Power System Analysis

Chapter 10 Unbalanced network: BFM

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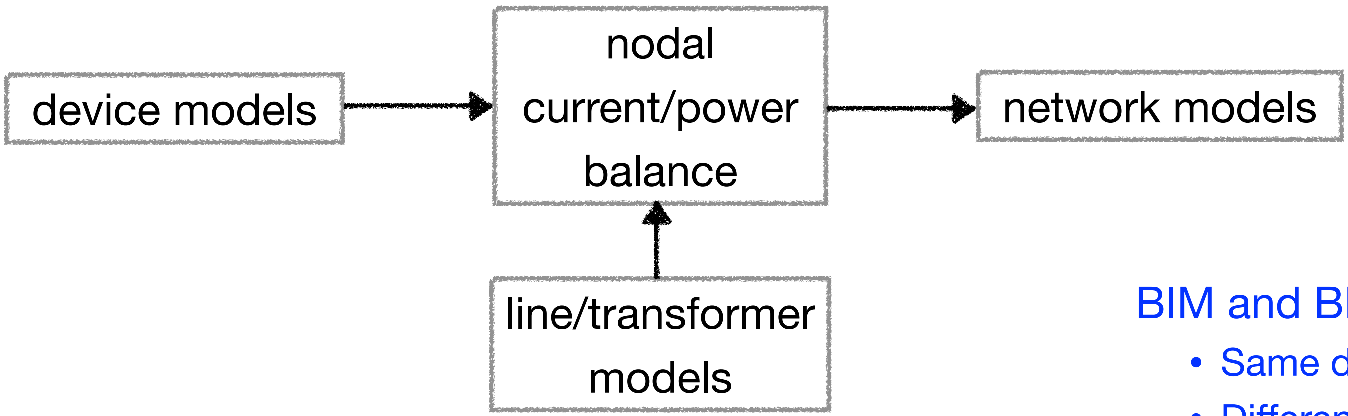
Outline

1. General network
2. Radial network
3. Overall network
4. Backward-forward sweep
5. Linear network

Overview



BIM and BFM : network models
- Same device models
- Different line models
- Equivalent, 1 or 3-phase

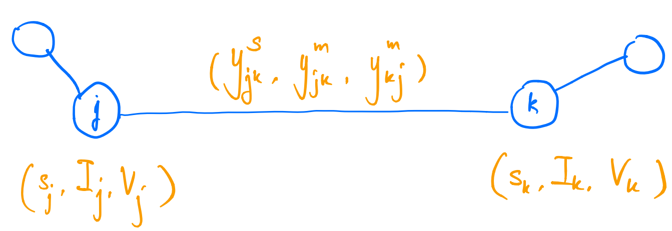
single-phase or 3-phase

Outline

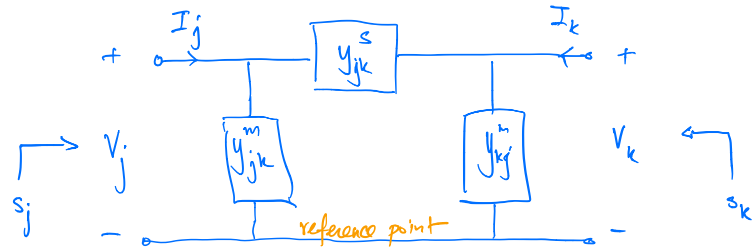
1. General network
    - Review: single-phase BFM
    - Three-phase model
    - Equivalence
2. Radial network
3. Overall network
4. Backward-forward sweep
5. Linear network

Review: single-phase BFM

1. Network $G := (\overline{N}, E)$
   - $\overline{N} := \{0\} \cup N := \{0\} \cup \{1,\ldots,N\}$: buses/nodes
   - $E \subseteq \overline{N} \times \overline{N}$: lines/links/edges
2. Each line (j, k) is parameterized by $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$
   - y^s_{jk} : series admittance
   - y^m_{jk}, y^m_{kj} : shunt admittances, generally different



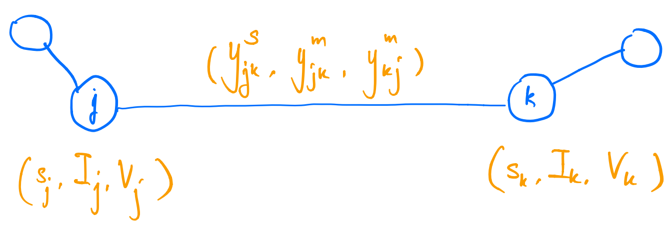
(a) Graph representation



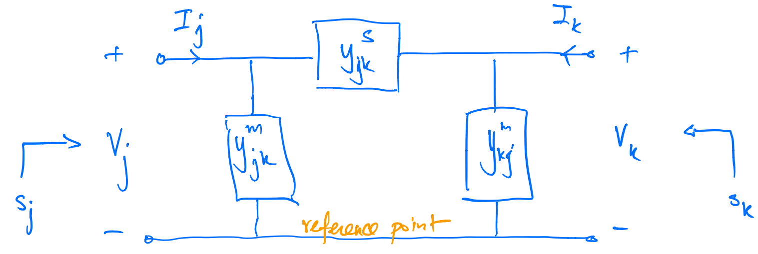
(b) Π equivalent circuit

Review: single-phase BFM

Branch flows



(a) Graph representation



(b) Π equivalent circuit

Sending-end currents

$$I_{jk} \;=\; y^s_{jk}(V_j - V_k) \;+\; y^m_{jk}\, V_j, \qquad I_{kj} \;=\; y^s_{jk}(V_k - V_j) \;+\; y^m_{kj}\, V_k,$$

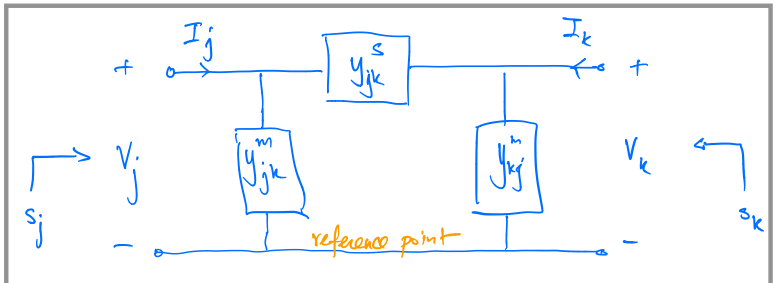
Bus injection model: relate nodal variables s and V

$$s_j = \sum_{k:j\sim k} \left(y^s_{jk}\right)^H \left(|V_j|^2 - V_j V_k^H\right) \;+\; \left(y^m_{jj}\right)^H |V_j|^2$$

Review: single-phase BFM
Branch flows

Branch flow model: includes branch vars as well

- Branch currents I_{jk} , branch power S_{jk}
- Adopt directed graph $\left(I_{kj} = - I_{jk}, \quad S_{kj} = - \left(S_{jk} - z_{jk} \left| I_{jk} \right|^2 \right) \right)$
- Assume $y_{jk}^m = y_{kj}^m = 0$



(b) Π equivalent circuit

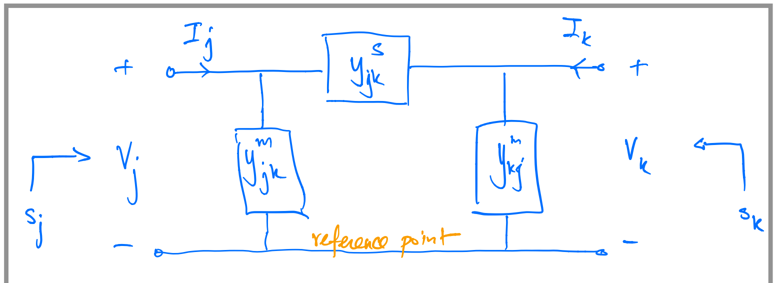
$$\sum_{k:j \to k} S_{jk} = \sum_{i:i \to j} \left(S_{ij} - z_{ij} \ell_{ij} \right) + s_j$$
$$V_j - V_k = z_{jk} I_{jk}$$
$$S_{jk} = V_j I_{jk}^{\mathsf{H}}$$
$$\ell_{jk} = |I_{jk}|^2$$

This model is equivalent to single-phase BIM

Three-phase BFM

Assumption:
3-phase Π circuit representation $\left(y^s_{jk} = y^s_{kj}\right)$
Applicable:
- Transmission or distribution lines
- Transformers in YY and $\Delta\Delta$ config. (not ΔY or $Y\Delta$)

Assumption:
Often assume $y^m_{jk} = y^m_{kj} = 0$



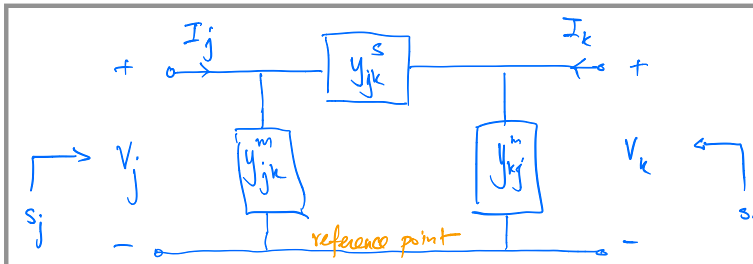
(b) Π equivalent circuit

$$z_{jk} := \left(y^s_{jk}\right)^{-1} \in \mathbb{C}^{3\times 3}, \quad V_j, s_j, I_{jk} \in \mathbb{C}^3$$

Three-phase BFM

Branch vars are outer products (rank-1 matrices)

- Branch current matrix: $\ell_{jk} := I_{jk} I_{jk}^{\mathsf{H}} \in \mathbb{C}^{3\times 3}$
- Branch power matrix: $S_{jk} := V_{jk} I_{jk}^{\mathsf{H}} \in \mathbb{C}^{3\times 3}$



(b) Π equivalent circuit

$$z_{jk} := \left(y^s_{jk}\right)^{-1} \in \mathbb{C}^{3\times 3}, \quad V_j, s_j, I_{jk} \in \mathbb{C}^3$$

Unbalanced 3-phase BFM (general network)

$$\begin{aligned}
\sum_{k:j\to k} \mathrm{diag}(S_{jk}) &= \sum_{i:i\to j} \mathrm{diag}\left(S_{ij} - z_{ij}\ell_{ij}\right) + s_j, & j \in \overline{N} \\
V_j - V_k &= z_{jk} I_{jk}, & j \to k \in E \\
S_{jk} &= V_j I_{jk}^{\mathsf{H}}, & j \to k \in E \\
\ell_{jk} &= I_{jk} I_{jk}^{\mathsf{H}}, & j \to k \in E
\end{aligned}$$

direct extension of single-phase BIM

Three-phase BFM

Recall 3-phase BIM (Ch 8)

$$\mathbb{V} \; := \; \left\{ (s, V) \in \mathbb{C}^{6(N+1)} \;\middle|\; s_j \; = \sum_{k:j\sim k} \text{diag}\left(V_j (V_j - V_k)^H \left(y_{jk}^s\right)^H \; + \; V_j V_j^H \left(y_{jk}^m\right)^H \right), \text{ given } V_0 \right\}$$

3-phase BFM (general network)

$$\tilde{\mathbb{X}} \; := \; \left\{ \tilde{x} := (s, V, I, \ell, S) \in \mathbb{C}^{6(N+1)+21M} \;\middle|\; \tilde{x} \text{ satisfies BFM, given } V_0 \right\}$$

<u>Theorem</u> (equivalence)

$$\mathbb{V} \equiv \tilde{\mathbb{X}}$$

Outline

1. General network
2. Radial network
   - Single-phase BFM
   - Three-phase model
   - Equivalence
3. Overall network
4. Backward-forward sweep
5. Linear network

Review: single-phase BFM

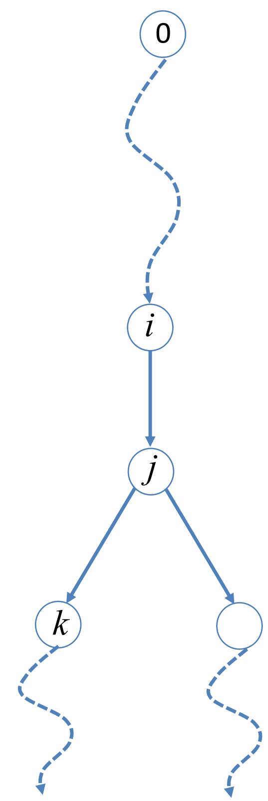
Without shunt admittances

DistFlow equations [Baran-Wu 1989] (radial network)

$$\sum_{k:j\to k} S_{jk} = S_{ij} - z_{ij}\ell_{ij} + s_j \qquad \text{power balance}$$

$$v_j - v_k = 2\,\mathrm{Re}\left(z_{jk}^H S_{jk}\right) - |z_{jk}|^2 \ell_{jk} \qquad \text{Ohm's law, KCL (magnitude)}$$

$$v_j \ell_{jk} = |S_{jk}|^2 \qquad \text{branch power magnitude}$$



All lines point away from bus 0

Three-phase BFM

BFM vars (radial network)

$$s_j \in \mathbb{C}^3, \qquad v_j \in \mathbb{S}^3_+, \qquad j \in \overline{N}$$
$$\ell_{jk} \in \mathbb{S}^3_+, \qquad S_{jk} \in \mathbb{C}^{3\times 3}, \qquad j \to k \in E$$

same set of vars but scalars in single-phase BFM

- $\mathbb{S}^n_+ \subseteq \mathbb{C}^{n\times n}$: $n \times n$ complex (Hermitian) positive definite matrices

Three-phase BFM

Three-phase BFM (radial network)

$$\sum_{k:j\to k} \text{diag}(S_{jk}) = \text{diag}\left(S_{ij} - z_{ij}\ell_{ij}\right) + s_j$$

$$v_j - v_k = \left(z_{jk} S_{jk}^{\mathsf{H}} + S_{jk} z_{jk}^{\mathsf{H}}\right) - z_{jk}\,\ell_{jk}\, z_{jk}^{\mathsf{H}}$$

$$\begin{bmatrix} v_j & S_{jk} \\ S_{jk}^{\mathsf{H}} & \ell_{jk} \end{bmatrix} \geq 0$$

$$\text{rank} \begin{bmatrix} v_j & S_{jk} \\ S_{jk}^{\mathsf{H}} & \ell_{jk} \end{bmatrix} = 1$$

Single-phase BFM (DistFlow)

$$\sum_{k:j\to k} S_{jk} = S_{ij} - z_{ij}\ell_{ij} + s_j$$

$$v_j - v_k = 2\,\text{Re}\left(z_{jk}^H S_{jk}\right) - |z_{jk}|^2 \ell_{jk}$$

$$v_j \ell_{jk} = |S_{jk}|^2$$

Three-phase BFM

Three-phase BFM (radial network)

$$\sum_{k:j\to k} \text{diag}(S_{jk}) = \text{diag}\left(S_{ij} - z_{ij}\ell_{ij}\right) + s_j$$

$$v_j - v_k = \left(z_{jk} S_{jk}^{\mathsf{H}} + S_{jk} z_{jk}^{\mathsf{H}}\right) - z_{jk}\ell_{jk} z_{jk}^{\mathsf{H}}$$

$$\begin{bmatrix} v_j & S_{jk} \\ S_{jk}^{\mathsf{H}} & \ell_{jk} \end{bmatrix} \succeq 0$$

$$\text{rank} \begin{bmatrix} v_j & S_{jk} \\ S_{jk}^{\mathsf{H}} & \ell_{jk} \end{bmatrix} = 1$$

Remark

1. BFM vars do not contain $V_j, I_{jk} \in \mathbb{C}^3$
2. psd rank-1 condition ensures $\exists \left(V_j, I_{jk}\right)$ s.t.

   $$v_j = V_j V_j^{\mathsf{H}}, \qquad \ell_{jk} = I_{jk} I_{jk}^{\mathsf{H}}, \qquad S_{jk} = V_j I_{jk}^{\mathsf{H}}$$
3. Given $\left(v_j, \ell_{jk}, S_{jk}\right)$, $\left(V_j, I_{jk}\right)$ is unique up to a ref angle

Three-phase BFM

3-phase BFM (general network)

$$\tilde{\mathbb{X}} \quad := \quad \left\{ \tilde{x} := (s, V, I, \ell, S) \in \mathbb{C}^{6(N+1)+21M} \;\middle|\; \tilde{x} \text{ satisfies BFM, given } V_0 \right\}$$

3-phase BFM (radial network)

$$\mathbb{X} \quad := \quad \left\{ x := (s, v, \ell, S) \in \mathbb{C}^{12(n+1)+18M} \;\middle|\; x \text{ satisfies radial BFM, given } V_0 \right\}$$

Theorem (equivalence)

If G is a tree, then $\mathbb{V} \equiv \tilde{\mathbb{X}} \equiv \mathbb{X}$

Outline

1. General network
2. Radial network
3. Overall network
   - Overall model
   - Examples
4. Backward-forward sweep
5. Linear network

Overall model

Device + network

1. Device model for each 3-phase device (same for BIM)
    - Internal model on $\left(V_j^{Y/\Delta}, I_j^{Y/\Delta}, s_j^{Y/\Delta}\right)$ + conversion rules
    - External model on $\left(V_j, I_j, s_j\right)$
    - Either can be used
    - Power source models are nonlinear; other devices are linear

2. Network model
    - BFM for radial networks on $x := (s, v, \ell, S)$
    - BFM for general networks on $\tilde{x} := (s, V, I, \ell, S)$
    - Both are nonlinear models
    - BFM is most useful for radial networks

Overall model

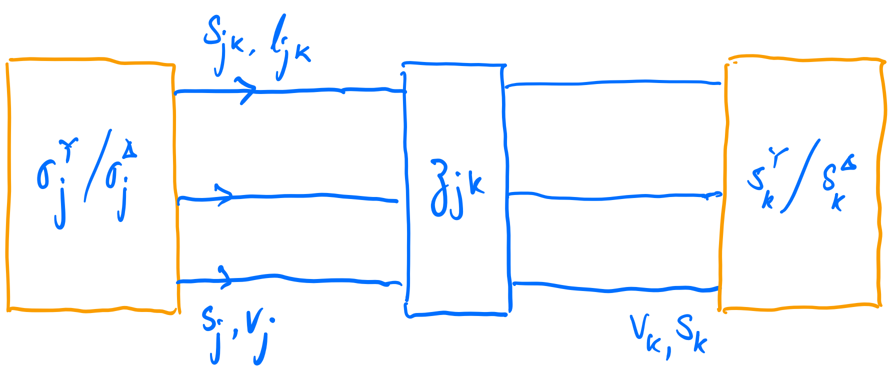
Device + network

Overall model is nonlinear whether or not power sources are present

- Network models are nonlinear for both radial or general networks
- Power sources, if present, are nonlinear

Example 1

Y configuration



Given:

- Constant-power source σ_j^Y with $\angle V_j^a := 0°$
- Impedance load z_k^Y
- Line parameters $\left(z_{jk}, y_{jk}^m = y_{kj}^m = 0\right)$
- Assumption C8.1 with $\gamma_j = V_j^n = \gamma_k = V_k^n = 0$

Calculate: $\left(s_k^Y, v_k, \ell_{jk}, S_{jk}\right)$

Network model (BFM radial):

$$\text{diag}(S_{jk}) \;=\; s_j, \qquad \text{diag}\left(S_{jk} - z_{jk}\ell_{jk}\right) \;=\; -\, s_k$$

$$v_j - v_k \;=\; \left(z_{jk}\, S_{jk}^{\mathsf{H}} + S_{jk} z_{jk}^{\mathsf{H}}\right) \;-\; z_{jk}\, \ell_{jk}\, z_{jk}^{\mathsf{H}}$$

$$\begin{bmatrix} v_j & S_{jk} \\ S_{jk}^{\mathsf{H}} & \ell_{jk} \end{bmatrix} \;\succeq\; 0, \quad \text{rank} \begin{bmatrix} v_j & S_{jk} \\ S_{jk}^{\mathsf{H}} & \ell_{jk} \end{bmatrix} \;=\; 1$$

Device model (internal model + conversion rule):

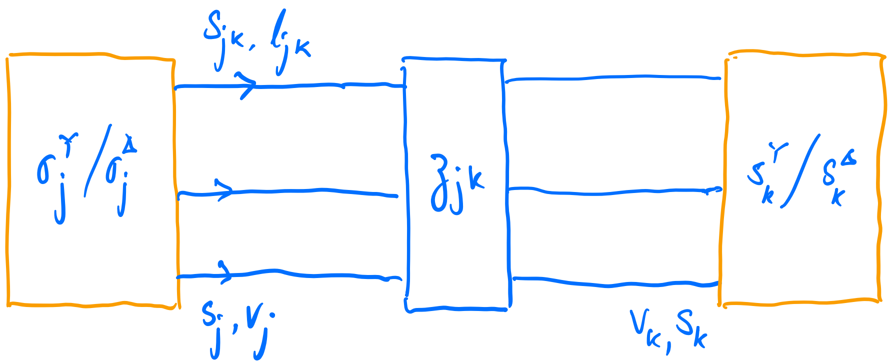
$$v_k \;=\; z_k^Y \ell_{jk}\, z_k^{Y\mathsf{H}}, \qquad s_k^Y \;=\; \text{diag}\left(z_k^Y \ell_{jk}\right)$$

$$s_j \;=\; -\,\sigma_j^Y, \qquad s_k \;=\; -\,s_k^Y$$

Solve numerically for $\left(s_k^Y, v_k, \ell_{jk}, S_{jk}\right)$

Example 1

Y configuration



Given:

- Constant-power source σ_j^Y with $\angle V_j^a := 0°$
- Impedance load z_k^Y
- Line parameters $\left(z_{jk}, y_{jk}^m = y_{kj}^m = 0\right)$
- Assumption C8.1 with $\gamma_j = V_j^n = \gamma_k = V_k^n = 0$

Calculate: $\left(s_k^Y, v_k, \ell_{jk}, S_{jk}\right)$

Simplification:

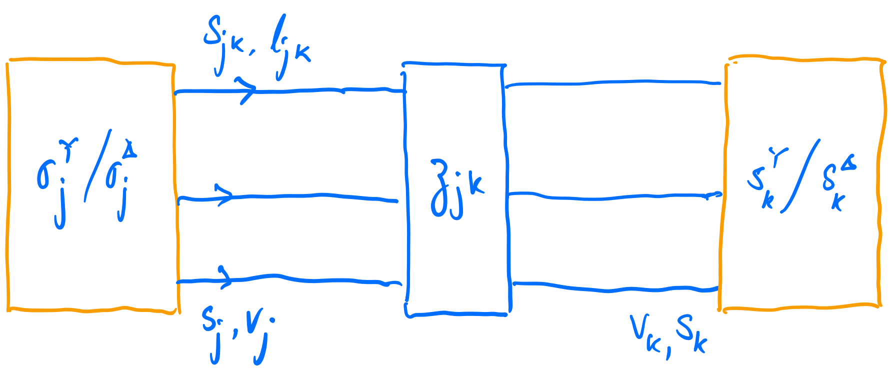
Combining $s_j = -\sigma_j^Y$, $s_k = -s_k^Y = -\text{diag}\left(z_k^Y \ell_{jk}\right)$ and $s_j = \text{diag}(S_{jk})$, $\text{diag}\left(S_{jk} - z_{jk}\ell_{jk}\right) = -s_k$ reduces equations to:

$$-\sigma_j^Y = \text{diag}\left(\left(z_k^Y + z_{jk}\right)\ell_{jk}\right) = \text{diag}\left(\begin{bmatrix} Z_k^{aa} & Z_k^{ab} & Z_k^{ac} \\ Z_k^{ba} & Z_k^{bb} & Z_k^{bc} \\ Z_k^{ca} & Z_k^{cb} & Z_k^{cc} \end{bmatrix} \begin{bmatrix} I_{jk}^a \\ I_{jk}^b \\ I_{jk}^c \end{bmatrix} \begin{bmatrix} I_{jk}^{a\mathsf{H}} & I_{jk}^{b\mathsf{H}} & I_{jk}^{c\mathsf{H}} \end{bmatrix}\right)$$

1. 3 quadratic equations in 3 unknowns $I_{jk} \in \mathbb{C}^3$
2. psd rank-1 cond ensures $\exists I_{jk}$
3. arbitrary reference angle of I_{jk} is fixed by given $\angle V_j^a = 0°$

Example 1

Y configuration



Given:

- Constant-power source σ_j^Y with $\angle V_j^a := 0°$
- Impedance load z_k^Y
- Line parameters $\left(z_{jk}, y_{jk}^m = y_{kj}^m = 0\right)$
- Assumption C8.1 with $\gamma_j = V_j^n = \gamma_k = V_k^n = 0$

Calculate: $\left(s_k^Y, v_k, \ell_{jk}, S_{jk}\right)$

Simplification:

Combining $s_j = -\sigma_j^Y$, $s_k = -s_k^Y = -\text{diag}\left(z_k^Y \ell_{jk}\right)$ and $s_j = \text{diag}(S_{jk})$, $\text{diag}\left(S_{jk} - z_{jk}\ell_{jk}\right) = -s_k$ reduces equations to:

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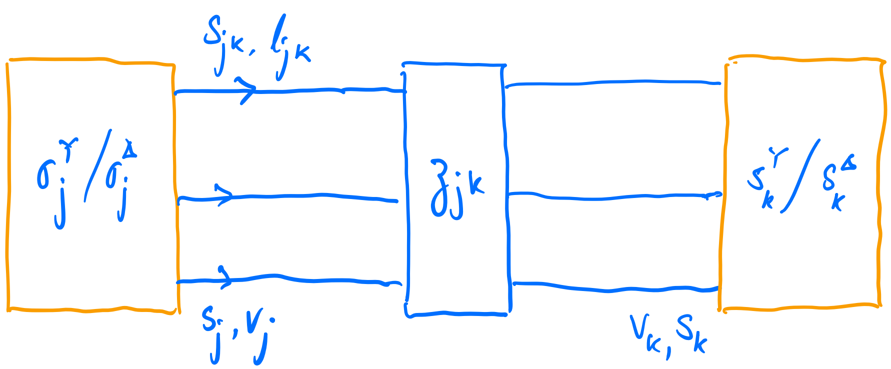
1. Solve for I_{jk} numerically
2. Derive analytically all other vars

Example 2

Δ configuration



Given:

- Constant-power source $\left(\sigma_j^\Delta, \gamma_j\right)$ with $\angle V_j^{ab} := 0°$
- Impedance load $\left(z_k^\Delta, \beta_k\right)$
- Line parameters $\left(z_{jk}, y_{jk}^m = y_{kj}^m = 0\right)$

Calculate: $\left(s_k^Y, v_k, \ell_{jk}, S_{jk}\right)$

Network model (same as previous example):

$$\text{diag}(S_{jk}) \;=\; s_j, \qquad \text{diag}\left(S_{jk} - z_{jk}\ell_{jk}\right) \;=\; -s_k$$

$$v_j - v_k \;=\; \left(z_{jk}S_{jk}^{\mathsf{H}} + S_{jk}z_{jk}^{\mathsf{H}}\right) \;-\; z_{jk}\,\ell_{jk}\,z_{jk}^{\mathsf{H}}$$

$$\begin{bmatrix} v_j & S_{jk} \\ S_{jk}^{\mathsf{H}} & \ell_{jk} \end{bmatrix} \;\geq\; 0, \quad \text{rank}\begin{bmatrix} v_j & S_{jk} \\ S_{jk}^{\mathsf{H}} & \ell_{jk} \end{bmatrix} \;=\; 1$$

Device model:

$$s_j \;:=\; \text{diag}\left(V_j I_j^{\mathsf{H}}\right), \qquad \sigma_j^\Delta \;=\; \text{diag}\left(\Gamma V_j I_j^{\Delta\mathsf{H}}\right)$$

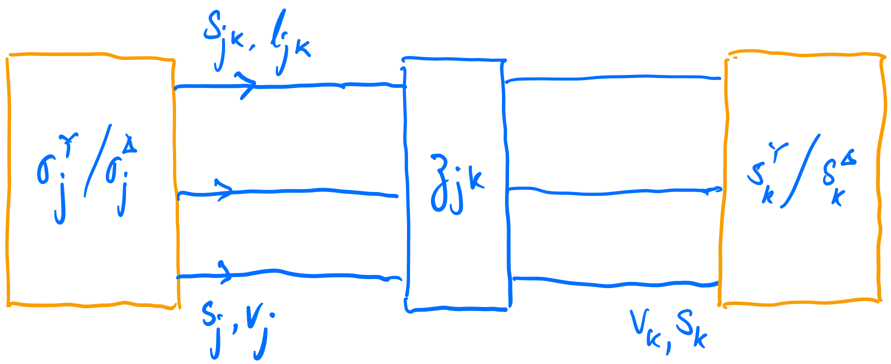
$$s_k \;:=\; \text{diag}\left(V_k I_k^{\mathsf{H}}\right), \qquad V_k \;=\; -Z^\Delta I_k \;+\; \gamma_k 1$$

$$1^{\mathsf{T}} I_k \;=\; 0$$

Solve numerically for $\left(s_k^Y, v_k, \ell_{jk}, S_{jk}\right)$

Example 2

Δ configuration



Given:

- Constant-power source $\left(\sigma_j^\Delta, \gamma_j\right)$ with $\angle V_j^{ab} := 0°$
- Impedance load $\left(z_k^\Delta, \beta_k\right)$
- Line parameters $\left(z_{jk}, y_{jk}^m = y_{kj}^m = 0\right)$

Calculate: $\left(s_k^Y, v_k, \ell_{jk}, S_{jk}\right)$

Simplification:

1. psd rank-1 condition ensures $\exists \left(V_j, V_k, I_{jk}\right)$ s. t.

$$V_j \;=\; \left(Z_k^\Delta + z_{jk}^s\right) I_{jk} + \gamma_k 1 \quad \text{and} \quad I_{jk} = I_j = -\Gamma^\mathsf{T} I_j^\Delta$$

2. Substitute into $\sigma_j^\Delta = \text{diag}\left(\Gamma V I^{\Delta \mathsf{H}}\right)$:

$$\sigma_j^\Delta \;:=\; -\,\text{diag}\left(\left(\Gamma \hat{Z}_k^\Delta \Gamma^\mathsf{T}\right) I_j^\Delta I_j^{\Delta \mathsf{H}}\right)$$

1. 3 quadratic equations in 3 unknowns $I_{jk}^\Delta \in \mathbb{C}^3$
2. Solve for $I_{jk}\Delta$ numerically
3. Derive analytically all other vars

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Backward forward sweep

Efficient solution method for power flow equations (1 or 3-phase networks)

- Applicable to radial networks

Partition solution (x, y) into two groups of variables x and y

- Typically, x are branch variables (e.g. line currents) and y are nodal variables (bus voltages)

Each round of spatial iteration consists of a backward sweep and a forward sweep

- Given y, compute each component x_j iteratively from leafs to root (backward)
- Given x, compute each component y_j iteratively from root to leaves (forward)

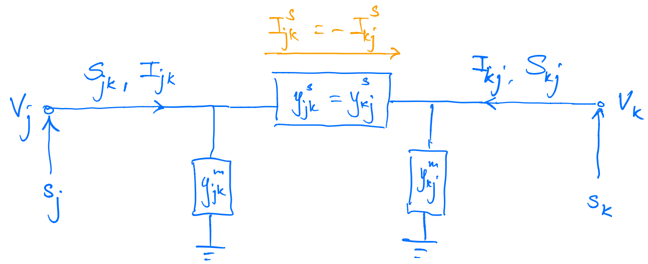
Iterate until stopping criterion

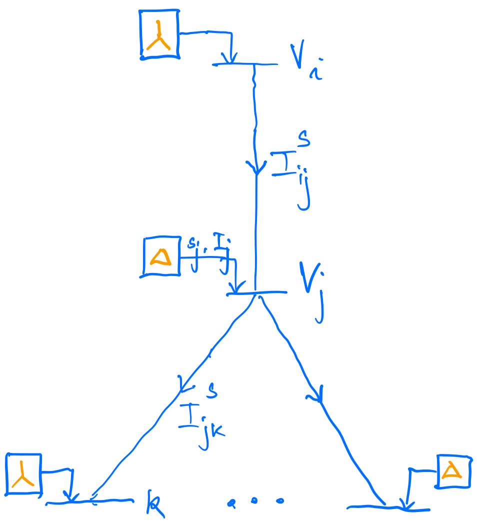
Different BFS methods differ in how to partition variables into x and y and the associated power flow equations

Example 1

Complex form BFM

Notation:

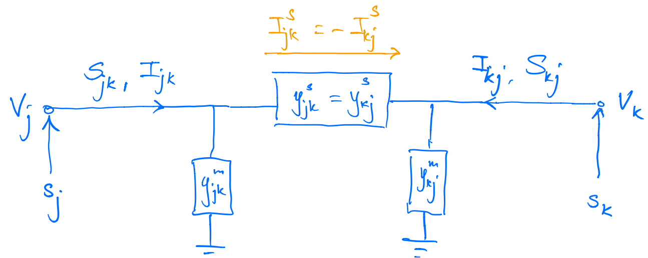


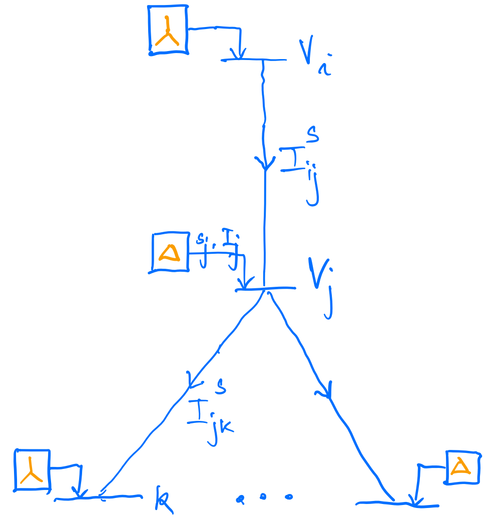


Example 1

Complex form BFM

Notation:





Given: V_0 and $s := \left(s_j, j \in N\right)$

Compute: $V := \left(V_j, j \in N\right)$ and currents $I^s := \left(I^s_{jk}, (j,k) \in E\right)$ through series impedance

- All other variables $I_{jk} = I^s_{jk} + y^m_{jk} V_j$, I_{kj}, S_{jk}, S_{kj} can then be computed
- Advantage: $I^s_{jk} = -I^s_{kj}$

Example 1

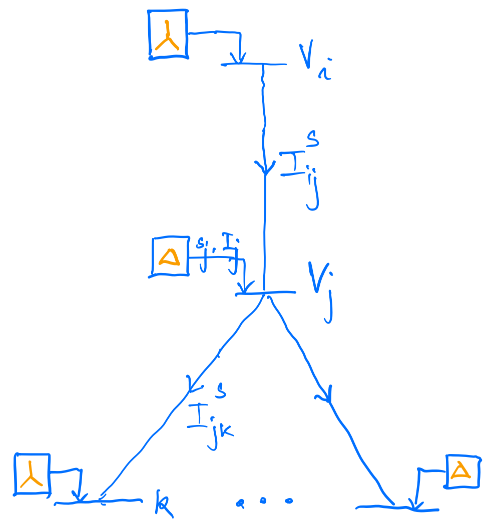
Complex form BFM

Network equations

$$I_{ij}^s = \sum_{k:j\to k} I_{jk}^s - \left(I_j - y_{jj}^m V_j\right), \qquad j \in N$$

$$V_j = V_i - z_{ij}^s I_{ij}^s, \qquad j \in N$$

where $y_{jj}^m := \sum_k y_{jk}^m$



Device models

Y configuration: $\sigma_j^Y = -\,\text{diag}\left(V_j I_j^{\mathsf{H}}\right)$

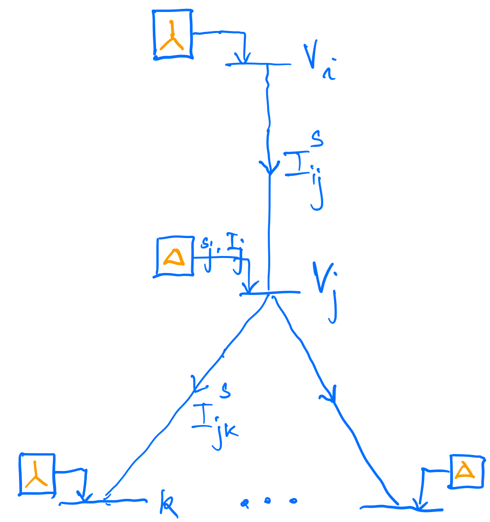
Δ configuration: $\sigma_j^\Delta = \text{diag}\left(\Gamma V_j I_j^{\Delta\mathsf{H}}\right), \quad I_j = -\Gamma^{\mathsf{T}} I_j^\Delta$

Example 1

Complex form BFM

BFS variables

$$x := \left(I_{ij}^s,\, j \in N\right), \qquad y := \left(V_j, I_j, I_j^{\Delta}\ j \in N\right)$$



Example 1

Complex form BFM

Backward sweep: start from leaf nodes and iterate towards root bus 0

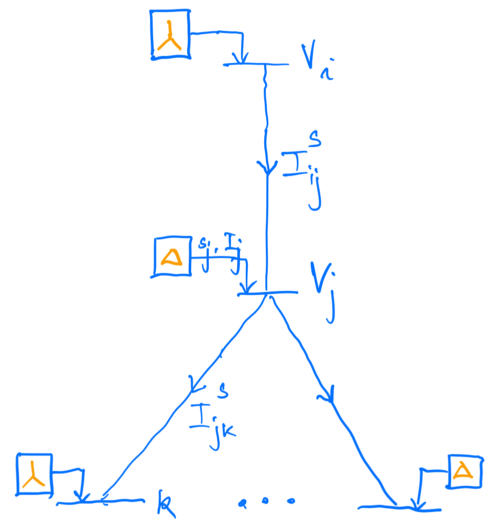
$$I_{ij}^{s}(t) \ \leftarrow \ \sum_{k:j\rightarrow k} I_{jk}^{s}(t) \ - \ \left(I_j(t-1) - y_{jj}^{m}\, V_j(t-1) \right), \quad i \rightarrow j \in E$$

Forward sweep: start from bus 0 and iterate towards leaf nodes

$$V_j(t) \leftarrow \ V_i(t) \ - \ z_{ij}^{s}\, I_{ij}^{s}(t)$$

$$Y: \qquad I_j(t) \leftarrow -\left(\text{diag}\ \overline{V}_j(t)\right)^{-1} \overline{\sigma}_j^{Y}$$

$$\Delta: \qquad I_j^{\Delta}(t) \leftarrow \left(\text{diag}\left(\Gamma \overline{V}_j(t)\right)\right)^{-1} \overline{\sigma}_j^{\Delta}, \qquad I_j(t) \leftarrow -\,\Gamma^{\mathsf{T}} I_j^{\Delta}(t)$$



Example 2

3-phase DistFlow model

Implicit description

$$\begin{bmatrix} v_j & S_{jk} \\ S_{jk}^{\mathsf{H}} & \ell_{jk} \end{bmatrix} \geq 0, \qquad \text{rank} \begin{bmatrix} v_j & S_{jk} \\ S_{jk}^{\mathsf{H}} & \ell_{jk} \end{bmatrix} = 1$$

Implies: $\exists (V, \tilde{I})$ s. t.

$$v_j = V_j V_j^{\mathsf{H}}, \qquad \ell_{jk} = I_{jk} I_{jk}^{\mathsf{H}}, \qquad S_{jk} = V_j I_{jk}^{\mathsf{H}}$$

Hence: design BFS based on $(V, v, \tilde{I}, S)$ instead of original 3-phase DistFlow equations

Example 2

3-phase DistFlow model

Network equations

$$S_{jk} = V_j \tilde{I}_{jk}^{\mathsf{H}}$$

$$V_j - V_k = z_{jk} \tilde{I}_{jk}, \qquad v_j = V_j V_j^{\mathsf{H}}, \qquad \tilde{I}_{jk} = \frac{1}{\text{tr } v_j} S_{jk}^H V_j$$

Device models (same as in Example 1)

Y configuration: $$\sigma_j^Y = -\,\text{diag}\left(V_j I_j^{\mathsf{H}}\right)$$

Δ configuration: $$\sigma_j^\Delta = \text{diag}\left(\Gamma V_j I_j^{\Delta\mathsf{H}}\right), \quad I_j = -\,\Gamma^{\mathsf{T}} I_j^\Delta$$

BFS variables

$$x := \left(S_{jk}, j \to k \in E\right), \qquad y := (V_j, v_j, \tilde{I}_{ij}, I_j, I_j^\Delta, j \in N)$$

Example 2

3-phase DistFlow model

Forward sweep: start from bus 0 and iterate towards leaf nodes

$$\tilde{I}_{ij}(t) \leftarrow \frac{1}{\text{tr } v_i(t)} S_{ij}^H(t-1)V_i(t)$$

$$V_j(t) \leftarrow V_i(t) - z_{ij}\tilde{I}_{ij}(t), \qquad v_j(t) \leftarrow V_j(t)V_j(t)^{\mathsf{H}}$$

$$Y: \qquad I_j(t) \leftarrow -\left(\text{diag } \overline{V}_j(t)\right)^{-1} \overline{\sigma}_j^Y$$

$$\Delta: \qquad I_j^{\Delta}(t) \leftarrow \left(\text{diag}\left(\Gamma\overline{V}_j(t)\right)\right)^{-1} \overline{\sigma}_j^{\Delta}, \qquad I_j(t) \leftarrow -\Gamma^{\mathsf{T}} I_j^{\Delta}(t)$$

Backward sweep: $S_{jk}(t) \leftarrow V_j(t)\,\tilde{I}_{jk}^{\mathsf{H}}(t)$

Outline

1. General network
2. Radial network
3. Overall network
4. Backward-forward sweep
5. Linear network
   - Assumptions
   - Network equations
   - Linear solution

Assumptions

1. Negligible line loss $\ell_{jk} = 0$
   - Small line loss relative to line flow: $z_{jk}\ell_{jk} \ll S_{jk}$

2. Balanced voltages

$$\frac{V_j^a}{V_j^b} = \frac{V_j^b}{V_j^c} = \frac{V_j^c}{V_j^a} = e^{i2\pi/3}$$

Network equations

Define $\gamma \;:=\; \begin{bmatrix} 1 & \alpha^2 & \alpha \\ \alpha & 1 & \alpha^2 \\ \alpha^2 & \alpha & 1 \end{bmatrix}$ where $\alpha := e^{-i2\pi/3}$

Linear 3-phase DistFlow model are linear equations in (v, s, λ, S):

$$\sum_{k: j \to k} \lambda_{jk} = \lambda_{ij} + s_j, \qquad j \in \overline{N}$$

nodal injections determine $\text{diag}\left(S_{jk}\right) =: \lambda_{jk}$

$$S_{jk} = \gamma \text{ diag}\left(\lambda_{jk}\right), \qquad j \to k \in E$$

this uses balanced voltage assumption to determine off-diagonal entries of S_{jk}

$$v_j - v_k = z_{jk} S_{jk}^{\mathsf{H}} + S_{jk} z_{jk}^{\mathsf{H}}, \qquad j \to k \in E$$

(λ_{jk} are diagonal entries of S_{jk})

Linear solution

Given $\left(v_0, s_j, j \in N\right)$, can determine $\left(s_0, v_j, j \in N\right)$ and $\left(\lambda_{jk}, S_{jk}, j \to k \in E\right)$:

$$s_0 = -\sum_{j \in N} s_j$$

$$\lambda_{ij} = -\sum_{k \in \mathsf{T}_j} s_k, \qquad S_{ij} = \gamma \text{ diag}\left(\lambda_{ij}\right), \qquad i \to j \in E$$

$$v_j = v_0 - \sum_{(i,k) \in \mathsf{P}_j} \left(z_{ik} S_{ik}^{\mathsf{H}} + S_{jk} z_{ik}^{\mathsf{H}}\right), \qquad j \in N$$

where

- T_j : subtree rooted at bus j, including j
- P_k : set of lines on the unique path from bus 0 to bus k