Power System Analysis

Chapter 3  Transformer models
Outline

1. Single-phase transformer
2. Three-phase transformer
3. Equivalent impedance
4. Per-phase analysis
5. Per-unit normalization
Outline

1. Single-phase transformer
   - Ideal transformer
   - Nonideal transformer
     - Circuit models: \( T \) eq circuit, simplified circuit, UVN

2. Three-phase transformer

3. Equivalent impedance

4. Per-phase analysis

5. Per-unit normalization
Ideal transformer

Voltage & current gains

\[
\frac{v_2(t)}{v_1(t)} = n \quad \frac{i_2(t)}{i_1(t)} = a
\]

voltage gain \( n := \frac{N_2}{N_1} \)

turns ratio \( a := \frac{N_1}{N_2} \)
Ideal transformer

 Voltage & current gains

\[ \frac{V_2}{V_1} = n \quad \frac{I_2}{I_1} = a \]

Transmission matrix

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} =
\begin{bmatrix}
a & 0 \\
0 & n
\end{bmatrix}
\begin{bmatrix}
V_2 \\
I_2
\end{bmatrix}
\]
**Ideal transformer**

![Diagram of an ideal transformer with symbols for voltage and current](image)

**Power transfer**

\[
-\frac{S_{21}}{S_{12}} := \frac{V_2 I_2^*}{V_1 I_1^*} = n \cdot a = 1
\]

i.e., deal transformer incurs no power loss

- **Voltage gain** \( n \) := \( \frac{N_2}{N_1} \)
- **Turns ratio** \( a \) := \( \frac{N_1}{N_2} \)
Nonideal transformer

Nonideal behavior
- Power losses (coil resistances, eddy currents, hysteresis losses)
- Leakage magnetic fluxes
- Finite permeability of magnetic cores
Nonideal transformer

Voltages
\[ v_1 = r_1 i_1 + \frac{d\lambda_1}{dt}, \quad v_2 = r_2 i'_2 + \frac{d\lambda_2}{dt} \]

Total flux linkages
\[ \lambda_1 = N_1 \Phi_m + \lambda_{l1}, \quad \lambda_2 = N_2 \Phi_m + \lambda_{l2} \]
\[ \lambda_{l1} = L_{l1} i_1, \quad \lambda_{l2} = L_{l2} i'_2 \]

Total magnetomotive force
\[ F = N_1 i_1 + N_2 i'_2 = R\Phi_m \]
Nonideal transformer

Voltages
\[ v_1 = r_1 i_1 + \frac{d\lambda_1}{dt}, \quad v_2 = r_2 i_2' + \frac{d\lambda_2}{dt} \]

Total flux linkages
\[ \lambda_1 = N_1 \Phi_m + \lambda_{l1}, \quad \lambda_2 = N_2 \Phi_m + \lambda_{l2} \]
\[ \lambda_{l1} = L_{l1} i_1, \quad \lambda_{l2} = L_{l2} i_2' \]

Total magnetomotive force
\[ F = N_1 i_1 + N_2 i_2' = R \Phi_m \]

Ideal transformer
- Zero power losses: \( r_1 = r_2 = 0 \)
- Zero leakage flux linkages: \( L_{l1} = L_{l2} = 0 \)
- Infinite permeability: \( R = 0 \)
Nonideal transformer

Voltages
\[ v_1 = r_1i_1 + L_{l1} \frac{di_1}{dt} + N_1 \frac{d\Phi_m}{dt} \]
\[ v_2 = r_2i_2' + L_{l2} \frac{di_2'}{dt} + N_2 \frac{d\Phi_m}{dt} \]

Primary magnetizing current \( \hat{i}_m \)
- primary current when secondary circuit is open \( i_2' = 0 \)
- \( N_1 \hat{i}_m = R\Phi_m \): let \( L_m := N_1^2/R \) and
  \[ \hat{u}_1 := N_1 \frac{d\Phi_m}{dt} = L_m \frac{d\hat{i}_m}{dt} \]
  \[ \hat{u}_2 := N_2 \frac{d\Phi_m}{dt} = \frac{N_2}{N_1} \hat{u}_1 \quad \text{ideal transformer} \]
Nonideal transformer

Nonideal elements

\[
\begin{align*}
v_1 &= r_1 i_1 + L_{ll} \frac{di_1}{dt} + \hat{u}_1, \\
\hat{u}_1 &= L_m \frac{d\hat{i}_m}{dt} \\
v_2 &= -r_2 i_2 - L_{l2} \frac{di_2}{dt} + \hat{u}_2
\end{align*}
\]

Ideal transformer

\[
\begin{align*}
\hat{u}_2 &= \frac{N_2}{N_1} \hat{u}_1, \\
i_2 &= \frac{N_1}{N_2} \left( i_1 - \hat{i}_m \right)
\end{align*}
\]
Nonideal transformer

Circuit model

Nonideal elements (phasor domain)

\[ V_1 = z_p I_1 + \hat{U}_1, \quad \hat{i}_m = y_m \hat{U}_1 \]

\[ \hat{U}_2 = z_s I_2 + V_2 \]

Ideal transformer (phasor domain)

\[ \hat{U}_2 = \frac{N_2}{N_1} \hat{U}_1, \quad I_2 = \frac{N_1}{N_2} \left( I_1 - \hat{i}_m \right) \]
Nonideal transformer

Circuit models

\[
\text{T.eq. circuit} = \text{Unitary voltage model}
\]

\[
\text{Simplified model}
\]
Refer series impedance $z_s$ to the primary side

$$
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} =
\begin{bmatrix}
a(1 + z_p y_m) & az_s(1 + z_p y_m) + nz_p \\
ay_m & n + az_s y_m
\end{bmatrix}
\begin{bmatrix}
V_2 \\
I_2
\end{bmatrix}
$$

where $n := N_2/N_1$, $a := 1/n$

"Equivalent model" means
- Same end-to-end behavior, e.g., transmission matrix, or admittance matrix;
- Internal variables may be different
The equivalent circuit is open, has finite permeability, i.e.,

\[ \mathcal{F} \text{ flux linkages and the mutual flux linkage:} \]

\[ R_1 = \frac{v_1}{i_1} = \frac{N_1}{L_1} \]

The leakage primary and secondary coils. The two dots indicate that the mutual flux components due to \( a \) and \( m \) are nonzero in general in Figure 3.2(b), as determined by (3.4), yielding

\[ \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a \left( 1 + z_p y_m \right) & az_s(1 + z_p y_m) + nz_p \\ ay_m & n + az_s y_m \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \]

where \( n := N_2/N_1, \ a := 1/n \)

Model parameters \((z_p, z_s, y_m)\) cannot be uniquely determined from just short-circuit & open-circuit tests

- Additional tests are needed

Refer series impedance \( z_s \) to the primary side

\[ \longrightarrow T \text{ equivalent circuit} \]
Nonideal transformer

Circuit models

\[ T_{eq.\text{ circuit}} = \text{Unitary voltage \& current} \]

\[ \text{Simplified model} \]
Simplified circuit

Interchange $a^2 z_s$ and $y_m$ and combine with $z_p$:

$$z_l := z_p + a^2 z_s$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a(1 + z_l y_m) & nz_l \\ ay_m & n \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

where $n := N_2/N_1$, $a := 1/n$
Interchange $a^2 z_s$ and $y_m$ and combine with $z_p$:

$$z_l := z_p + a^2 z_s$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a (1 + z_l y_m) & nz_l \\ a y_m & n \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

where $n := N_2/N_1$, $a := 1/n$

Good approximation of $T$ equivalent circuit when $|y_m| \ll 1/|a^2 z_s|$

$$\frac{||M - T||}{||T||} < |e| \ll 1$$

$M$ : transmission matrix of simplified model

$T$ : transmission matrix of simplified model

$e := a^2 z_s y_m$
Parameter determination.

We now justify the model in Figure 3.5(a) with the equivalent circuit in Figure 3.4. For instance, when the secondary circuit is shorted, i.e., setting \( y_m \) to zero, the model reduces to (3.1) for an ideal transformer.

Two simple tests are often used to determine transformer model parameters:

1. When the series impedance \( z_s \) is treated as a vector in (3.7), can be uniquely determined from two simple tests:
   \[
   V_1 = \begin{bmatrix} a & z_s \\ ay_m & n \end{bmatrix} V_2
   \]

   where \( n := N_2/N_1, \ a := 1/n \)

   Good approximation when \( |y_m| \ll 1/|a^2z_s| \)

   \[
   \frac{||M - T||}{||T||} < |e| \ll 1
   \]

   If \( y_m = 0 \) : \( T \) equivalent circuit and simplified model are equivalent, \( M = T \)

Interchange \( a^2z_s \) and \( y_m \) and combine with \( z_p \):

\[
z_l := z_p + a^2z_s
\]

Approximation to \( T \) equivalent circuit.
Parameter determination

Short & open-circuit tests

Parameters \((z_l, y_m)\) can be determined from open and short-circuit tests

- **Short-circuit test** \((V_2 := 0)\):
  \[
  z_l = \frac{V_{sc}}{I_{sc}}
  \]

- **Open-circuit test** \((I_2 := 0)\):
  \[
  \frac{1}{y_m} = \frac{V_{oc}}{I_{oc}} - \frac{V_{sc}}{I_{sc}}
  \]

Most popular model (at least for transmission systems)
Parameter determination

Zero shunt admittance $y_m = 0$

When $y_m = 0$, parameter $z_l$ can be determined from standard 3-phase transformer ratings:

- Rated primary line-to-line voltage $V_{pri}$
- Rated primary line current $I_{pri}$
- Impedance voltage $\beta$ on the primary side, per phase, as % of rated primary voltage

$\beta$: voltage needed on the primary side to produce rated primary current across each single-phase transformer is $\beta \times$ rated primary voltage
Parameter determination

Zero shunt admittance $y_m = 0$

For both $Y$ and $\Delta$ configurations

$$z_i = \frac{V_{sc}}{I_{sc}}$$

- $\Delta$ config:

$$|V_{sc}| = |V_{ab}| = \beta |V_{pri}|$$

$$|I_{sc}| = |I_{ab}| = \left| \frac{I_{pri}}{\sqrt{3}} e^{i\pi/6} \right|$$

- $Y$ config:

$$|V_{sc}| = |V_{an}| = \beta \left| \frac{V_{pri}}{\sqrt{3} e^{i\pi/6}} \right|$$

$$|I_{sc}| = |I_{an}| = |I_{pri}|$$
Parameter determination
Zero shunt admittance $y_m = 0$

For both $Y$ and $\Delta$ configurations

$$z_l = \frac{V_{sc}}{I_{sc}}$$

- $\Delta$ config:
  $$|z_l| = \frac{\sqrt{3} \beta |V_{pri}|}{|I_{pri}|}$$

- $Y$ config:
  $$|z_l| = \frac{\beta |V_{pri}|}{\sqrt{3} |I_{pri}|}$$

$V_{pri}$ denotes line-to-line voltage even for $Y$ configuration

* Otherwise, $|z_l| = \frac{\beta |V_{pri}|}{|I_{pri}|}$ for $Y$ configuration if $V_{pri}$ is line-to-neutral

---

Steven Low EE/CS/EST 135 Caltech
Parameter determination

Zero shunt admittance \( y_m = 0 \)

Sometimes \( |S_{3\phi}| \) instead of \( |I_{pri}| \) is specified:

- Rated primary line-to-line voltage \( |V_{pri}| \)
- Rated 3-phase power \( |S_{3\phi}| \)
- Impedance voltage \( \beta \) on the primary side, per phase, as % of rated primary voltage
Parameter determination

Zero shunt admittance $y_m = 0$

- **Δ config:**
  \[
  |S_{3\phi}| = 3 |S_\phi| = 3 |V_{ab}| |I_{ab}|
  \]
  \[
  |V_{sc}| = |V_{ab}| = \beta |V_{pri}|
  \]
  \[
  |I_{sc}| = |I_{ab}| = \frac{|S_{3\phi}|}{3 |V_{pri}|}
  \]

- **Y config:**
  \[
  |S_{3\phi}| = 3 |S_\phi| = 3 |V_{an}| |I_{an}|
  \]
  \[
  |V_{sc}| = |V_{an}| = \beta \left| \frac{V_{pri}}{\sqrt{3} e^{i\pi/6}} \right|
  \]
  \[
  |I_{sc}| = |I_{an}| = \frac{|S_{3\phi}|}{3 \left| \frac{V_{pri}}{\sqrt{3} e^{i\pi/6}} \right|} = \frac{|S_{3\phi}|}{\sqrt{3} |V_{pri}|}
  \]
Parameter determination

Zero shunt admittance \( y_m = 0 \)

For both \( Y \) and \( \Delta \) configurations

\[
z_l = \frac{V_{sc}}{I_{sc}}
\]

- \( \Delta \) config:

\[
|z_l| = \frac{3\beta |V_{pri}|^2}{|S_{3\phi}|}
\]

- \( Y \) config:

\[
|z_l| = \frac{\beta |V_{pri}|^2}{|S_{3\phi}|}
\]

\( V_{pri} \) denotes line-to-line voltage even for \( Y \) configuration

- Otherwise, \( |z_l| = \frac{3\beta |V_{pri}|^2}{|I_{pri}|} \) for \( Y \) configuration if \( V_{pri} \) is line-to-neutral
Parameter determination

Example

3-phase transformer ratings (primary):
- Rated 3-phase power $|S_{3\phi}| = 150$ kVA
- Rated primary line-to-line voltage $|V_{pri}| = 480$ V
- Rated primary line current $|I_{pri}| = 180$ A
- Impedance voltage $\beta = 5.45\%$ on primary side

Primary in $\Delta$ configuration:

$$|S_{3\phi}| = 3 |S_{ab}| = 3 |V_{ab} \bar{I}_{ab}| = 3 |V_{pri}| |I_{ab}|$$

Since $I_a = I_{ab} - I_{ca} = I_{ab} \cdot \sqrt{3} e^{-i\pi/6}$, we have

$$|I_{pri}| = \sqrt{3} |I_{ab}|$$

Hence

$$|S_{3\phi}| = \sqrt{3} |V_{pri}| |I_{pri}|$$

Verify:

- $\sqrt{3} |V_{pri}| |I_{pri}| = \sqrt{3} \cdot 480 \cdot 180 = 149.65$ kVA = $|S_{3\phi}|$
- $|z_l| = \frac{\sqrt{3} \beta |V_{pri}|}{|I_{pri}|} = \frac{\sqrt{3} \cdot 5.45\% \cdot 480}{180} = 0.2517\Omega$
Parameter determination

Example

Secondary in $Y$ configuration:

$$|S_{3\phi}| = 3 |S_{an}| = 3 |V_{an} \bar{I}_{an}| = 3 \left| \frac{V_{sec}}{\sqrt{3}e^{i\pi/6}} \right| |I_{sec}|$$

Hence

$$|S_{3\phi}| = \sqrt{3} |V_{sec}| |I_{sec}|$$

Verify:

- $\sqrt{3} |V_{sec}| |I_{sec}| = \sqrt{3} \cdot 208 \cdot 416 = 149.87 \text{ kVA} = |S_{3\phi}|$
Distribution transformer
Examples

| line-to-line voltage (kV) $|V_{ab}|$ | phase voltage (kV) $|V_{an}|$ | total power (MVA) $|S_{3\phi}|$ |
|---|---|---|
| 4.8 | 2.8 | 3.3 |
| 12.47 | 7.2 | 8.6 |
| 22.9 | 13.2 | 15.9 |
| 34.5 | 19.9 | 23.9 |
Distribution transformer

Examples

Common deployment in US
- Single phase
- Split-phase 120/240 V

Figure 3.3: A common single-phase distribution transformer in the US.

Figure 3.4: Primary and secondary windings in Y and D configurations respectively. The thick lines in the schematic diagrams represent transformer windings.
Nonideal transformer

Circuit models

\[
\begin{align*}
T_{\text{eq. circuit}} & \quad \equiv \quad \text{Unitary voltage \textit{w.r.}} \\
\text{Simplified model} & \quad \equiv \quad \text{Unitary voltage \textit{w.r.}}
\end{align*}
\]
Unitary voltage network
Single-phase 2-winding transformer

UVN-based model
- Unitary voltage network (UVN) connecting 2 ideal transformers
- Equivalent to $T$ equivalent circuit
- Simplified model is an approximation

Advantages
- UVN can be generalized to incorporate multiple windings, e.g., split-phase transformers
- Ideal transformers on both ends can be connected in various ways, e.g., 3-phase transformers in $Y/\Delta$ configurations, non-standard transformers
Single-phase transformer

Unitary voltage network

\[ \hat{j}_1 = y_1(\hat{U}_1 - \hat{U}_0), \quad \hat{j}_2 = y_2(\hat{U}_2 - \hat{U}_0) \]

\[ y_0\hat{U}_0 = \hat{j}_0 + \hat{j}_1 + \hat{j}_2 \]

Admittance matrix

\[
\begin{bmatrix}
\hat{j}_0 \\
\hat{j}_1 \\
\hat{j}_2
\end{bmatrix} =
\begin{bmatrix}
y_0 + y_1 + y_2 & -y_1 & -y_2 \\
-y_1 & y_1 & 0 \\
-y_2 & 0 & y_2
\end{bmatrix}
\begin{bmatrix}
\hat{U}_0 \\
\hat{U}_1 \\
\hat{U}_2
\end{bmatrix}
\]

Since \( \hat{j}_0 = 0 \), can eliminate \( \hat{U}_0 \) to obtain Kron reduced admittance matrix

\[
\begin{bmatrix}
\hat{j}_1 \\
\hat{j}_2
\end{bmatrix} =
\frac{1}{\sum_i y_i}
\begin{bmatrix}
y_1(y_0 + y_2) & -y_1y_2 \\
-y_1y_2 & y_2(y_0 + y_1)
\end{bmatrix}
\begin{bmatrix}
\hat{U}_1 \\
\hat{U}_2
\end{bmatrix}
\]

\( y_{uvn} \)
Single-phase transformer

External model: admittance matrix

Let

\[
I := \begin{bmatrix} I_1 \\ -I_2 \end{bmatrix}, \quad V := \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}
\]

\[
M := \begin{bmatrix} 1/N_1 & 0 \\ 0 & 1/N_2 \end{bmatrix}
\]

Conversion between internal vars & terminal vars across ideal transformers

\[
\hat{U} = MV, \quad \hat{J} = M^{-1}I
\]

Hence, external model:

\[
I = (MY_{uvn}M) V
\]
Three-phase transformers
Standard configurations

External model:

\[ I = D^T(MY_{uvn}M)D \left( V - \gamma \right) \]

where \( \gamma := (V_1^n, V_2^n) \in \mathbb{C}^6 \) are neutral voltages in YY configuration, and

- **YY config**: \( D := \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{bmatrix} \)
- **\( \Delta \Delta \) config**: \( D := \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix} \)
- **\( \Delta Y \) config**: \( D := \begin{bmatrix} \Gamma & 0 \\ 0 & \mathbb{I} \end{bmatrix} \)
- **\( Y\Delta \) config**: \( D := \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \Gamma \end{bmatrix} \)

\[ Y_{uvn} := \left( \mathbb{I}_2 \otimes \left( \sum_{i=0}^2 y_i \right)^{-1} \right) \begin{bmatrix} y_j(y_0 + y_k) & -y_jy_k \\ -y_jy_k & y_k(y_0 + y_j) \end{bmatrix} \]

Let

\[ I := \begin{bmatrix} I_{abc} \\ -I_{abc} \end{bmatrix} \in \mathbb{C}^6, \quad V := \begin{bmatrix} V_{abc}^1 \\ V_{abc}^2 \end{bmatrix} \in \mathbb{C}^6 \]

\[ M := \begin{bmatrix} 1/N_{abc}^1 & 0 \\ 0 & 1/N_{abc}^2 \end{bmatrix} \in \mathbb{C}^{6 \times 6} \]
Multi-winding transformers

Example: split-phase transformer

The model (3.11) generalizes the single-phase model (3.10) in three ways. First the unitary voltage network can be generalized to model non-standard transformers with more than two windings. As an illustration we now use this approach to model multi-winding transformers. The model (3.11) has the same structure as the configuration in either the primary or the secondary circuit is represented by conversion matrices and configuration:  

\[ Y_n \]

in (3.10) has the same structure as the configuration:

\[ Y_3 \]

We consider a split-phase transformer. The internal voltages for the split-phase transformer are neutral voltages for each phase and the voltages are defined in the figure. The admittance matrix that maps these voltages to currents is given by:

\[ Y_n (1 J_{\hat{0}}, J_{\hat{1}}, J_{\hat{2}}, J_{\hat{3}}) = Y_{uvn} \]

Example: split-phase transformer

\[ \hat{J}_0 = \sum_{i=0}^{3} (-y_1 y_i - y_2 y_i - y_3) \begin{bmatrix} \hat{U}_0 \\ \hat{U}_1 \\ \hat{U}_2 \\ \hat{U}_3 \end{bmatrix} \]

\[ \hat{J}_1 = \begin{bmatrix} y_1 (y_0 + y_2 + y_3) \\ -y_2 y_1 \\ -y_3 y_1 \end{bmatrix} \]

\[ \hat{J}_2 = \begin{bmatrix} -y_1 y_2 \\ y_2 (y_0 + y_1 + y_3) \\ -y_3 y_2 \end{bmatrix} \]

\[ \hat{J}_3 = \begin{bmatrix} -y_1 y_3 \\ -y_2 y_3 \\ y_3 (y_0 + y_1 + y_2) \end{bmatrix} \]

\[ Y_{uvn} = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} \]
Multi-winding transformers

Example: split-phase transformer

Let

\[ I := \begin{bmatrix} I_1 \\ -I_2 \\ -I_3 \end{bmatrix}, \quad V := \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \]

\[ M := \begin{bmatrix} \frac{1}{N_1} & 0 & 0 \\ 0 & \frac{1}{N_2} & 0 \\ 0 & 0 & \frac{1}{N_3} \end{bmatrix} \]

Conversion between internal vars & terminal vars across ideal transformers: \( \hat{U} = MV \) and

\[ \hat{J} = M^{-1} \begin{bmatrix} I_1 \\ -I_2 \\ -I_2 - I_3 \end{bmatrix} =: M^{-1}A I \quad \text{where} \quad A := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \]
Multi-winding transformers
Example: split-phase transformer

Let

\[
I := \begin{bmatrix} I_1 \\ -I_2 \\ -I_3 \end{bmatrix}, \quad V := \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}
\]

\[
M := \begin{bmatrix} 1/N_1 & 0 & 0 \\ 0 & 1/N_2 & 0 \\ 0 & 0 & 1/N_3 \end{bmatrix}
\]

Eliminate internal vars \((\hat{J}, \hat{U})\) from

\[
\hat{U} = Y_{uvn}\hat{J}, \quad \hat{U} = MV, \quad \hat{J} = M^{-1}AI
\]

External model:

\[
I = A^{-1} (MY_{uvn}M) V
\]
Outline

1. Single-phase transformer
2. Three-phase transformer
   • Ideal transformer
   • Equivalent circuit
3. Equivalent impedance
4. Per-phase analysis
5. Per-unit normalization
Ideal transformer

Connectivity

(a) Primary winding in $Y$ configuration
Ideal transformer
Connectivity

(a) Primary winding in $Y$ configuration
(b) Secondary winding in $\Delta$ configuration
Ideal transformer
Configurations

**YY** configuration.

The winding of an ideal three-phase transformer in **YY** configuration and its schematic diagram are shown in Figure 3.5(a). The parallel lines in the schematic diagram indicate corresponding primary and secondary windings in the single-phase transformers. From the figure, the **YY** configuration is characterized by the following voltage and current gains:

\[
V_{an} = n, \quad I_{a0} = I_{a0}n
\]

**Figure 3.5**: Ideal three-phase transformers in **YY** and **DD** configurations. The parallel lines in the schematic diagram indicate corresponding primary and secondary windings.
Ideal transformer
Configurations

YY configuration.

ΔΔ configuration.

We now derive the properties of ideal three-phase transformers in YY, YY, DΔ, ΔY, YΔ configurations. We will first derive the terminal behavior. This means the line-to-neutral voltage gain for YY configuration and line-to-line voltage gain for other configurations, as well as the line current gains in these configurations.

We then derive the line-to-neutral voltage and current gains of their YY equivalent models, which yields their per-phase circuits. We will see that, as expected, the terminal behavior of a three-phase transformer has the same gains as those in its per-phase circuit. The derivation proceeds in three steps:

1. Derive voltage and current gains for each single-phase transformer.
2. Derive line voltage and current gains for the three-phase transformer.
3. Derive phase voltage and line current gains for the single-phase equivalent circuit.

The voltage and current gains in step 1 are defined by the pairing of primary and secondary windings in each single-phase ideal transformer and depends on the configuration. In per-phase analysis later, we will convert each DΔ configuration into an equivalent YY configuration.

YY configuration.

ΔΔ configuration.

Figure 3.5: Ideal three-phase transformers in YY and ΔΔ configurations. The parallel lines in the schematic diagram indicate corresponding primary and secondary windings.
Ideal transformer
Configurations

\[ \Delta Y \]
Ideal transformer
Configurations

\[ \Delta Y \]

\[ Y \Delta \]
What is external model?

- Line-to-line voltage gain $\frac{V_{a'b'}}{V_{ab}}$
- Line current gain $\frac{I_{a'}}{I_a}$

What is $YY$ equivalent circuit?

- Yields per-phase circuit
Ideal transformer
Configurations

Derivation
- Single-phase voltage & current gains
- Derive external model
- Derive $YY$ equivalent circuit

Use conversion between phase & line vars
- $V_{ab} = \sqrt{3} e^{i\pi/6} V_{an}$, $V_{a'b'} = \sqrt{3} e^{i\pi/6} V_{a'n'}$
- $I_a = \sqrt{3} e^{-i\pi/6} I_{ab}$, $I'_a = -\sqrt{3} e^{-i\pi/6} I_{a'b'}$
Ideal transformer

YY configuration

\[ \begin{align*}
V_{an} & = n, \quad I_{an} = \frac{1}{n} \\
V_{ab} & = n, \quad I_{ab} = \frac{1}{n}
\end{align*} \]

- Single-phase gains
- External model

external model = internal model
The external model of the ideal three-phase transformer in $\Delta\Delta$ configuration is characterized by the following voltage and current gains:

\[
\frac{V_{a'b'}}{V_{ab}} = n, \quad \frac{-I_{a'b'}}{I_{ab}} = \frac{1}{n}
\]

\[
\frac{I_{a'}}{I_a} = \frac{\sqrt{3} \ e^{-i\pi/6} I_{a'b'}}{\sqrt{3} \ e^{-i\pi/6} I_{ab}} = \frac{1}{n}
\]

The external model is equal to the internal model.

---

**Ideal transformer**

$\Delta\Delta$ configuration

- Single-phase gains
  \[
  \frac{V_{a'b'}}{V_{ab}} = n, \quad \frac{-I_{a'b'}}{I_{ab}} = \frac{1}{n}
  \]

- External model
  \[
  \frac{I_{a'}}{I_a} = \frac{\sqrt{3} \ e^{-i\pi/6} I_{a'b'}}{\sqrt{3} \ e^{-i\pi/6} I_{ab}} = \frac{1}{n}
  \]

The external model is equal to the internal model.
Ideal transformer

ΔΔ configuration

- Single-phase gains
  \[
  \frac{V_{a'b'}}{V_{ab}} = n, \quad \frac{-I_{a'b'}}{I_{ab}} = \frac{1}{n}
  \]

- Equivalent YY circuit
  \[
  \frac{V_{an'}}{V_{an}} = \left(\sqrt{3} e^{i\pi/6}\right)^{-1} \frac{V_{a'b'}}{V_{ab}} = n
  \]
  \[
  \frac{-I_{an'}}{I_{an}} = \frac{I_{a'}}{I_a} = \frac{1}{n}
  \]
Ideal transformer

\(\Delta\Delta\) configuration

- Single-phase gains

\[
\frac{V_{a'b'}}{V_{ab}} = n, \quad \frac{-I_{a'b'}}{I_{ab}} = \frac{1}{n}
\]

- Equivalent YY circuit

\[
\frac{V_{a'b'}}{V_{ab}} = \frac{V_{a'n'}}{V_{an}} = n
\]
\[
\frac{I_{a'}}{I_a} = \frac{-I_{a'n'}}{I_{an}} = \frac{1}{n}
\]

external model = YY equivalent = internal model
Ideal transformer
\(\Delta Y\) configuration

- Single-phase gains
  \[
  \frac{V_{a'n'}}{V_{ab}} = n, \quad \frac{-I_{a'n'}}{I_{ab}} = \frac{1}{n}
  \]

- External model
  \[
  \frac{V_{a'b'}}{V_{ab}} = \frac{\sqrt{3}e^{i\pi/6}V_{a'n'}}{V_{ab}} = \sqrt{3}e^{i\pi/6}n
  \]
  \[
  \frac{I_{a'}}{I_a} = \frac{-I_{a'n'}}{\sqrt{3}e^{-i\pi/6}I_{ab}} = \frac{1}{\sqrt{3}e^{-i\pi/6}n}
  \]
Ideal transformer

\( \Delta Y \) configuration

- Single-phase gains

\[
\frac{V_{a'n'}}{V_{ab}} = n, \quad \frac{-I_{a'n'}}{I_{ab}} = \frac{1}{n}
\]

- Complex voltage gain

\[
K_{\Delta Y}(n) := \sqrt{3}e^{i\pi/6}n
\]
Quantities in a per-phase equivalent circuit. Similarly for phase equivalent circuit of a balanced three-phase system. We will sometimes write since Equivalent YY configuration. Similarly on other lines.

Boosts the voltage gain by where the complex voltage gain To obtain the terminal behavior we have (assuming balanced positive sequence) single-phase transformer:

The winding of an ideal three-phase transformer in diagram are shown in Figure 3.6(a). It is characterized by the following voltage and current gains in the Y configuration.

In general

The line current gain is (using (3.2b))

By definition of

We have on the primary side

\[ V_{a'n'} = n, \quad \frac{-I_{a'n'}}{I_{ab}} = \frac{1}{n} \]

• Single-phase gains

• Complex voltage gain

\[ K_{\Delta Y}(n) := \sqrt{3} e^{i\pi/6} n \]

• External model

\[ V_{a'b'} = K_{\Delta Y}(n) V_{ab} \]

\[ I_{a'} = \frac{I_a}{K_{\Delta Y}(n)} \]
Ideal transformer
ΔY configuration

- Single-phase gains
  \[
  \frac{V_{a'n'}}{V_{ab}} = n, \quad \frac{-I_{a'n'}}{I_{ab}} = \frac{1}{n}
  \]

- Equivalent YY circuit
  \[
  \frac{V_{a'n'}}{V_{an}} = \frac{V_{a'n'}}{\left(\sqrt{3}e^{i\pi/6}\right)^{-1}V_{ab}} = K_{\Delta Y}(n)
  \]
  \[
  \frac{I_{a'}}{I_a} = \frac{-I_{a'n'}}{\sqrt{3}e^{-i\pi/6}I_{ab}} = \frac{1}{K_{\Delta Y}^*(n)}
  \]

terminal behavior = YY equivalent
Ideal transformer

\( Y\Delta \) configuration

- Single-phase gains
  \[
  \frac{V_{a'c'}}{V_{an}} = n, \quad \frac{I_{c'a'}}{I_{an}} = \frac{1}{n}
  \]

- External model
  \[
  \frac{V_{a'c'}}{V_{ac}} = \frac{V_{a'c'}}{\sqrt{3} e^{-i\pi/6}V_{an}} = \frac{n}{\sqrt{3}} e^{i\pi/6}
  \]
  \[
  \frac{I_{a'}}{I_{a}} = \frac{\sqrt{3} e^{i\pi/6} I_{c'a'}}{I_{an}} = \frac{\sqrt{3} e^{i\pi/6}}{n}
  \]
**Ideal transformer**

\(Y\Delta\) configuration

- Single-phase gains
  \[
  \frac{V_{a'c'}}{V_{an}} = n, \quad \frac{I_{c'a'}}{I_{an}} = \frac{1}{n}
  \]

- Complex voltage gain
  \[
  K_{Y\Delta}(n) := \frac{n}{\sqrt{3}} e^{i\pi/6}
  \]

- External model
  \[
  V_{a'b'} = K_{Y\Delta}(n) V_{ab} \\
  I_{a'} = K_{Y\Delta}^*(n) I_a
  \]
### Ideal transformer

#### Summary

<table>
<thead>
<tr>
<th>Property</th>
<th>Gain</th>
<th>Configuration</th>
<th>Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Voltage gain</td>
<td>$K(n)$</td>
<td>$YY$</td>
<td>$K_{YY}(n) := n$</td>
</tr>
<tr>
<td>Current gain</td>
<td>$\frac{1}{K^*(n)}$</td>
<td>$\Delta\Delta$</td>
<td>$K_{\Delta\Delta}(n) := n$</td>
</tr>
<tr>
<td>Power gain</td>
<td>$1$</td>
<td>$\Delta Y$</td>
<td>$K_{\Delta Y}(n) := \sqrt{3}n \ e^{i\pi/6}$</td>
</tr>
<tr>
<td>Sec $Z_l$ referred to pri</td>
<td>$\frac{Z_l}{</td>
<td>K(n)</td>
<td>^2}$</td>
</tr>
</tbody>
</table>

These properties of an ideal three-phase transformer in balanced operation are summarized in Table 3.2. For each configuration, $K(n)$ denotes the complex voltage gain of an ideal three-phase transformer.
Equivalent circuit

**YY configuration**

\[ K_{YY}(n) = n \]

![Equivalent circuit diagram](image)

**Per-phase circuit**

\[ V_{an} \]

\[ Z_l \]

\[ Y_m \]

\[ 1:n \]
Equivalent circuit

ΔΔ configuration

Equivalent circuit

per-phase circuit


Equivalent circuit

$\Delta Y$ configuration

\[ K_{\Delta Y}(n) \]

\[ Z_i \]

\[ Y_m \]

\[ V_{an} \]

\[ e^{i\pi/6} \]

\[ 1: \sqrt{3}n \]
Equivalent circuit

$Y\Delta$ configuration

![Equivalent circuit and per-phase circuit](image)
Outline

1. Single-phase transformer
2. Three-phase transformer
3. Equivalent impedance
   • Equivalence
   • Transmission matrix
   • Driving-point impedance
4. Per-phase analysis
5. Per-unit normalization
Motivation

Short cut in analyzing circuits containing transformers

- Thevenin equivalent of impedances in series and in parallel
- Equivalent impedances in primary or secondary circuits
Equivalent impedances

• referring $Z_s$ in secondary to primary

$$Z_p = \frac{Z_s}{|K(n)|^2}$$

“It is equivalent to replace $Z_s$ in the secondary circuit by $Z_p$ in the primary circuit”

• referring $Z_p$ in primary to secondary

$$Z_s = |K(n)|^2 \ Z_p$$

“It is equivalent to replace $Z_p$ in the primary circuit by $Z_s$ in the secondary circuit”
Equivalent admittances

- referring $Y_s$ in secondary to primary
  
  $$Y_p = |K(n)|^2 Y_s$$

  “It is equivalent to replace $Y_s$ in the secondary circuit by $Y_p$ in the primary circuit”

- referring $Y_p$ in primary to secondary
  
  $$Y_s = \frac{Y_p}{|K(n)|^2}$$

  “It is equivalent to replace $Y_p$ in the primary circuit by $Y_s$ in the secondary circuit”
Equivalent impedances

What is equivalence?

- Same transmission matrices
- Same driving-point impedance

This is a simple consequence of Kirchhoff’s and Ohm’s laws
External models (transmission matrices) of 2 circuits are equal if and only if
\[ Z_p = \frac{Z_s}{|K(n)|^2} \]
Transmission matrix

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} =
\begin{bmatrix}
1 & Z_s \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
V_2 \\
I_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} =
\begin{bmatrix}
K^{-1}(n) & 0 \\
0 & K^*(n)
\end{bmatrix}
\begin{bmatrix}
V \\
I
\end{bmatrix}
\]
Transmission matrix

\[
\begin{bmatrix}
V_1 \\ I_1
\end{bmatrix} = \begin{bmatrix}
K^{-1}(n) & K^{-1}(n)Z_s \\ 0 & K^*(n)
\end{bmatrix} \begin{bmatrix}
V_2 \\ I_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
V_1 \\ I_1
\end{bmatrix} = \begin{bmatrix}
K^{-1}(n) & K^*(n)Z_p \\ 0 & K^*(n)
\end{bmatrix} \begin{bmatrix}
V_2 \\ I_2
\end{bmatrix}
\]

External models (transmission matrices) of 2 circuits are equal if and only if

\[Z_p = \frac{Z_s}{|K(n)|^2}\]
Transmission matrix

External models (transmission matrices) of 2 circuits are equal if and only if \( Y_p = |K(n)|^2 Y_s \)
Example 3.2. A combination of a series impedance $Z_s$ and a shunt admittance $Y_s$ in the secondary circuit, as shown in Figure 3.14(a), can be referred to the primary one element at a time, starting from the element that is closest to the ideal transformer. The transformer gain is $K(n) = \frac{n}{a}$. Referring the series impedance $V_1 + - V_2 + - I_2 I_1 Y_s N_2 N_1$ ideal transformer to the primary yields the equivalent circuit in Figure 3.14(b) with an equivalent primary impedance $a^2 Z_s$. Referring then the shunt admittance $V_1 + - V_2 + - I_2 I_1 n^2 Y_s N_2 N_1$ ideal transformer to the primary yields the equivalent circuit in Figure 3.14(c) with an equivalent shunt admittance $n^2 Y_s$.

3.3.2 Driving-point impedance

In the second case the terminal behavior is the driving point impedances on one side of the transformer when the other side is connected to an impedance. In general suppose we apply a voltage $V_{ac}$ across two terminals that are connected to a network of impedances and transformers. Suppose a current $I$ flows between these two terminals through the network. The ratio $V/I$ is called the driving-point impedance at these terminals. For networks consisting of a cascade of impedances in series and in parallel, the driving-point impedance is also called the Thévenin equivalent impedance. The Thévenin equivalent impedance of such a network can be derived by repeatedly applying simple reduction rules for the two basic configurations shown in Figure 3.15. For two impedances $Z_1, Z_2$ in series depicted in Figure 3.15(a), the Thévenin equivalent $V_{eq} = Z_1 + Z_2$. For two impedances $Z_1, Z_2$ in parallel depicted in Figure 3.15(b), the Thévenin equivalent $V_{eq} = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}} = \frac{Z_1 Z_2}{Z_1 + Z_2}$. (a) $(Z_s, Y_s)$ in the secondary circuit. (b) Refer $Z_s$ to the primary. (c) Refer $Y_s$ to the primary.
Driving-point impedance

Thevenin equivalent

\[ I \]

\[ V \]

(a) Impedances in series

\[ I \]

\[ V \]

\[ Z_{eq} = Z_1 + Z_2 \]

(b) Impedances in parallel

Thevenin equivalent is a short cut in analyzing circuits with impedances only
Driving-point impedance
Thevenin equivalent

What if circuits contain both impedance and transformers?
Driving-point impedance
Referring impedance from secondary to primary

Both circuits have same driving-point impedance $V_1/I_1$ on primary side
- Can verify using Kirchhoff's and Ohm's laws

\[
\frac{V_1}{I_1} = Z_{eq} = \frac{1}{|K(n)|^2} Z_{2,eq}
\]

Figure 3.16: Driving-point impedances
Driving-point impedance
Referring impedance from primary to secondary

Both circuits have same driving-point impedance \( V_2/I_2 \) on secondary side
- Can verify using Kirchhoff’s and Ohm’s laws
To find $V_1/I_1$, can analyze using Kirchhoff’s and Ohm’s laws
Driving-point impedance

Example

\[ \frac{V_1}{I_1} = Z_{1,\text{eq}} + \left( Y_{1,\text{eq}} + \frac{1}{Z_{2,\text{eq}}/|K(n)|^2} \right)^{-1} \]
Driving-point impedance

Example

To find $\frac{V_2}{I_2}$, can analyze using Kirchhoff’s and Ohm’s laws
Driving-point impedance

### Example

- **Transformer Circuit**
- **Equivalent Circuit seen on the secondary side**

\[
\frac{V_2}{I_2} = \left( Y_{2, eq} + \frac{1}{Z_{2, eq} + |K(n)|^2 \cdot Z_{1, eq}} \right)^{-1}
\]
Driving-point impedance

Reference from one circuit to the other is not always applicable

- Example: circuits containing parallel paths (see example later)
- Generally applicable in a radial network without parallel paths
- Can always analyze original circuit using Kirchhoff’s and Ohm’s laws
Outline

1. Single-phase transformer
2. Three-phase transformer
3. Equivalent impedance
4. Per-phase analysis
   • Example
   • Normal systems
5. Per-unit normalization
Per-phase analysis

Procedure

1. Convert all sources and loads in Δ configurations into their Y equivalents
2. Convert all ideal transformers in Δ configurations into their Y equivalents
3. Obtain phase \( a \) equivalent circuit by connecting all neutrals
4. Solve for desired phase-\( a \) variables
   - Use Thevenin equivalent of series impedances and shunt admittances in networks containing transformers whenever applicable, e.g., for a radial network
5. Obtain variables for phases \( b \) and \( c \) by subtracting 120° and 240° from phase \( a \) variables (positive sequence sources)
   - If variables in the internal of Δ configurations are desired, derive them from original circuits
Per-phase analysis

Example

Balanced 3φ system

- Generator with line voltage $V_{\text{line}}$
- Step-up $\Delta Y$ transformer
- Transmission line with series impedance $Z_{\text{line}}$
- Step-down $\Delta Y$ transformer (primary on right)
- Load with impedance $Z_{\text{load}}$
- Single-phase transformer with voltage gain $n$ and series impedance $3Z_l$ on primary side
Per-phase analysis

Example

Balanced $3\phi$ system

- Generator with line voltage $V_{\text{line}}$
- Step-up $\Delta Y$ transformer
- Transmission line with series impedance $Z_{\text{line}}$
- Step-down $\Delta Y$ transformer (primary on right)
- Load with impedance $Z_{\text{load}}$
- Single-phase transformer with turns ratio $n$ and series impedance $3Z_l$ on primary side

$V_{1} = \frac{V_{\text{line}}}{\sqrt{3}} e^{i\pi/6}$

$Z^Y = Z_l$
Per-phase analysis

Example

Calculate

- Generator current $I_1$
- Transmission line current $I_2$
- Load current $I_3$
- Load voltage $V_3$
- Power delivered to load: $V_3 I_3^*$

\[
V_1 = \frac{V_{line}}{\sqrt{3} e^{i\pi/6}} \\
Z^Y = Z_l
\]
Per-phase analysis

Example

Solution strategy
- Refer all impedances to primary side of step-up transformer
- Derive driving-point impedance $V_1/I_1$
- Derive generator current $I_1$
- Propagate calculation towards load

\[ V_1 = \frac{V_{\text{line}}}{\sqrt{3} e^{i\pi/6}} \quad Z^Y = Z_l \]
Per-phase analysis

Example

\[ \frac{V_1}{I_1} = 2Z_l + \frac{Z_{\text{line}}}{|K(n)|^2} + Z_{\text{load}} \]

transformer gains on \(Z_{\text{load}}\) is canceled
Per-phase analysis

Example

\[ V_1 + I_1 Z_l + I_2 Z_{\text{line}} + I_3 Z_l = 0 \]

\[ V_2 + I_2 Z_{\text{line}} + I_3 Z_l = 0 \]

\[ V_3 + I_3 Z_{\text{load}} = 0 \]

\[ I_1 = \frac{V_{\text{line}}}{2 Z_l + \frac{Z_{\text{line}}}{|K(n)|^2} + Z_{\text{load}}} \]

\[ I_2 = \frac{I_1}{K^*(n)} \]

\[ I_3 = K^*(n) I_2 = I_1 \]

\[ V_3 = Z_{\text{load}} I_3 = Z_{\text{load}} I_1 \]
Per-phase analysis

Example

\[ I_1 = \frac{V_{\text{line}}}{2Z_l + \frac{Z_{\text{line}}}{|K(n)|^2} + Z_{\text{load}}} \]

\[ I_3 = I_1 \]

\[ V_3 = Z_{\text{load}} I_1 \]

- External behavior does not depend on connection-induced phase shift \( e^{i\pi/6} \)
- Only internal variables \( I_{\text{line}} \) does
Terminal behavior does not depend on $e^{i\pi/6}$

- The simplified model has the same transmission matrix.
Normal systems

A system is normal if, in its per-phase circuit, the product of complex ideal transformer gains around every loop is 1.

Equivalently, on each parallel path,
1. Product of ideal transformer gain magnitudes is the same, and
2. Sum of ideal transformer phase shifts is the same.
Inverting the matrix, we obtain
or
equations express current gains of the transformers. Eliminating where the first equation expresses KCL, the second and third equations express the load voltage seen on Ohm's laws.

Solution.

Normal systems

Example

Generator & load connected by two 3φ transformers in parallel (forming a loop)
Normal systems
Example

Calculate
- Load current $I_{\text{load}}$
- Line currents $I'_1$, $I'_2$

in terms of $V_{\text{gen}}$, $Z_l$, $Z_{\text{load}}$

Implications when
- $K_2 = K_1$ (normal system)
- $K_2 = K_1 e^{i\theta}$
- $K_2 = k \cdot K_1$

Per-phase circuit
Normal systems

Example

\[ K_2 = K_1 \] (normal system):

- \[ I'_1 = I'_2 \]
- \[ \frac{I_{\text{load}}}{I'_1} = \frac{I_{\text{load}}}{I'_2} = 2 \]
Normal systems

Example

\[ K_2 = K_1 e^{i\theta} : \]
- \( I'_1 \neq I'_2 \)
- \[ \frac{|I_{load}|}{|I'_1|} = \frac{|1 + e^{i\theta}|}{|\alpha_1|}, \quad \frac{|I_{load}|}{|I'_2|} = \frac{|1 + e^{i\theta}|}{|\alpha_2|} \]

Example: \( K_2 = K_1 e^{i\pi/6} : \)
- \[ \frac{|I_{load}|}{|I'_1|} = 20.6\%, \quad \frac{|I_{load}|}{|I'_2|} = 17.1\% \]

Most current loops between transformers without entering load

Per-phase circuit
Normal systems

Example

\[ K_2 = K_1 e^{i\theta} : \]

- \( I'_1 \neq I'_2 \)
- \( \frac{|I_{\text{load}}|}{|I'_1|} = \frac{|1 + e^{i\theta}|}{|\alpha_1|}, \quad \frac{|I_{\text{load}}|}{|I'_2|} = \frac{|1 + e^{i\theta}|}{|\alpha_2|} \)

Example: \( K_2 = K_1 e^{i\pi/6} : \)

- \( S_{\text{gen}} = 183 \angle 71^\circ, \quad S_{\text{load}} = 60 \angle 0^\circ \) MVA

Most current loops between transformers without entering load
Normal systems

Example

\[ K_2 = k \cdot K_1 : \]

- \[ I'_1 \neq I'_2 \]
- \[ \left| \frac{I_{\text{load}}}{I'_1} \right| = \frac{1 + k^{-1}}{|\alpha_1|} \]
- \[ \left| \frac{I_{\text{load}}}{I'_2} \right| = \frac{1 + k}{|\alpha_2|} \]

Example: \[ K_2 = 2K_1 : \]

- \[ \left| \frac{I_{\text{load}}}{I'_1} \right| = 29.4 \% \]
- \[ \left| \frac{I_{\text{load}}}{I'_2} \right| = 29.9 \% \]

Most current loops between transformers without entering load

Per-phase circuit
Outline

1. Single-phase transformer
2. Three-phase transformer
3. Equivalent impedance
4. Per-phase analysis
5. Per-unit normalization
   - Kirchhoff’s and Ohm’s laws
   - Across ideal transformer
   - $3\phi$ quantities
   - Per-unit per-phase analysis
Per-unit normalization

- Quantities of interest: voltages $V$, currents $I$, power $S$, impedances $Z$

  - Quantity in p.u. = \( \frac{\text{actual quantity}}{\text{base value of quantity}} \)

- Base values
  - Real positive values
  - Same units as actual quantities

- Choose base values to satisfy same physical laws
  - Kirchhoff’s and Ohm’s laws
  - Across ideal transformer
  - Relationship between $3\phi$ and $1\phi$ quantities
Per-unit normalization

General procedure

1. Choose voltage base value $V_{1B}$ for (say) area 1
2. Choose power base value $S_B$ for entire network
3. Calculate all other base values from physical laws

Example: Choose

1. $V_{1B} =$ nominal voltage magnitude of area 1
2. $S_B =$ rated apparent power of a transformer in area 1

How to calculate the other base values $(V_{iB}, I_{iB}, Z_{iB})$?

• Consider single-phase or per-phase circuit of balanced 3φ system
Kirchhoff’s and Ohm’s laws

Given base values \((V_{1B}, S_B)\), within area 1:

\[
I_{1B} := \frac{S_B}{V_{1B}} \text{ A}, \quad Z_{1B} := \frac{V_{1B}^2}{S_B} \text{ \Omega}
\]

Then: physical laws are satisfied by both the base values and p.u. quantities

\[
V_{1B} = Z_{1B} I_{1B}, \quad V_{1\text{pu}} = Z_{1\text{pu}} I_{1\text{pu}}
\]

\[
S_B = V_{1B} I_{1B}, \quad S_{1\text{pu}} = V_{1\text{pu}} I_{1\text{pu}}
\]

Can perform circuit analysis using pu quantities instead of actual quantities
Kirchhoff’s and Ohm’s laws

Other quantities

These quantities \((V_{1B}, S_B, I_{1B}, Z_{1B})\) serve as base values for other quantities within area 1, with appropriate units

- \(S_B\) is base value for real power in W, reactive power in var

  \[P_{1pu} := \frac{P_1}{S_B}, \quad Q_{1pu} := \frac{Q_1}{S_B}, \quad S_{1pu} = P_{1pu} + iQ_{1pu}\]

- \(Z_{1B}\) is base value for resistances & reactances in \(\Omega\)

  \[R_{1pu} := \frac{R_1}{Z_{1B}}, \quad X_{1pu} := \frac{X_1}{Z_{1B}}, \quad Z_{1pu} = R_{1pu} + iX_{1pu}\]

- \(Y_{1B} := 1/Z_{1B}\) in \(\Omega^{-1}\)is base value for conductances, susceptances, & admittances

  \[G_{1pu} := \frac{G_1}{Y_{1B}}, \quad B_{1pu} := \frac{B_1}{Y_{1B}}, \quad Y_{1pu} = G_{1pu} + iB_{1pu} = \frac{1}{Z_{1pu}}\]
Across ideal transformer

Choose \((V_{2B}, I_{2B}, Z_{2B})\) according to

\[ V_{2B} := |K(n)| V_{1B} \quad V \]
\[ I_{2B} := \frac{I_{1B}}{|K(n)|} \quad A \]
\[ Z_{2B} := |K(n)|^2 Z_{1B} \quad \Omega \]

Base values remain real positive

\(S_B\) remains base value for power
Across ideal transformer

\begin{align*}
\hat{V}_{1\text{pu}} &= \frac{\hat{V}_1}{V_{1B}} = \frac{V_2}{K(n)} \frac{|K(n)|}{V_{2B}} = V_{2\text{pu}} e^{-j\angle K(n)} \\
\hat{I}_{1\text{pu}} &= \frac{\hat{I}_1}{I_{1B}} = \frac{K^*(n)I_2}{|K(n)|I_{2B}} = I_{2\text{pu}} e^{-j\angle K(n)}
\end{align*}

If \( \angle K(n) = 0 \) then
\[ \hat{V}_{1\text{pu}} = V_{2\text{pu}}, \quad \hat{I}_{1\text{pu}} = I_{2\text{pu}} \]
Across ideal transformer

If \( \angle K(n) = 0 \) then
\[
\tilde{V}_{1pu} = V_{2pu}, \quad \tilde{I}_{1pu} = I_{2pu}
\]
Ideal transformer has disappeared!
Across ideal transformer

\[\Delta K(n) = 0\] if

- 1\(\phi\) or balanced 3\(\phi\) in \(YY\) or \(\Delta\Delta\)
- Normal systems where connection-induced phase shifts can be ignored
Across ideal transformer

\[ V_1 \rightarrow Z_x \rightarrow V_1' \]

\[ Y_m \]

\[ I_1, I_2 \]

area 1

area 2

\[ V_2 \]

\[ K(n) \]

Otherwise

- pu circuit contains an off-nominal phase-shifting transformer
Across ideal transformer

Example

Given nameplate rating of generator

• Voltage $v$ V
• Apparent power $s$ VA

Calculate base values

Voltage base $V_{1B} := v$, power base $S_B := s$

• Area 1: $I_{1B} := s/v$, $Z_{1B} := v^2/s$
• Area 2: $V_{2B} := n_1v$, $I_{2B} := s/(n_1v)$, $Z_{2B} := (n_1v)^2/s$, $Y_{2B} := s/(v_1v)^2$
• Area 3: $V_{3B} := n_1v/n_2$, $I_{3B} := n_2s/(n_1v)$, $Z_{3B} := (n_1v)^2/(n_2^2s)$, $Y_{3B} := (n_2^2s)/(v_1v)^2$
3φ quantities

Given 1φ devices (generators, lines, loads) with

• with 1φ quantities \((S^{1φ}, V^{1φ}, I^{1φ}, Z^{1φ})\)
• and their base values

Construct balanced 3φ devices from these 1φ devices

• What are 3φ quantities of interest?
• What are base values so that 3φ quantities equal to 1φ quantities in p.u.?

Base values should satisfy the same 3φ relationships as actual quantities
Values depend on the configuration, \(Y\) or \(Δ\)
In terms of

\((S^{1\phi}, V^{1\phi}, I^{1\phi}, Z^{1\phi})\)

and their base values

\(3\phi\) quantities

\(Y\) configuration

- \(3\phi\) power (total power to/from 3 1\(\phi\) devices):
  \[ S^{3\phi} = 3S^{1\phi}, \]
- Line-to-line voltage
  \[ V^{\text{ll}} = \sqrt{3} e^{i\pi/6} V^{\text{ln}}, \]
- Line current
  \[ I^{3\phi} = I_{an} = I^{1\phi}, \]
- Line-to-neutral voltage
  \[ V^{\text{ln}} = V^{1\phi}, \]
- Impedance
  \[ Z^{3\phi} = Z^{1\phi}, \]
3φ quantities
Y configuration

In terms of
\( (S^{1φ}, V^{1φ}, I^{1φ}, Z^{1φ}) \)
and their base values

- 3φ power (total power to/from 3 1φ devices):
  \[ S^{3φ} = 3S^{1φ}, \quad S_B^{3φ} = 3S_B^{1φ} \]

- Line-to-line voltage
  \[ V_{ll} = \sqrt{3}e^{i\pi/6} V^{ln}, \quad V_B^{ll} = \sqrt{3} V_B^{ln} \]

- Line current
  \[ I_{3φ} = I_{an} = I^{1φ}, \quad I_B^{3φ} = I_B^{1φ} \]

- Line-to-neutral voltage
  \[ V_{ln} = V^{1φ}, \quad V_B^{ln} = V_B^{1φ} \]

- Impedance
  \[ Z^{3φ} = Z^{1φ}, \quad Z_B^{3φ} = Z_B^{1φ} \]

Calculation
Base values satisfy the same relationship
$3\phi$ quantities
$\Delta$ configuration

In terms of 
$(S^1\phi, V^1\phi, I^1\phi, Z^1\phi)$
and their base values

- $3\phi$ power (total power to/from $3$ $1\phi$ devices):
  \[ S^{3\phi} = 3S^{1\phi}, \]

- Line-to-line voltage
  \[ V^{ll} = \sqrt{3} e^{i\pi/6} V^{ln}, \]

- Line current
  \[ I^{3\phi} = I_{ab} - I_{ca} = \sqrt{3} e^{-i\pi/6} I^{1\phi}, \]

- Line-to-neutral voltage
  \[ V^{ln} = \left(\sqrt{3} e^{i\pi/6}\right)^{-1} V^{1\phi}, \]

- Impedance
  \[ Z^{3\phi} = Z^{1\phi}/3, \]

Note:
$V^{ln}, Z^{3\phi}$ are voltage and & impedance in $Y$ equivalent circuit
$3\phi$ quantities

$\Delta$ configuration

In terms of

$(S^{1\phi}, V^{1\phi}, I^{1\phi}, Z^{1\phi})$

and their base values

- $3\phi$ power (total power to/from $3$ $1\phi$ devices):
  \[ S^{3\phi} = 3S^{1\phi}, \quad S_B^{3\phi} = 3S_B^{1\phi} \]

- Line-to-line voltage
  \[ V_{ll} = \sqrt{3} e^{i\pi/6} V_{ln}, \quad V_{ll}^B = \sqrt{3} V_{ln}^B \]

- Line current
  \[ I^{3\phi} = I_{ab} - I_{ca} = \sqrt{3} e^{-i\pi/6} I^{1\phi}, \quad I_B^{3\phi} = \sqrt{3} I_B^{1\phi} \]

- Line-to-neutral voltage
  \[ V_{ln} = \left( \sqrt{3} e^{i\pi/6} \right)^{-1} V^{1\phi}, \quad V_{ln}^B = (\sqrt{3})^{-1} V_{ln}^B \]

- Impedance
  \[ Z^{3\phi} = Z^{1\phi}/3, \quad Z_B^{3\phi} = Z_B^{1\phi}/3 \]

Note:

$V_{ln}^{3\phi}$, $Z_{ln}^{3\phi}$ are voltage and & impedance in $Y$ equivalent circuit.
Per-unit quantities

Per-unit quantities satisfy

\[
S_{3\phi}^{pu} = S_{1\phi}^{pu}, \quad V_{\text{ll}}^{pu} = V_{\text{ln}}^{pu}, \quad Z_{3\phi}^{pu} = Z_{1\phi}^{pu}
\]

\[
\left| V_{\text{ln}}^{pu} \right| = \left| V_{\text{pu}}^{1\phi} \right|, \quad \left| I_{\text{pu}}^{3\phi} \right| = \left| I_{\text{pu}}^{1\phi} \right|
\]

- \(3\phi\) quantities equal \(1\phi\) quantities in p.u.
- modulo phase shifts in \(\Delta\) configuration:

\[
V_{\text{ln}}^{pu} := \frac{V_{\text{ln}}^{B}}{V_{\text{ln}}^{pu}} = \left(\sqrt{3}e^{i\pi/6}\right)^{-1} V_{\text{ln}}^{1\phi} = e^{-i\pi/6} V_{\text{pu}}^{1\phi}
\]
Per-unit per-phase analysis

1. For single-phase system, pick power base $S_B^{1\phi}$ for entire system and voltage base $V_{1B}^{1\phi}$ in area 1, e.g., induced by nameplate ratings of transformer.

2. For balanced 3$\phi$ system, pick 3$\phi$ power base $S_B^{3\phi}$ and line-to-line voltage base $V_{1B}^{ll}$ induced by nameplate ratings of 3$\phi$ transformer. Then choose power & voltage bases for per-phase equivalent circuit:

$$S_B^{1\phi} := S_B^{3\phi} / 3, \quad V_{1B}^{1\phi} := V_{1B}^{ll} / \sqrt{3}$$

$S_{1B}^{1\phi}$ will be power base for entire per-phase circuit.

3. Calculate current and impedance bases in that area:

$$I_{1B} := \frac{S_B^{1\phi}}{V_{1B}^{1\phi}}, \quad Z_{1B} := \left(\frac{V_{1B}^{1\phi}}{S_B^{1\phi}}\right)^2$$
Per-unit per-phase analysis

4. Calculate base values for voltages, currents, and impedances in areas $i$ connected to area 1 using the magnitude $n_i$ of transformer gains (assume area 1 is primary):

\[
V_{iB}^{1\phi} := n_i V_{1B}^{1\phi}, \quad V_{iB}^\| := n_i V_{1B}^\|, \quad I_{iB} := \frac{1}{n_i} I_{1B}, \quad Z_{iB} := n_i^2 Z_{1B}
\]

Continue this process to calculate the voltage, current, and impedance base values for all areas
Per-unit per-phase analysis

5. For real, reactive, apparent power in entire system, use $S_{B}^{1\phi}$ as base value.
   For resistances and reactances, use $Z_{iB}$ as base value in area $i$.
   For admittances, conductances, and susceptances, use $Y_{iB} := 1/Z_{iB}$ as base value in area $i$

6. Draw impedance diagram of entire system, and solve for desired per-unit quantities

7. Convert back to actual quantities if desired
Summary

1. Single-phase transformer
   • Ideal transformer gain $n$, equivalent circuit

2. Three-phase transformer
   • $YY$, $ΔΔ$, $ΔY$, $YΔ$: external behavior, $YY$ equivalent

3. Equivalent impedance
   • Short cut for analyzing circuits containing transformers
     • Transmission matrix, driving-point impedance

4. Per-phase analysis

5. Per-unit normalization
   • Physical laws, across transformer, $3ϕ$ quantities, per-unit per-phase analysis