculated equilibrium point of each source for each scenario. Fig. 2 and 3 show the simulation results of Scenario 1 and 2, respectively, where the solid lines indicate the *ns-2* results and the broken lines indicate the MATLAB results. The results from MATLAB (broken lines) in Fig. 2 and 3 show that TCP Vegas modeled by (9) and (10) converges to the calculated equilibrium point in finite time irrespective of the homogeneity of network sources. Moreover, *ns-2* simulation results in Fig. 2 and 3 verify that TCP Vegas, even in the presence of feedback information delay, follows the similar trend to the model without feedback information delay. Consequently, the simulation results illustrate that, even in the presence of feedback information delay, TCP Vegas globally converges in finite time to its equilibrium point.

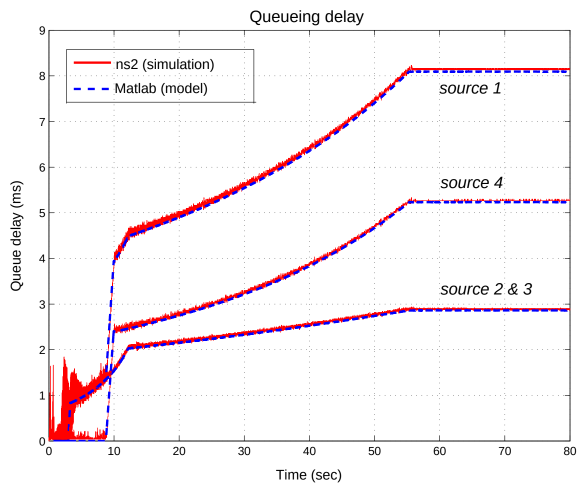
Regarding how the congestion control parameter $\alpha$ affects the performance of TCP Vegas, it is shown from Fig. 2 and 3 that the convergent time increases as the parameter $\alpha$ increases.




(a) Window size

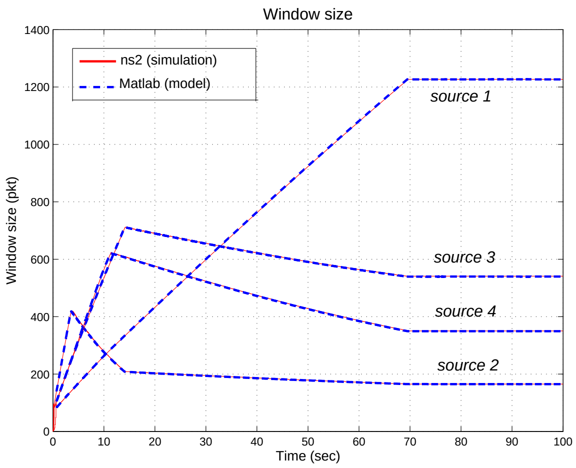



(b) Sending rate

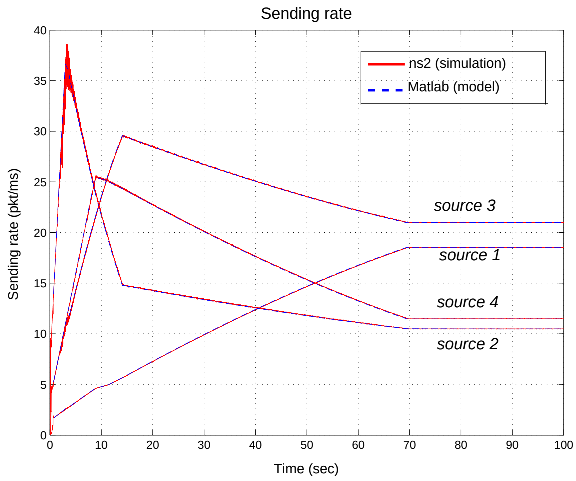



(c) Queuing delay

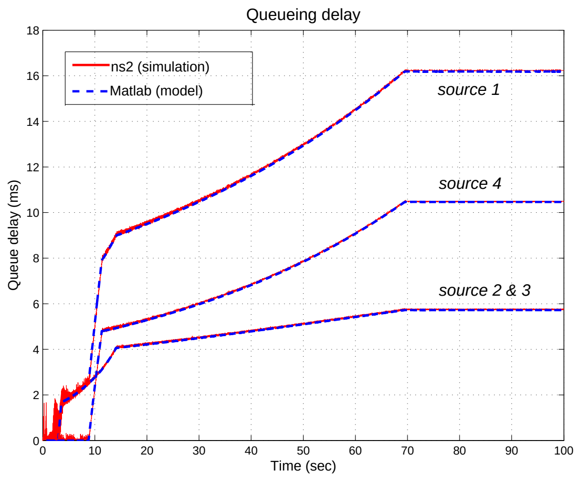
Fig 2: Window size, sending rate, and queuing delay when $\alpha = 3$pkts/RTT, $d_1 = 50$ms, $d_2 = 10$ms, $d_3 = 20$ms, $d_4 = 20$ms, $c_1 = 50$pkts/ms, and $c_2 = 30$pkts/ms




(a) Window size




(b) Sending rate




(c) Queuing delay

Fig 3: Window size, sending rate, and queuing delay when $\alpha = 6$pkts/RTT, $d_1 = 50$ms, $d_2 = 10$ms, $d_3 = 20$ms, $d_4 = 20$ms, $c_1 = 50$pkts/ms, and $c_2 = 30$pkts/ms

## 6. CONCLUSIONS

Considering a multi-link multi-source network with TCP Vegas sources without feedback information delay, we analyze the dynamic properties of TCP Vegas based on a continuous-time model without any approximation of the model, namely, keeping the high nonlinearity and discontinuous signum function of TCP Vegas model. We prove that assuming no feedback information delay, TCP Vegas is globally finite-time convergent to its equilibrium point. As illustrated in the simulation results, the dynamic behavior of TCP Vegas in the presence of feedback information delay shows similar phase to that of TCP Vegas without feedback information delay. This observation presents a significance that dynamic properties of TCP Vegas without feedback information delay can be used to estimate those of TCP Vegas in the presence of feedback information delay.

As a future work, we will consider establishing a method to obtain an accurate and useful estimate of the finite convergence time of TCP Vegas by using only a priori available system information.

## 7. APPENDIX

### 7.1. Proof of Lemma 1

**Proof:** We prove this lemma in the same way as in [13]. If there exist two equilibrium points $(x^*, p^*)$ and $(\hat{x}, \hat{p})$, then we will derive $(x^*, p^*) = (\hat{x}, \hat{p})$. Since $x_i^* q_i^* = \alpha_i d_i$ and $\hat{x}_i \hat{q}_i = \alpha_i d_i$ from (7),

$$x_i^* q_i^* - \hat{x}_i \hat{q}_i = (x_i^* - \hat{x}_i) q_i^* + \hat{x}_i (q_i^* - \hat{q}_i) = 0 \quad \text{and}$$

$$(x_i^* - \hat{x}_i) \frac{q_i^*}{\hat{x}_i} + (q_i^* - \hat{q}_i) = 0 \qquad \text{for all } 1 \leqslant i \leqslant N. \quad (16)$$

Multiplying $x_i^* - \hat{x}_i$ on both sides of (16) and summing from $i = 1$ to $N$ then yields

$$\begin{aligned} 0 &= \sum_{i=1}^{N} \left( (x_i^* - \hat{x}_i) \frac{q_i^*}{\hat{x}_i} + (q_i^* - \hat{q}_i) \right) (x_i^* - \hat{x}_i) \\ &= \left( \sum_{i=1}^{N} (x_i^* - \hat{x}_i)^2 \frac{q_i^*}{\hat{x}_i} \right) + (q^* - \hat{q})^T (x^* - \hat{x}) \\ &= \left( \sum_{i=1}^{N} (x_i^* - \hat{x}_i)^2 \frac{q_i^*}{\hat{x}_i} \right) + (p^* - \hat{p})^T R (x^* - \hat{x}) \\ &= \left( \sum_{i=1}^{N} (x_i^* - \hat{x}_i)^2 \frac{q_i^*}{\hat{x}_i} \right) + \sum_{l=1}^{L} (p_l^* - \hat{p}_l) R^l (x^* - \hat{x}). \end{aligned} \quad (17)$$

Next, since $(R^l x^* - c_l)_{p_l}^+ = 0$ and $(R^l \hat{x} - c_l)_{p_l}^+ = 0$ from (7), one of the following cases occurs for each link $l$: (i) $R^l x^* = c_l$ and $R^l \hat{x} = c_l$; (ii) $R^l x^* = c_l$ and $R^l \hat{x} < c_l$, $\hat{p}_l = 0$; (iii) $R^l x^* < c_l, p_l^* = 0$ and $R^l \hat{x} = c_l$; (iv) $R^l x^* < c_l, p_l^* = 0$ and $R^l \hat{x} < c_l$, $\hat{p}_l = 0$. In case (i), we have $R^l (x^* - \hat{x}) = 0$;

in case (ii), $(p_l^* - \hat{p}_l)R^l(x^* - \hat{x}) \geqslant 0$ because $p_l^* - \hat{p}_l \geqslant 0$ and $R^l(x^* - \hat{x}) > 0$; in case (iii), $(p_l^* - \hat{p}_l)R^l(x^* - \hat{x}) \geqslant 0$ because $p_l^* - \hat{p}_l \leqslant 0$ and $R^l(x^* - \hat{x}) < 0$; and in case (iv), $p_l^* - \hat{p}_l = 0$. Summarizing all these cases, we conclude that

$$(p_l^* - \hat{p}_l)R^l(x^* - \hat{x}) \geqslant 0 \quad \text{for all } 1 \leqslant l \leqslant L. \tag{18}$$

Then, from (17) and (18), we have

$$\sum_{i=1}^{N}(x_i^* - \hat{x}_i)^2 \frac{q_i^*}{\hat{x}_i} = 0 \tag{19}$$

and, since $q_i^*/\hat{x}_i > 0$, (19) yields

$$x_i^* = \hat{x}_i \quad \text{for all } 1 \leqslant i \leqslant N. \tag{20}$$

On the other hand, it follows from (16) and (20) that

$$(q_i^* - \hat{q}_i) = -\frac{q_i^*}{\hat{x}_i}(x_i^* - \hat{x}_i) = 0 \quad \text{for all } 1 \leqslant i \leqslant N,$$

from which we derive, using (1), that

$$\left(RR^T\right)(p^* - \hat{p}) = R(q^* - \hat{q}) = \sum_{i=1}^{N} R_i(q_i^* - \hat{q}_i) = 0.$$

$RR^T$ being nonsingular, we then have

$$p^* = \hat{p} \tag{21}$$

and combining (20) and (21) results in $(x^*, p^*) = (\hat{x}, \hat{p})$. □

### 7.2. Proof of Lemma 4

**Proof:** $(i)$ Assume that there exists an unbounded $x_i(t)$ among $1 \leqslant i \leqslant N$ and derive a contradiction. Since $x_i(t)$ is unbounded, there exists a $t_K \geqslant 0$ for every $K > 0$ such that $x_i(t_K) > K$. If we choose a $K$ such that $K > x_i(0)$, then, $x_i(t)$ being continuous from Lemma 2, there exists a $t_K^* \geqslant 0$ such that $x_i(t_K^*) = K$ and $x_i(t) \geqslant K$ for all $t \in [t_K^*, t_K]$. Now choose a $K$ such that

$$K > \max\left(x_i(0),\ 2\max_{l\in R_i^T} c_l,\ \frac{1}{d_i}\right). \tag{22}$$

Since $x_i(t) \geqslant K > 2\max_{l\in R_i^T} c_l$ for all $t \in [t_K^*, t_K]$,

$$\dot{p}_l(t) = \frac{1}{c_l}(y_l(t) - c_l) \geqslant \frac{1}{c_l}(x_i(t) - c_l) > 1$$

for all $t \in [t_K^*, t_K]$ and $l \in R_i^T$, and, since $\dot{q}_i = R_i^T \dot{p}$,

$$\dot{q}_i(t) > 1 \quad \text{for all } t \in [t_K^*, t_K]. \tag{23}$$

On the other hand, note from (9) that

$$\begin{aligned}\dot{x}_i(t) &= -\frac{\dot{q}_i(t)}{T_i(t)}x_i(t) + \frac{1}{T_i^2(t)}\mathrm{sgn}(s_i(t)) \\ &\leqslant -\frac{1}{T_i(t)}\left(\dot{q}_i(t)x_i(t) - \frac{1}{d_i}\right)\ \forall\, t \geqslant 0. \end{aligned}\tag{24}$$

Since $\dot{q}_i(t) > 1$ for all $t \in [t_K^*, t_K]$ from (23) and $x_i(t) \geqslant K > \dfrac{1}{d_i}$ for all $t \in [t_K^*, t_K]$ from (22), then (24) yields

$$\dot{x}_i(t) \leqslant -\frac{1}{T_i(t)}\left(1 \cdot \frac{1}{d_i} - \frac{1}{d_i}\right) = 0 \tag{25}$$

for all $t \in [t_K^*, t_K]$. But this contradicts the fact that $x_i(t_K) > K$ and $x_i(t)$ is bounded above by $K$ for all $t \geq 0$ and for all $1 \leqslant i \leqslant N$.

$(ii)$ Assume that there exists an unbounded $w_i(t)$ among $1 \leqslant i \leqslant N$ and derive a contradiction. Since $w_i(t)$ is unbounded, there exists a $t_K \geqslant 0$ for every $K > 0$ such that $w_i(t_K) > K$. If we choose a $K$ such that $K > w_i(0)$, then, $w_i(t)$ being continuous from Lemma 2, there exists a $t_K^* \geqslant 0$ such that $w_i(t_K^*) = K$ and $w_i(t) \geqslant K$ for all $t \in [t_K^*, t_K]$. Now choose a $K$ such that

$$K > \max\left(w_i(0),\ x_i^{UB}(d_i + 1),\ \alpha_i d_i (d_i + 1)\right) \tag{26}$$

where $x_i^{UB}$ is an upper bound of $x_i(t)$, which was shown in $(i)$. From the definition of $x_i(t)$, we have

$$q_i(t) = \frac{w_i(t)}{x_i(t)} - d_i \quad \text{for all } t \in [t_K^*, t_K], \tag{27}$$

which implies that

$$q_i(t) \geqslant \frac{x_i^{UB}}{x_i(t)}(d_i + 1) - d_i > (d_i + 1) - d_i = 1 \tag{28}$$

for all $t \in [t_K^*, t_K]$ because $w_i(t) \geqslant K > x_i^{UB}(d_i + 1)$ for all $t \in [t_K^*, t_K]$ from (26). On the other hand, noting that $w_i(t) > \alpha_i d_i (d_i + 1)$ for all $t \in [t_K^*, t_K]$ from (26) and $\dfrac{q_i(t)}{T_i(t)} = \dfrac{q_i(t)}{d_i + q_i(t)} > \dfrac{1}{d_i + 1}$ for all $t \in [t_K^*, t_K]$ from (28), we have

$$\begin{aligned} s_i(t) &= \frac{1}{w_i(t)} - \frac{q_i(t)}{\alpha_i d_i T_i(t)} \\ &< \frac{1}{\alpha_i d_i (d_i + 1)} - \frac{1}{\alpha_i d_i (d_i + 1)} = 0 \end{aligned}$$

for all $t \in [t_K^*, t_K]$, and,

$$\dot{w}_i(t) = \frac{1}{T_i(t)}\mathrm{sgn}(s_i(t)) < 0$$

for all $t \in [t_K^*, t_K]$. But this contradicts the fact that $w_i(t) > K$ for all $t \in [t_K^*, t_K]$ and $w_i(t)$ is bounded above by $K$ for all $t \geq 0$ and for all $1 \leqslant i \leqslant N$.

$(iii)$ Assume that there exists an unbounded $q_i(t)$ among $1 \leqslant i \leqslant N$ and derive a contradiction. Since $q_i(t)$ is unbounded, there exists a $t_K \geqslant 0$ for every $K > 0$ such that $q_i(t_K) > K$. If we choose an $K$ such that $K > q_i(0)$, then, $q_i(t)$ being continuous from Lemma 2, there exists a $t_K^* \geqslant 0$ such that $q_i(t_K^*) = K$ and $q_i(t) \geqslant K$ for all

$t \in [t_K^*, t_K]$. Now choose an $K$ such that

$$K > \max\left( q_i(0), \frac{\max_{l \in R_i^T} |R^l|}{\min_{l \in R_i^T} c_l} \max_{i \in R^l} w_i^{UB} \right),$$

where $w_i^{UB}$ is the upper bound of $w_i(t)$ as shown in $(ii)$ and $\left|R^l\right|$ is the number of users passing through link $l$. Then we obtain

$$\begin{aligned} \dot{q}_i(t_K^*) &= \sum_{l \in R_i^T} \frac{1}{c_l} \left( \sum_{i \in R^l} \frac{w_i}{T_i} - c_l \right) \Bigg|_{t=t_K^*} \\ &\leqslant \sum_{l \in R_i^T} \frac{1}{c_l} \left( \sum_{i \in R^l} \frac{w_i^{UB}}{K} - c_l \right) < 0, \end{aligned}$$

which contradicts the fact that $q_i(t) \geqslant K$ for all $t \in [t_K^*, t_K]$ and $q_i(t)$ is bounded above by $K$ for all $t \geq 0$ and for all $1 \leqslant i \leqslant N$.

$(iv)$ From the definition of $s_i(t)$, we obtain

$$\begin{aligned} |s_i| &= \left| \frac{1}{w_i} - \frac{q_i}{\alpha_i d_i T_i} \right| \\ &\leqslant \left| \frac{1}{w_i} \right| + \left| \frac{q_i}{\alpha_i d_i T_i} \right| \\ &\leqslant \frac{1}{w_i^{LB}} + \frac{1}{\alpha_i d_i}, \end{aligned}$$

where $w_i^{LB}$ is the lower bound of $w_i(t)$ as shown in $(i)$ of Lemma 3.

$(v)$ Assume that there exists an unbounded $p_l(t)$ among $1 \leqslant l \leqslant L$ and derive a contradiction. Since $p_l(t)$ is unbounded, there exists a $t_M \geqslant 0$ for every $M > 0$ such that $p_l(t_M) > M$. If we choose an $M$ such that $M > p_l(0)$, then, $p_l(t)$ being continuous from Lemma 2, there exists a $t_M^* \geqslant 0$ such that $p_l(t_M^*) = M$ and $p_l(t) \geqslant M$ for all $t \in [t_M^*, t_M]$. Now choose an $M$ such that

$$M > \max\left( p_l(0), \frac{\left|R^l\right|}{c_l} \max_{i \in R^l} w_i^{UB} \right),$$

where $w_i^{UB}$ ( for all $1 \leqslant i \leqslant N$ ) is the upper bound of $w_i(t)$ as shown in $(ii)$ and $\left|R^l\right|$ is the number of users passing through link $l$. Then we obtain

$$\begin{aligned} \dot{p}_l(t_M^*) &= \frac{1}{c_l} \left( \sum_{i \in R^l} \frac{w_i}{T_i} - c_l \right) \Bigg|_{t=t_M^*} \\ &\leqslant \frac{1}{c_l} \left( \sum_{i \in R^l} \frac{w_i^{UB}}{M} - c_l \right) < 0, \end{aligned}$$

which contradicts the fact that $p_l(t) \geqslant M$ for all $t \in [t_M^*, t_M]$ and $p_l(t)$ is bounded above by $M$ for all $t \geq 0$ and for all $1 \leqslant l \leqslant L$. □

## REFERENCES


[1] Lawrence S. Brakmo and Larry L. Peterson, "TCP Vegas: End to End Congestion Avoidance on a Global Internet," *IEEE Journal on Selected Areas in Communications*, Vol. 13, No. 8, pp. 1465–1480, 1995.

[2] J. S. Ahn, P. B. Danzig, Z. Liu and L. Yan, "Evaluation of TCP Vegas: emulation and experiment," In *Proceedings of SIGCOMM'95*, 1995.

[3] J. Mo, R. La, V. Anantharam and J. Walrand, "Analysis and comparison of TCP Reno and Vegas," In *Proceedings of IEEE Infocom*, March 1999.

[4] U. Hengartner, J. Bolliger and Th. Gross, "TCP Vegas Revisited," In *Proceedings of IEEE Infocom*, March 2000.

[5] Steven H. Low, Larry L. Peterson, and Limin Wang, "Understanding TCP Vegas: A Duality Model," *Journal of the ACM*, Vol. 49, No. 2, pp. 207–235, 2002.

[6] Hyojeong Choe and Steven H. Low, "Stabilized Vegas," In *Proceedings of IEEE Infocom*, April 2003.

[7] Fernando Paganini, "A Global Stability Result in Network Flow Control," *System & Control Letters*, Vol. 46, No. 3, pp. 153–163, 2002.

[8] Supratim Deb and R. Srikant, "Global Stability of Congestion Controllers for the Internet," *IEEE Tran. on Automatic Control*, Vol. 48, No. 6, pp. 1055–1060, June 2003.

[9] Tansu Alpcan and Tamar Basar, "Global Stability of an End-to-End Congestion Control Scheme for General Topology Networks with Delay," In *Proceedings of the 42nd IEEE Conference on Decision and Control*, Maui, Hawaii, Dec. 2003.

[10] Shao Liu, Tamar Basar and R. Srikant, "Controlling the Internet: A survey and some new results," In *Proceedings of the 42nd IEEE Conference on Decision and Control*, Maui, Hawaii, Dec. 2003.

[11] Zhikui Wang and Fernando Paganini, "Global Stability with Time-Delay of a Primal-Dual Congestion Control," In *Proceedings of the 42nd IEEE Conference on Decision and Control*, Maui, Hawaii, Dec. 2003.

[12] Lei Ying, G. E. Dullerud and R. Srikant, "Global Stability of Internet Congestion Controllers with Heterogeneous Delays," *IEEE/ACM Transactions on Networking*, Vol. 14, No. 3, pp. 579–590, June 2006.

[13] Jin S. Lee and Kyungmo Koo, "Global Stability of TCP Vegas without Network Delay," In *Proc. 43rd IEEE Conf. Decision and Control*, Atlantis, Bahamas, Dec. 2004.

[14] Hassan K. Khalil, *Nonlinear Systems*, 3rd ed., Prentice Hall, 2002.

[15] A. F. Filippov, "Differential equations with discontinuous right hand side," *Amer. Math. Soc. Translations*, Vol. 42, Ser. 2, pp. 199–231, 1964.

[16] J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer-Verlag Berlin Heidelberg, 1984.