Power Systems Analysis

Chapter 6  Branch flow models
Outline

1. General network
2. Radial network
3. Equivalence
4. Backward forward sweep
5. Linearized model
Outline

1. General network
   • Complex form power flow equations
   • Real form power flow equations

2. Radial network

3. Equivalence

4. Backward forward sweep

5. Linearized model
General network

1. Network $G := (\bar{N}, E)$
   - $\bar{N} := \{0\} \cup N := \{0\} \cup \{1, \ldots, N\}$ : buses/nodes
   - $E \subseteq \bar{N} \times \bar{N}$ : lines/links/edges

2. Each line $(j, k)$ is parameterized by $(y^s_{jk}, y^m_j, y^m_k)$
   - $y^s_{jk}$ : series admittance
   - $y^m_j, y^m_k$ : shunt admittances, generally different

![Graph representation](a)

![Π equivalent circuit](b)
General network

1. We will introduce several BFM models
   - General: complex form, real form
   - Radial: with/without shunt admits.

2. Each model defined by
   - Set of variables
   - Set of power flow equations relating these variables

3. These models are equivalent to each other, and to BIM
General network
Branch currents

Sending-end currents

\[ I_{jk} = y_{jk}^s (V_j - V_k) + y_{jk}^m V_j, \quad I_{kj} = y_{jk}^s (V_k - V_j) + y_{kj}^m V_k, \]

Bus injection model: relate nodal variables \( s \) and \( V \)

\[ s_j = \sum_{k:j \sim k} \left( y_{jk}^s \right)^H \left( |V_j|^2 - V_j V_k^H \right) + \left( y_{jj}^m \right)^H |V_j|^2 \]
General network

Complex form

Branch flow model: includes branch vars as well

- Branch currents \( (I_{jk}, I_{kj}) \)
- Branch power \( (S_{jk}, S_{kj}) \)

\[
S_j = \sum_{k : j \sim k} S_{jk}
\]

\[
S_{jk} = V_j I_{jk}^H, \quad S_{kj} = V_k I_{kj}^H
\]

\[
I_{jk} = y_{jk}^s (V_j - V_k) + y_{jk}^m V_j
\]

\[
I_{kj} = y_{kj}^s (V_k - V_j) + y_{kj}^m V_k
\]

This model is equivalent to BIM (later)
- Serves as a bridge to BIM

(b) \Pi \text{ equivalent circuit}
General network
Real form

Key feature of original Dist Flow equations (branch flow model) of Baran-Wu1989

• No voltage/current phase angles
• Suitable for radial networks (tree topology)
• We generalize to meshed networks

For each bus \( j \)

• \( s_j := (p_j, q_j) \) or \( s_j := p_j + iq_j \) : power injection
• \( v_j \) : squared voltage magnitude

For each branch \((j, k)\)

• \( S_{jk} := \left(P_{jk}, Q_{jk}\right) \) or \( S_{jk} := P_{jk} + iQ_{jk} \) : sending-end power \( j \to k \); also \( S_{kj} \) from \( k \to j \)
• \( \left(\ell_{jk}, \ell_{kj}\right) \) : squared magnitude of sending-end current \( j \to k \), and \( k \to j \)
General network

Real form

The variables $v_i$ and $\left(\ell_{jk}, \ell_{kj}\right)$ contain no angle information

Angle info must be recoverable from a power flow solution $x := (s, v, \ell, S) \in \mathbb{R}^{3(N+1)+6M}$

- This is easy for radial networks
- Trickier for meshed networks
For each line $(j, k)$ let:

$$z_{jk}^s := \left(y_{jk}^s\right)^{-1} =: z_{kj}^s$$

$$\alpha_{jk} := 1 + z_{jk}^s y_{jk}^m, \quad \alpha_{kj} := 1 + z_{kj}^s y_{kj}^m$$

$$\alpha_{jk} = \alpha_{kj} \text{ if and only if } y_{jk}^m = y_{kj}^m$$

$$\alpha_{jk} = \alpha_{kj} = 1 \text{ if and only if } y_{jk}^m = y_{kj}^m = 0$$
General network

Real form

For each line \((j, k)\) let:

\[
z^s_{jk} := \left(y^s_{jk}\right)^{-1} =: z^s_{kj}
\]

\[
\alpha_{jk} := 1 + z^s_{jk} y^m_{jk}, \quad \alpha_{kj} := 1 + z^s_{kj} y^m_{kj}
\]

Given \(x := (s, v, \ell, S) \in \mathbb{R}^{3(N+1)+6M}\) define nonlinear functions:

\[
\beta_{jk}(x) := \angle\left(\alpha^H_{jk} v_j - \left(z^s_{jk}\right)^H S_{jk}\right)
\]

\[
\beta_{kj}(x) := \angle\left(\alpha^H_{kj} v_k - \left(z^s_{jk}\right)^H S_{kj}\right)
\]

If \(x\) is a power flow solution, then \(\left(\beta_{jk}(x), \beta_{kj}(x)\right)\) are angle differences across \((j, k)\)
General network

Real form

\[ s_j = \sum_{k:j \sim k} S_{jk} \]

power balance
General network

Real form

\[ s_j = \sum_{k:j \sim k} S_{jk} \]

\[ |S_{jk}|^2 = v_j \ell_{jk}, \quad |S_{kj}|^2 = v_k \ell_{kj} \]

power balance

branch power magnitude

The complex notation is only shorthand for real equations

\[ p_j = \sum_k P_{jk}, \quad q_j = \sum_k Q_{jk} \]

\[ v_j \ell_{jk} = P_{jk}^2 + Q_{jk}^2, \quad v_k \ell_{kj} = P_{kj}^2 + Q_{kj}^2 \]
General network
Real form

\[ s_j = \sum_{k:j \sim k} S_{jk} \]

\[ \left| S_{jk} \right|^2 = v_j \ell_{jk}, \quad \left| S_{kj} \right|^2 = v_k \ell_{kj} \]

\[ \left| \alpha_{jk} \right|^2 v_j - v_k = 2 \text{Re} \left( \alpha_{jk} \left( z_{jk}^s \right)^H S_{jk} \right) - \left| z_{jk}^s \right|^2 \ell_{jk} \]

\[ \left| \alpha_{kj} \right|^2 v_k - v_j = 2 \text{Re} \left( \alpha_{kj} \left( z_{kj}^s \right)^H S_{kj} \right) - \left| z_{kj}^s \right|^2 \ell_{kj} \]

power balance
branch power magnitude
Ohm’s law, KCL (magnitude)
General network

Real form

\[ s_j = \sum_{k:\, j \sim k} S_{jk} \]

there exists \( \theta \in \mathbb{R}^{N+1} \) s.t.

\[ \beta_{jk}(x) = \theta_j - \theta_k \]
\[ \beta_{kj}(x) = \theta_k - \theta_j \]
General network

Real form

Cycle condition on $x$ is highly nonlinear

$$\beta_{jk}(x) := \angle \left( \alpha_{jk}^H v_j - \left( z_{jk}^s \right)^H S_{jk} \right)$$

$$\beta_{kj}(x) := \angle \left( \alpha_{kj}^H v_k - \left( z_{jk}^s \right)^H S_{kj} \right)$$

$$\beta(x) = \begin{bmatrix} C^T \\ -C^T \end{bmatrix} \theta \text{ for some } \theta \in \mathbb{R}^{N+1}$$

It ensures angle consistency of a power flow solution $x$
General network
Real form

Any \( x := (s, v, \ell, S) \in \mathbb{R}^{3(N+1)+6M} \) that satisfies power flow equations with

\[ v \geq 0, \quad \ell \geq 0 \]

is a power flow solution

Branch flow models have been most useful for radial networks
All BFMs for radial networks are special cases of this model

- Tree topology
- Tree topology with zero shunt admittances \( y_{jk}^m = y_{kj}^m = 0 \)
- Tree topology with linear approximations
Outline

1. General network

2. Radial network
   • With shunt admittances
   • Without shunt admittances

3. Equivalence

4. Backward forward sweep

5. Linearized model
Radial network

Cycle condition

Major simplification for radial network: nonlinear cycle condition becomes linear in $x$

$$\beta_{jk}(x) := \angle \left( \alpha_{jk}^H v_j - \left( z_{jk}^s \right)^H S_{jk} \right)$$

$$\beta_{kj}(x) := \angle \left( \alpha_{kj}^H v_k - \left( z_{jk}^s \right)^H S_{kj} \right)$$

$$\beta(x) = \begin{bmatrix} C^T \\ -C^T \end{bmatrix} \theta \text{ for some } \theta \in \mathbb{R}^{N+1}$$

radial network

general network
Radial network
With shunt admittances

\[ s_j = \sum_{k:j \sim k} S_{jk} \]

\[ |S_{jk}|^2 = v_j \ell_{jk}, \quad |S_{kj}|^2 = v_k \ell_{kj} \]

\[ |\alpha_{jk}|^2 v_j - v_k = 2 \text{Re} \left( \alpha_{jk} \left( z_{jk}^s \right)^H S_{jk} \right) - \left| z_{jk}^s \right|^2 \ell_{jk} \]

\[ |\alpha_{kj}|^2 v_k - v_j = 2 \text{Re} \left( \alpha_{kj} \left( z_{kj}^s \right)^H S_{kj} \right) - \left| z_{kj}^s \right|^2 \ell_{kj} \]

\[ \alpha_{jk}^H v_j - \left( z_{jk}^s \right)^H S_{jk} = \left( \alpha_{kj}^H v_k - \left( z_{kj}^s \right)^H S_{kj} \right)^H \]

power balance

branch power magnitude

Ohm’s law, KCL (magnitude)

cycle condition

2(N + 1) + 6M real equations in 3(N + 1) + 6M real vars (M = N)
Radial network
With shunt admittances

Any $x := (s, v, \ell, S) \in \mathbb{R}^{3(N+1)+6M}$ that satisfies power flow equations with

$v \geq 0, \quad \ell \geq 0$

is a power flow solution

All equations are linear in $x$, except the quadratic equalities

$\left| S_{jk} \right|^2 = v_j \ell_{jk}, \quad \left| S_{kj} \right|^2 = v_k \ell_{kj}$

This can be relaxed to second-order cone constraint in OPF (later)
General network

Angle recovery

Treat network $G := (\overrightarrow{N}, E)$ as directed graph with arbitrary orientation

- (Re-)Define branch variables $\begin{pmatrix} S_{jk}, \ell_{jk} \end{pmatrix}$ only in direction of lines $(j, k)$
- (Re-)Define $\beta(x) := \begin{pmatrix} \beta_{jk}(x), (j, k) \in E \end{pmatrix}$ where

$$
\beta_{jk}(x) := \angle \left( \alpha_{jk}^H v_j - \begin{pmatrix} z_{jk}^S \\ S_{jk} \end{pmatrix}^H \right)
$$

Any power flow solution $x := (s, v, \ell, S) \in \mathbb{R}^{3(N+1)+6M}$ implies

$$
\beta(x) = C^T \theta \quad \text{for some } \theta \in \mathbb{R}^{N+1}
$$

Angle recovery:

$$
V_j = \sqrt{v_j} e^{i\theta_j}, \quad I_{jk} = \sqrt{\ell_{jk}} e^{i(\theta_j - \angle S_{jk})}
$$
Radial network
Without shunt admittances

When shunt admittances \( y_{jk}^m = y_{kj}^m = 0 \)

- \( \alpha_{jk} = \alpha_{kj} = 1 \)
- \( \ell_{kj} = \ell_{jk} \) and \( S_{kj} + S_{jk} = z_{jk}^s \ell_{jk} \)

Can use directed graph with vars \( (\ell_{jk}, S_{jk}) \) defined only in direction of lines \((j, k)\)

Substitute \( (\ell_{kj}, S_{kj}) \) in terms of \( (\ell_{jk}, S_{jk}) \) into previous power flow equations yields original DistFlow equations of [Baran-Wu 1989]
Radial network
Without shunt admittances

DistFlow equations [Baran-Wu 1989] (down direction):

\[
\sum_{k:j \to k} S_{jk} = S_{ij} - z_{ij} \ell_{ij} + s_j \quad \text{power balance}
\]

\[
v_j - v_k = 2 \text{ Re} \left( z_{jk}^H S_{jk} \right) - |z_{jk}|^2 \ell_{jk} \quad \text{Ohm's law, KCL (magnitude)}
\]

\[
v_j \ell_{jk} = |S_{jk}|^2 \quad \text{branch power magnitude}
\]

2\((N + 1) + 2M\) real equations in 3\((N + 1) + 3M\) real vars \((M = N)\)

- Given \((v_0, s_j, j \in N)\), there are 4\(N + 2\) equations in 4\(N + 2\) vars \((s_0, v_j, j \in N, \ell, S)\)

All lines point away from bus 0
Radial network
Without shunt admittances

DistFlow equations [Baran-Wu 1989] (down direction):
\[
\sum_{k:j \rightarrow k} S_{jk} = S_{ij} - z_{ij} \ell_{ij} + s_j \\
v_j - v_k = 2 \text{Re} \left( z_{jk}^H S_{jk} \right) - |z_{jk}|^2 \ell_{jk} \\
v_j \ell_{jk} = |S_{jk}|^2
\]

All equations are linear in \( x \), except the quadratic equalities
\[
v_j \ell_{jk} = \left| S_{jk} \right|^2
\]

All lines point away from bus 0
Outline

1. General network
2. Radial network
3. Equivalence
4. Backward forward sweep
5. Linearized model
Equivalence

Recap

Bus injection model
- General networks: complex form, polar form, Cartesian form

Branch flow model
- General networks: complex form, real form
- Radial networks: with / without shunt admittances

All these models are equivalent
- In what sense?
- They consist of different equations with different variables in different domains
**Equivalence**

**Solution set**

Bus injection model

\[
    s_j = \sum_{k:j \sim k} \left( y^s_{jk} \right)^H \left( |V_j|^2 - V_j V^H_k \right) + \left( y^m_{jj} \right)^H |V_j|^2
\]

Solution set

\[
    \mathbb{V} := \{(s, V) \in \mathbb{C}^{2(n+1)} \mid V \text{ satisfies BIM}\}
\]
Equivalence

Solution set

Branch flow models: solution sets

\[ \tilde{\Xi} := \{ \tilde{x} : (s, V, I, S) \in \mathbb{C}^{2(N+1)+4M} \mid \tilde{x} \text{ satisfies BFM complex} \} \]

\[ \Xi_{\text{meshed}} := \{ x : (s, v, \ell, S) \in \mathbb{R}^{3(N+1)+6M} \mid x \text{ satisfies BFM real} \} \]

\[ \Xi_{\text{tree}} := \{ x : (s, v, \ell, S) \in \mathbb{R}^{9N+3} \mid x \text{ satisfies BFM radial} \} \]

\[ \mathbb{T}_0 := \{ x : (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies BFM radial zero } y_{jk}^m \} \]

Definition: Two sets \( A \) and \( B \) are equivalent \((A \equiv B)\) if there is a bijection between them
Equivalence
Solution set

Theorem
Suppose $G$ is connected
1. $\forall \equiv \tilde{X} \equiv X_{\text{meshed}}$
2. If $G$ is a tree, then $X_{\text{meshed}} \equiv X_{\text{tree}}$
3. If $G$ is a tree and $y_{jk}^m = y_{kj}^m = 0$, then $X_{\text{tree}} \equiv T_0 \equiv \hat{T}_0$
Equivalence

Bus injection models and branch flow models are equivalent

- Any result proved in one model holds also in another model

Some results are easier to formulate / prove in one model than the other

- BIM: semidefinite relaxation of OPF (later)
- BFM: some exact relation proofs

Should freely use whichever is more convenient for problem at hand

BFM is particularly suitable for modeling distribution systems

- Tree topology allows efficient computation of power flows (BFS)
- Seems to be much more numerically stable than BIM for large networks
- Models and relaxations extend to unbalanced 3φ networks
Outline

1. General network
2. Radial network
3. Equivalence
4. Backward forward sweep
   • For radial networks
5. Linearized model
Backward forward sweep

Efficient solution method for power flow equations

- Applicable for radial networks

Partition solution \((x, y)\) into two groups of variables \(x\) and \(y\)

Each round of spatial iteration consists of a backward sweep and a forward sweep

- Given \(y\), compute each component \(x_j\) iteratively from leafs to root (backward)
- Given \(x\), compute each component \(y_j\) iteratively from root to leaves (forward)

Iterate until stopping criterion

Different BFS methods differ in how to partition variables into \(x\) and \(y\) and the associated power flow equations
Backward forward sweep

**Example**

Use complex form BFM

**Given:** $V_0$ and $s := \left( s_j, j \in N \right)$

**Compute:** $V := \left( V_j, j \in N \right)$ and currents $I^s := \left( I^s_{jk}, (j, k) \in E \right)$ through series impedance

- All other variables $I_{jk} = I^s_{jk} + y^m_{jk} V_j, I_{kj}, S_{jk}, S_{kj}$ can then be computed
- Can also compute $V_j$ and $I_{jk}$ instead (exercise)
- Advantage: $I^s_{jk} = - I^s_{kj}$
Backward forward sweep

Example

Power flow equation

\[ s_j = V_j^{H} + \sum_{k:j \rightarrow k} V_j^{H} = V_j \left( \left( I_{ji}^s + y_{ji}^m V_j \right)^H + \sum_{k:j \rightarrow k} \left( I_{jk}^s + y_{jk}^m V_j \right)^H \right) \]

Substitute \( I_{kj}^s = - I_{jk}^s \) to write all vars in direction of line \( j \rightarrow k \):

\[ \left( \frac{s_j}{V_j} \right)^H = - I_{ij}^s + y_{ij}^m V_j + \sum_{k:j \rightarrow k} I_{jk}^s \]

where \( y_{jj}^m := \sum_k y_{jk}^m \)
Backward forward sweep

Example

Rewrite in spatially recursive structure

\[ I^s_{ij} = \sum_{k:j \rightarrow k} I^s_{jk} - \left( \left( \frac{s_j}{V_j} \right)^H - y^m_{jj} V_j \right) \]

Spatial iteration: propagating from leaves towards root (bus 0) in reverse BFS order

- Given all voltages \( V := \left( V_j, j \in \mathbb{N} \right) \)
- Given all currents \( I^s_{jk} \) in previous layer
- Compute currents \( I^s_{ij} \) in current layer
Backward forward sweep

Example

Write in \textit{spatially recursive} structure

\[ V_j = V_i - z_{ij}^s I^s_{ij}, \]

\textbf{Spatial iteration}: propagating from root (bus 0) towards leaves in BFS order

- Given all currents \( I^s := \left( I^s_{ij}, (i,j) \in E \right) \)
- Given all voltages \( V_i \) in previous layer
- Compute voltages \( V_j \) in current layer

Consider bus \( j \) whose parent node is \( i \) (so \( i \neq j \) is a directed edge) and whose children nodes are \( k \) (so \( j \neq k \) are directed edges). Use (6.1c) to eliminate branch power \( S_{jk} \) from (6.1a)(6.1b) to obtain

\[ s_j = S_{ji} + \sum_{k} I^s_{jk}, \]

\[ V_j I_{Hji} + \sum_{k} V_j I_{Hjk} = V_j I_{sji} + y_{mji} V_j, \]

with the understanding that \( S_{jk} = I_{jk} = I^s_{jk} = 0 \) if \( j \) is a leaf node. Substituting \( I^s_{ji} = I^s_{ij} \) we can write power balance in terms of currents \( I^s_{jk} \) in the direction of the lines:

\[ ✓ s_j V_j ◆ H = I^s_{ij} + y_{mjj} V_j + \sum_{k} I^s_{jk}, \]

where \( y_{mjj} := y_{mji} + \sum_{k} y_{mjk} \) is the total shunt admittance incident on bus \( j \). This can be rearranged to highlight the recursive structure as:

\[ I^s_{ij} = \sum_{k} I^s_{jk}, \]

\[ ✓ s_j V_j ◆ H y_{mjj} V_j, \]

\( i \neq j \in E \) (6.15a)
Backward forward sweep

Summary

*Input*: voltage $V_0 = 1$ pu and injections $(s_i, i \in N)$.
*Output*: currents $(I^s_{jk}, j \rightarrow k \in E)$ and voltages $(V_i, i \in N)$.

1. *Initialization.*

   - $V_0(t) := 1$ pu at bus $j = 0$ for all iterations $t = 1, 3, \ldots$
   - $V_j(0) := 1$ pu at all buses $j \in N$ for iteration $t = 0$. 

\[ \text{Figure 6.4: Spatially recursive structure of the backward forward sweep.} \]
Backward forward sweep

Summary

2. Backward forward sweep: iterate \( t = 1, 3, 5, \ldots \) till stopping criterion

(a) Backward sweep. Starting from the leaf nodes and working towards bus 0, compute

\[
I_{ij}^s(t) \leftarrow \sum_{k:j \rightarrow k} I_{jk}^s(t) - \left( \left( \frac{s_j}{V_j(t-1)} \right)^H - y_{jj}^m V_j(t-1) \right), \quad i \rightarrow j \in E
\]

(b) Forward sweep. Starting from bus 0 and working towards the leaf nodes, compute

\[
V_j(t+1) = V_i(t+1) - z_{ij}^s I_{ij}^s(t), \quad j \in N
\]

3. Output: \( I_{jk}^s := I_{jk}^s(t), \quad V_i := V_i(t + 1) \)
**Backward forward sweep**

**General formulation**

Backward sweep: let

- $T_i^o := \{ \text{buses in subtree rooted at } i, \text{ excluding } i \}$
- $x_{T_i^o} := \left( x_j, j \in T_i^o \right)$

$x$ satisfies a **spatially recursive** structure if

$$x_i = f_i \left( x_{T_i^o}, y \right)$$
**Backward forward sweep**

**General formulation**

Forward sweep: let

- $P_i := \{\text{buses in path from root to } i, \text{ inc. 0 but exc. } i\}$
- $y_{P_i} := (y_j, j \in P_i)$

$y$ satisfies a **spatially recursive** structure if

$$y_i = g_i(y_{P_i}; x)$$

Spatial initialization

$$y_j = g_j(y_{0}; x)$$
Backward forward sweep
General formulation

while stopping criterion not met do

(a) \( t \leftarrow t + 1; \ y_0(t) \leftarrow y_0; \)

(b) Backward sweep: for \( i \) starting from the leaf nodes and iterating towards bus 0 do

\[
x_i(t) \leftarrow f_i \left( x_{\mathcal{T}_i}(t); y(t-1) \right), \quad i \in \overline{N}
\]

c) Forward sweep: for \( i \) starting from bus 0 and iterating towards the leaf nodes do

\[
y_i(t + 1) \leftarrow g_i \left( y_{\mathcal{P}_i}(t + 1); x(t) \right), \quad i \in N
\]
Backward forward sweep
Open question

Convergence analysis
Outline

1. General network
2. Radial network
3. Equivalence
4. Backward forward sweep
5. Linearized model
   • Analytical solution
   • Bounds on nonlinear solutions
   • Application: decentralized volt/var control
Linearized model

Linear DistFlow equations

For radial networks

Set \( y_{jk}^m = y_{kj}^m = 0, \ell_{jk} = 0 \)

Linear DistFlow equations [Baran-Wu 1989]

\[
\sum_{k:j \to k} S_{jk} = \sum_{i:i \to j} S_{ij} + s_j
\]

\[
v_j - v_k = 2 \text{Re} \left( z_{jk}^H S_{jk} \right)
\]
Linearized model

Linear DistFlow equations

In vector form:

bus-by-line incidence matrix

\[ C_{jl} = \begin{cases} 
1 & \text{if } l = j \to k \text{ for some bus } k \\
-1 & \text{if } l = i \to j \text{ for some bus } i \\
0 & \text{otherwise} 
\end{cases} \]

Linear DistFlow equations

\[ s = CS \]
\[ CTv = 2(D_rP + D_xQ) \]

where \( D_r := \text{diag}(r_l, l \in E) \), \( D_x := \text{diag}(x_l, l \in E) \)

\[ \sum_{k:j\to k} S_{jk} = \sum_{i:i\to j} S_{ij} + s_j \]
\[ v_j - v_k = 2 \Re(z_{jk}^H S_{jk}) \]
Linearized model

Linear DistFlow equations

Linear DistFlow can be solved explicitly

Given: \( v_0 = 1 \text{ pu} \), injection \( \hat{s} := \left( s_j, j \in N \right) \)

Determine: line power \( S := \left( S_{jk}, j \rightarrow k \in E \right) \), voltage \( \hat{v} := \left( v_j, j \in N \right) \), injection \( s_0 \)

Key observation: Reduced incidence matrix has full rank

\( G \) connected \( \implies (N + 1) \times N \) incidence matrix \( C \) has rank \( N \)

Decompose \( C =: \begin{bmatrix} -c_0^T & - \end{bmatrix} \hat{C} \)

\( G \) tree topology \( \implies N \times N \) reduced incidence matrix \( \hat{C} \) is invertible
Linearized model

Linear DistFlow solution

Linear DistFlow:

\[
\hat{s} = \hat{C} S \\
v_0 c_0 + \hat{C}^T \hat{v} = 2 \left( D_r P + D_x Q \right) \\
\text{and } s_0 = c_0^T S
\]

Solution:

\[
S = \hat{C}^{-1} \hat{s} \\
\hat{v} = v_0 1 + 2 \left( R \hat{p} + X \hat{q} \right) \quad \text{voltages } = v_0 + \text{correction term } (\hat{p}, \hat{q})
\]

where \( R := \hat{C}^{-T} D_r \hat{C}^{-1}, \) \( X := \hat{C}^{-T} D_x \hat{C}^{-1} \) are positive definite matrices
Linearized model
Bounds on nonlinear solution

Corollary

Fix $v_0$ and injections $\hat{s} \in \mathbb{R}^{2N}$ at non-reference buses. Let $(v, \ell, S)$ and $(v_{\text{lin}}, \ell_{\text{lin}}, S_{\text{lin}})$ be a solution of nonlinear and linear DistFlow equations respectively (in the down direction).

1. $S_{ij} \geq S_{ij}^{\text{lin}}$
   
   Linear DistFlow ignores line losses and underestimates required power to supply loads

2. $v_j \leq v_j^{\text{lin}}$
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Linearized model
Application: volt/var control

Volv/var control: control reactive power injections $q$ to stabilize voltages $\hat{v}$

How should $q$ adapt as voltages fluctuate?
Linearized model
Application: volt/var control

Local memoryless feedback control:

\[ q_j(t + 1) = \left[ u_j \left( v_j(t) - v_j^{\text{ref}} \right) \right]_{U_j} \]

Adapt reactive power \( q_i(t) \) to drive voltage \( v_j(t) \) towards target \( v_j^{\text{ref}} \)

Control \( q_j(t + 1) \) depends only on:

- Feedback: measured system state \( v(t) \)
- Memoryless: latest voltage \( v(t) \) at time \( t \), not history \( v(s), s < t \)
- Local: local voltage \( v_j(t) \) at bus \( j \), not other voltages \( v_k(t) \)
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Local memoryless feedback control:

\[ q_j(t + 1) = \left[u_j(v_j(t) - v_j^{\text{ref}})\right]_{U_j} \]

Adapt reactive power \( q_i(t) \) to drive voltage \( v_j(t) \) towards target \( v_j^{\text{ref}} \)

How does the closed-loop system behave?

• under this simple local control
• if network is described by Linear DistFlow
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Linear DistFlow model describes how voltages (linearly) depend on control $q$:

$$v(q) = v_0 1 + 2 (Rp + Xq) = 2Xq + \tilde{v}$$

where $\tilde{v} := v_0 1 + 2Rp$

Since

$$\frac{\partial v_j}{\partial q_j} = 2X_{jj} = \sum_{(i,k) \in P_j} x_{ik} > 0$$

it justifies choosing $u_j$ to be a decreasing function of $v_j(t) - v_j^{\text{ref}}$
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Assume measured voltage is given by Linear DistFlow, i.e., $v_j(t) = v_j(q(t))$

Closed-loop system is discrete-time dynamical system:

$$q_j(t + 1) = \left[ u_j \left( v_j(q(t)) - v_j^{\text{ref}} \right) \right]_{U_j}$$

where $v(q) = 2Xq + \tilde{v}$

**Definition:** $q^*$ is a fixed point or equilibrium point if $q^* = \left[ u \left( v(q^*) - v^{\text{ref}} \right) \right]_{U_j}$
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What are convergence and optimality properties of closed-loop system?
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Closed-loop system is discrete-time dynamical system:

\[ q_j(t + 1) = \left[ u_j \left( v_j(q(t)) - v_j^{\text{ref}} \right) \right] U_j \]

where \( v(q) = 2Xq + \tilde{v} \)

Assumptions

- \( u_j \) are differentiable and \( \left| u_j'(v_j) \right| \leq \alpha_j \)
- \( u_j \) are strictly decreasing
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**Theorem** (Convergence)

If largest singular value $\sigma_{\text{max}}(AX) < 1/2$ then

1. $\exists$ ! equilibrium point $q^* \in U$

2. Closed-loop system converges to $q^*$ geometrically, i.e.,

$$\|q(t) - q^*\| \leq \beta^t \|q(0) - q^*\| \rightarrow 0$$

for some $\beta \in (0,1)$

$$A := \text{diag}\left(\alpha_j, j \in N\right)$$
Linearized model
Application: volt/var control

\textbf{Theorem} (Optimality)

The unique equilibrium point $q^* \in U$ solves

$$
\min_{q \in U} \sum_j c_j(q_j) + q^T X q + q^T \Delta \tilde{v}
$$

where $c_j(q_j) := - \int_0^{q_j} u_j^{-1}(\hat{q}_j) \, d\hat{q}_j$ and $\Delta \tilde{v} := \tilde{v} - v^{\text{ref}}$

\textbf{Reverse engineering}: by choosing a control function $u_j$, we implicitly choose a cost function $c_j(q_j)$ that the closed-loop equilibrium optimizes
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**Forward engineering**: Choose a cost function \( c_j(q_j) \) and derive control functions \( u_j \) as a distributed algorithm to solve the optimization problem
Summary

1. General network
   • Complex form, real form

2. Radial network
   • With and without shunt admittances

3. Equivalence

4. Backward forward sweep

5. Linearized model
   • Analytical solution, bounds, local volt/var control