Power System Analysis

Chapter 3 Transformer models

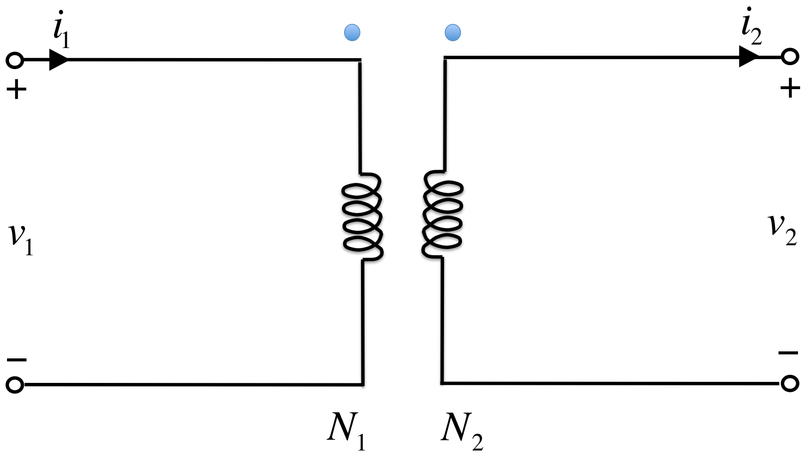
Outline

1. Single-phase transformer
2. Three-phase transformer
3. Equivalent impedance
4. Per-phase analysis
5. Per-unit normalization

Outline

1. Single-phase transformer
    - Ideal transformer
    - Nonideal transformer
    - Circuit models: T eq circuit, simplified circuit, UVN
2. Three-phase transformer
3. Equivalent impedance
4. Per-phase analysis
5. Per-unit normalization

Ideal transformer



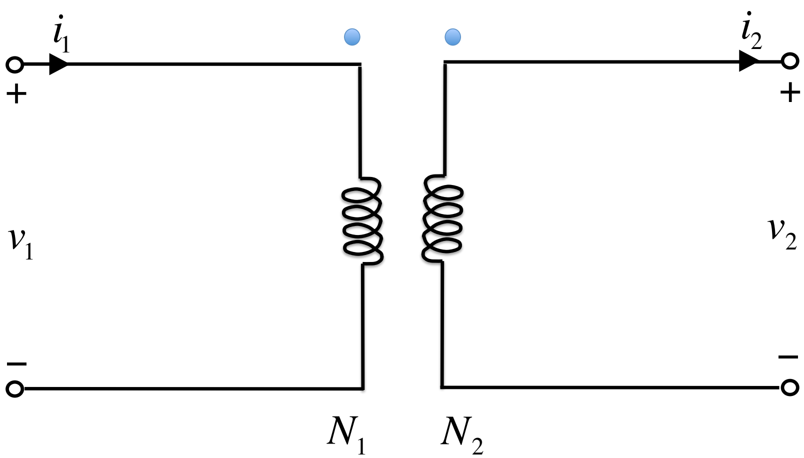
voltage gain $n := \frac{N_2}{N_1}$

turns ratio $a := \frac{N_1}{N_2}$

Voltage & current gains

$$\frac{v_2(t)}{v_1(t)} = n \qquad \frac{i_2(t)}{i_1(t)} = a$$

Ideal transformer



voltage gain $n := \frac{N_2}{N_1}$

turns ratio $a := \frac{N_1}{N_2}$

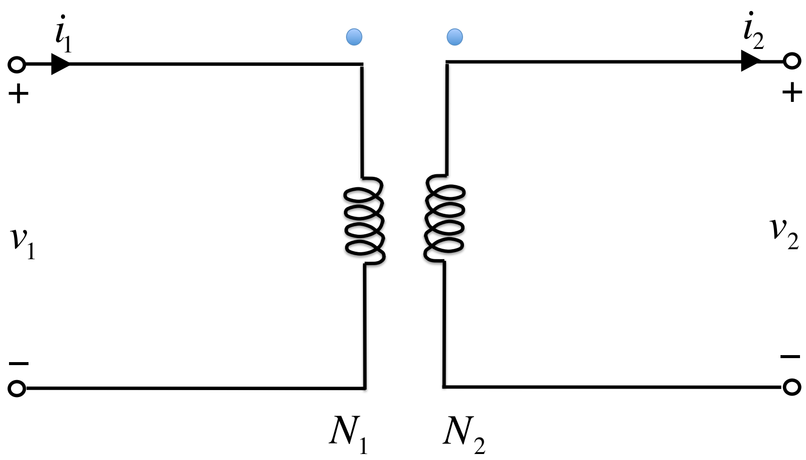
Voltage & current gains

$$\frac{V_2}{V_1} = n \qquad \frac{I_2}{I_1} = a$$

Transmission matrix

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & n \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

Ideal transformer



voltage gain $n := \frac{N_2}{N_1}$

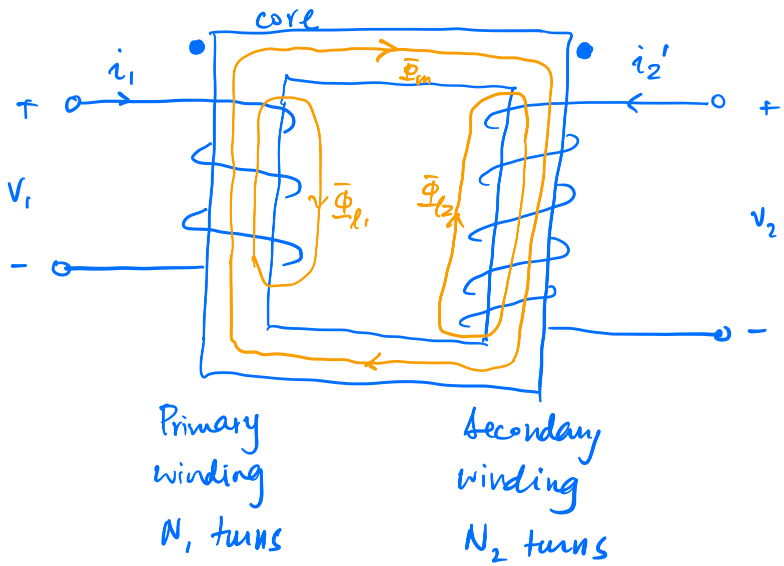
turns ratio $a := \frac{N_1}{N_2}$

Power transfer

$$\frac{-S_{21}}{S_{12}} := \frac{V_2 I_2^*}{V_1 I_1^*} = n \cdot a = 1$$

i.e., deal transformer incurs no power loss

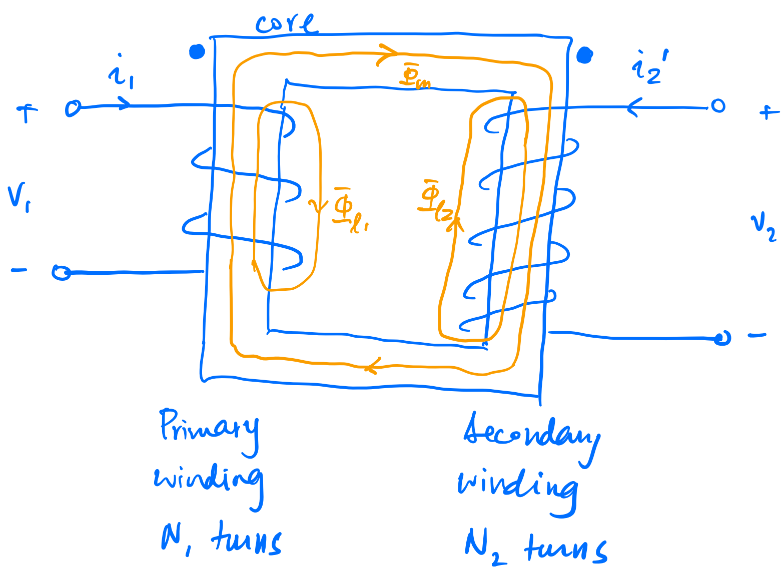
Nonideal transformer



Nonideal behavior

- Power losses (coil resistances, eddy currents, hysteresis losses)
- Leakage magnetic fluxes
- Finite permeability of magnetic cores

Nonideal transformer



Mutual flux: Φ_m

Leakage fluxes: $\lambda_{l1},\ \lambda_{l2}$

Voltages

$$v_1 = r_1 i_1 + \frac{d\lambda_1}{dt}, \qquad v_2 = r_2 i_2' + \frac{d\lambda_2}{dt}$$

Total flux linkages

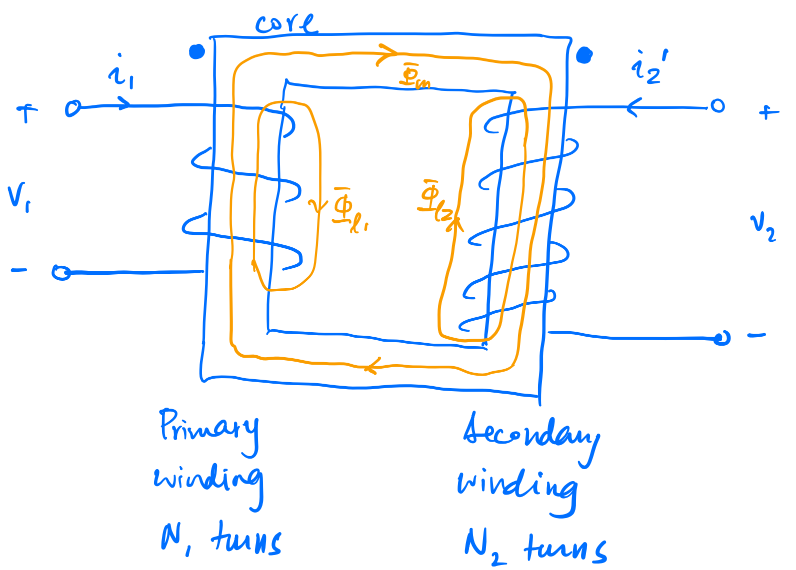
$$\lambda_1 = N_1\Phi_m + \lambda_{l1}, \qquad \lambda_2 = N_2\Phi_m + \lambda_{21}$$

$$\lambda_{l1} = L_{l1}\, i_1, \qquad \lambda_{l2} = L_{l2}\, i_2'$$

Total magnetomotive force

$$F = N_1 i_1 + N_2 i_2' = R\Phi_m$$

Nonideal transformer



Voltages

$$v_1 \;=\; r_1 i_1 + \frac{d\lambda_1}{dt}, \qquad v_2 \;=\; r_2 i_2' + \frac{d\lambda_2}{dt}$$

Total flux linkages

$$\lambda_1 \;=\; N_1\Phi_m + \lambda_{l1}, \qquad \lambda_2 \;=\; N_2\Phi_m + \lambda_{21}$$

$$\lambda_{l1} \;=\; L_{l1}\, i_1, \qquad \lambda_{l2} \;=\; L_{l2}\, i_2'$$

Total magnetomotive force

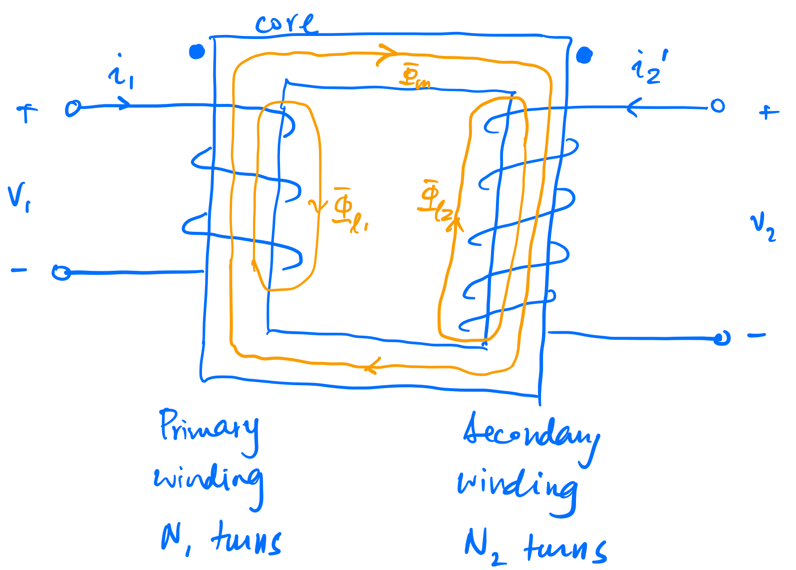
$$F \;=\; N_1 i_1 + N_2 i_2' \;=\; R\Phi_m$$

Ideal transformer

- Zero power losses: $r_1 = r_2 = 0$
- Zero leakage flux linkages: $L_{l1} = L_{l2} = 0$
- Infinite permeability: $R = 0$

$$\Longrightarrow \qquad v_1 = N_1 \frac{d\Phi_m}{dt}, \quad v_2 = N_2 \frac{d\Phi_m}{dt}, \quad 0 = N_1 i_1 + N_2 i_2'$$

Nonideal transformer



Voltages

$$v_1 \;=\; r_1 i_1 + L_{l1}\frac{di_1}{dt} + N_1\frac{d\Phi_m}{dt}$$

$$v_2 \;=\; r_2 i_2' + L_{l2}\frac{di_2'}{dt} + N_2\frac{d\Phi_m}{dt}$$

Primary magnetizing current $\hat{i}_m$

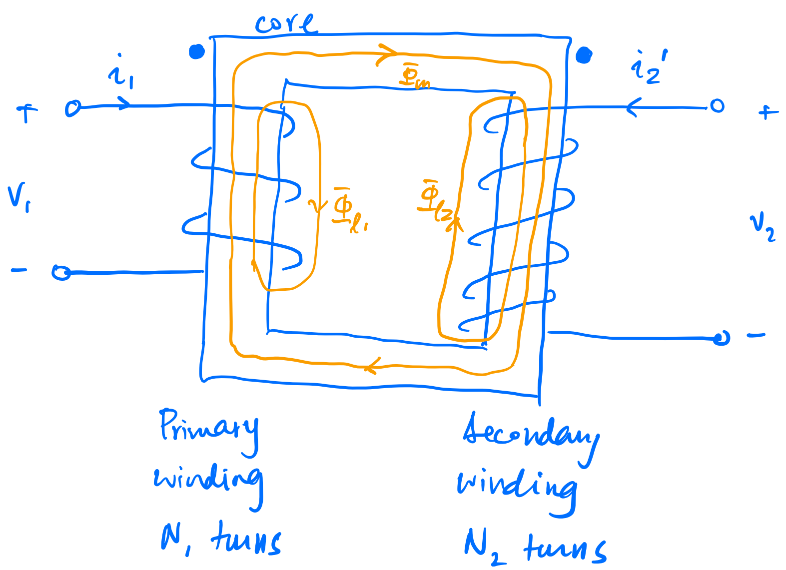
- primary current when secondary circuit is open $i_2' := 0$
- $N_1 \hat{i}_m = R\Phi_m$: let $L_m := N_1^2/R$ and

$$\hat{u}_1 \;:=\; N_1\frac{d\Phi_m}{dt} \;=\; L_m\frac{d\hat{i}_m}{dt}$$

$$\hat{u}_2 \;:=\; N_2\frac{d\Phi_m}{dt} \;=\; \frac{N_2}{N_1}\hat{u}_1$$

ideal transformer

Nonideal transformer



Nonideal elements

$$v_1 = r_1 i_1 + L_{l1}\frac{di_1}{dt} + \hat u_1, \qquad \hat u_1 = L_m \frac{d\hat i_m}{dt}$$

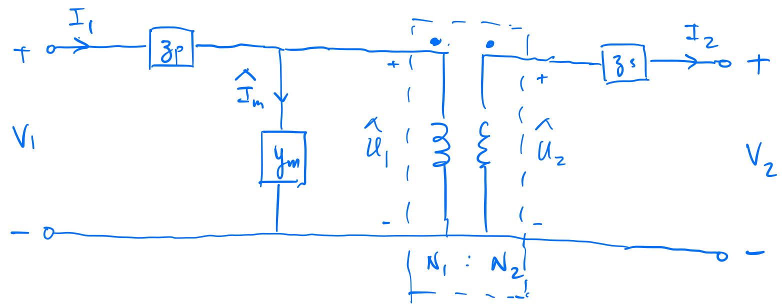
$$v_2 = -r_2 i_2 - L_{l2}\frac{di_2}{dt} + \hat u_2$$

Ideal transformer

$$\hat u_2 = \frac{N_2}{N_1}\hat u_1, \qquad i_2 = \frac{N_1}{N_2}\left(i_1 - \hat i_m\right)$$

Nonideal transformer

Circuit model



Nonideal elements (phasor domain)

$$V_1 = z_p I_1 + \hat{U}_1, \qquad \hat{I}_m = y_m \hat{U}_1$$

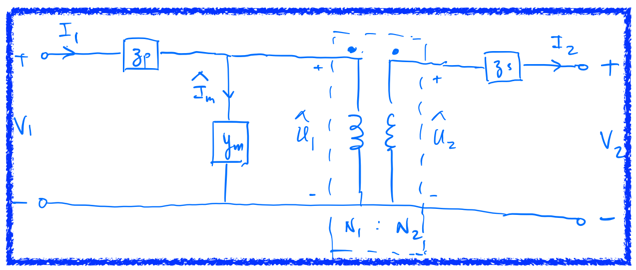
$$\hat{U}_2 = z_s I_2 + V_2$$

Ideal transformer (phasor domain)

$$\hat{U}_2 = \frac{N_2}{N_1}\hat{U}_1, \qquad I_2 = \frac{N_1}{N_2}\left(I_1 - \hat{I}_m\right)$$

Nonideal transformer

Circuit models

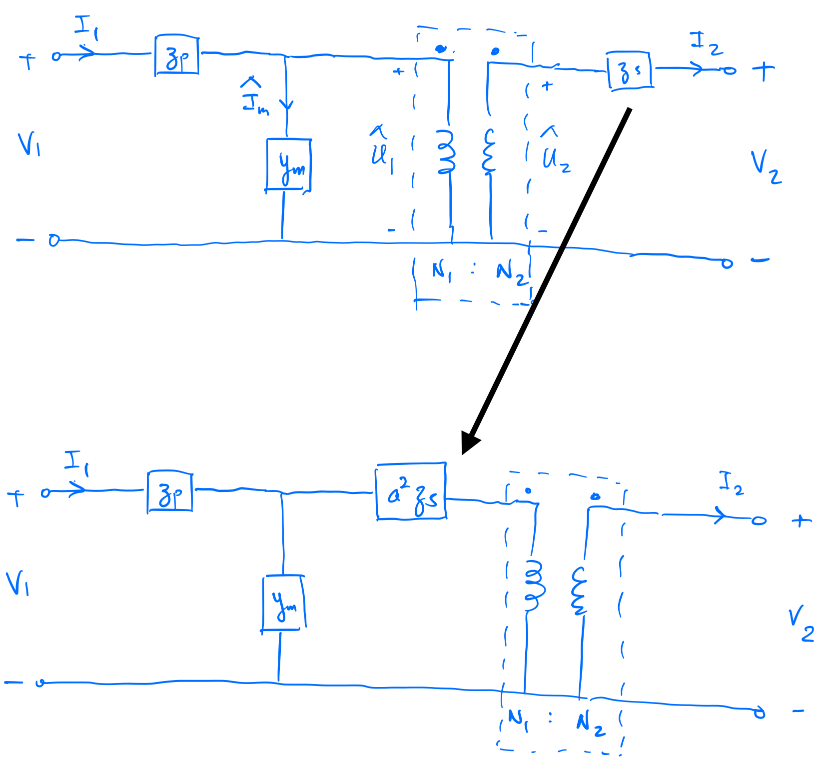


T eq. circuit $\equiv$ Unitary voltage nk

$\cong$

Simplified model

T equivalent circuit



Refer series impedance z_s to the primary side
$\longrightarrow$ T equivalent circuit

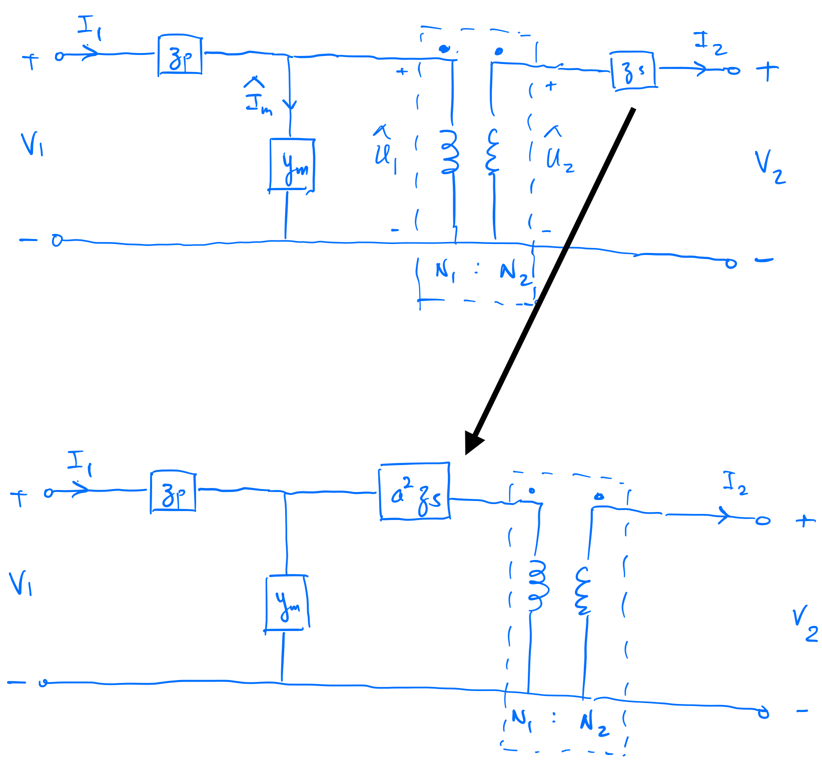
$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a\left(1 + z_p y_m\right) & a z_s(1 + z_p y_m) + n z_p \\ a y_m & n + a z_s y_m \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

where $n := N_2/N_1,\ a := 1/n$

"Equivalent model" means

- Same end-to-end behavior, e.g., transmission matrix, or admittance matrix;
- Internal variables may be different

T equivalent circuit



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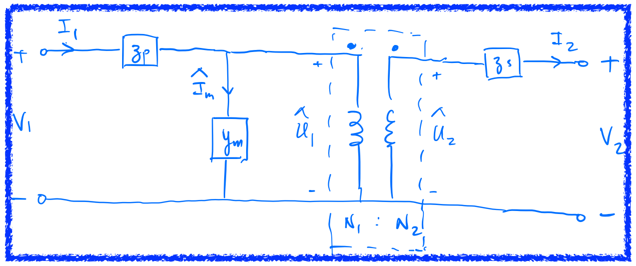
where $n := N_2/N_1$, $a := 1/n$

Model parameters (z_p, z_s, y_m) cannot be uniquely determined from just short-circuit & open-circuit tests

- Additional tests are needed

Nonideal transformer

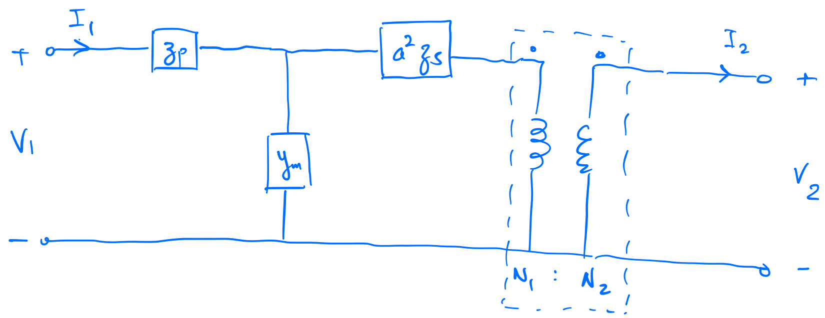
Circuit models

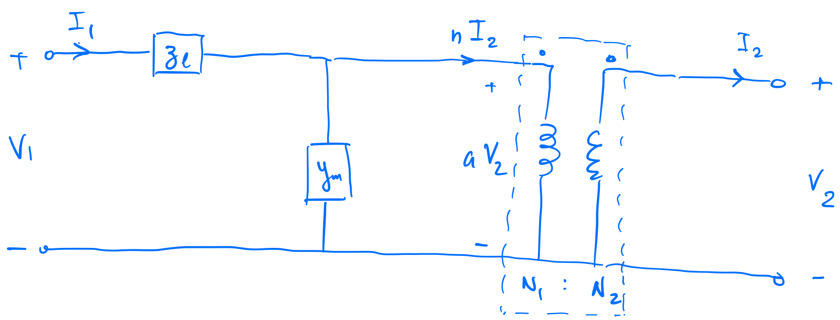


T eq. circuit $\equiv$ Unitary voltage nk

Simplified model

Simplified circuit





Interchange $a^2 z_s$ and y_m and combine with z_p :

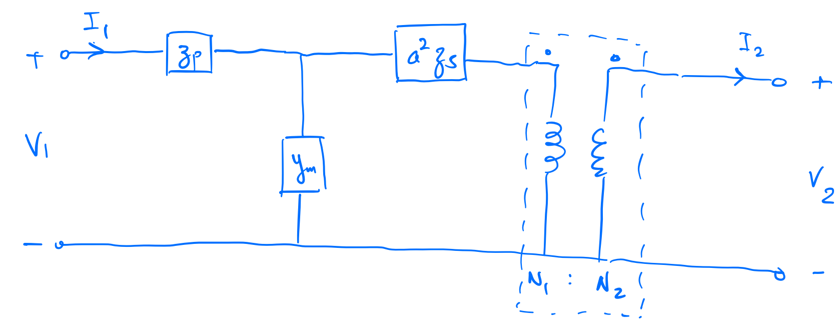
$z_l := z_p + a^2 z_s$

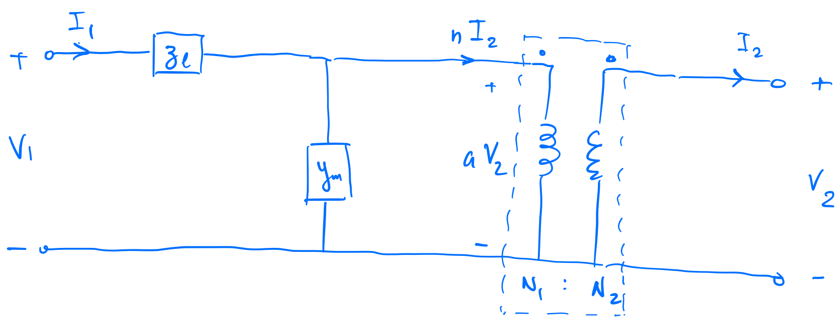
$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a\left(1 + z_l y_m\right) & n z_l \\ a y_m & n \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

where $n := N_2/N_1$, $a := 1/n$

Simplified circuit

Approximation to T eq circuit





Interchange $a^2 z_s$ and y_m and combine with z_p :

$z_l := z_p + a^2 z_s$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a\left(1 + z_l y_m\right) & n z_l \\ a y_m & n \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

where $n := N_2/N_1$, $a := 1/n$

Good approximation of T equivalent circuit when $|y_m| \ll 1/|a^2 z_s|$

$$\frac{\|M - T\|}{\|T\|} < |\epsilon| \ll 1$$

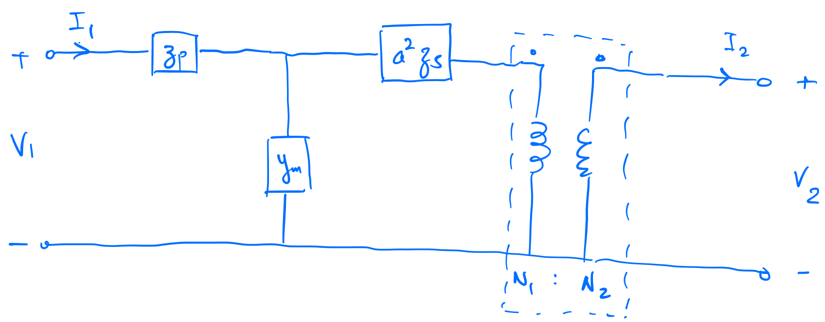
M : transmission matrix of simplified model

T : transmission matrix of simplified model

$\epsilon := a^2 z_s y_m$

Simplified circuit

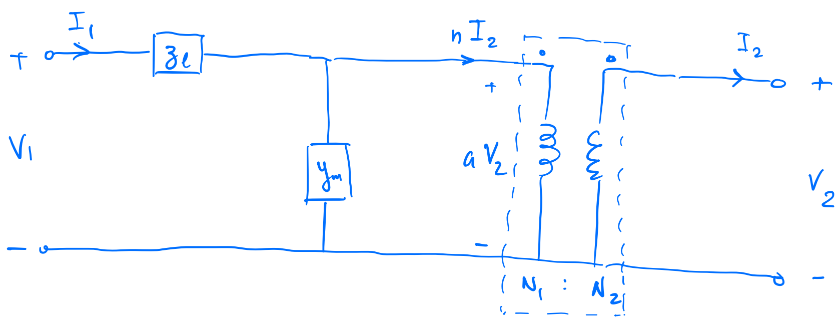
Approximation to T eq circuit



Interchange $a^2 z_s$ and y_m and combine with z_p :
$z_l := z_p + a^2 z_s$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a\left(1 + z_l y_m\right) & n z_l \\ a y_m & n \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

where $n := N_2/N_1$, $a := 1/n$



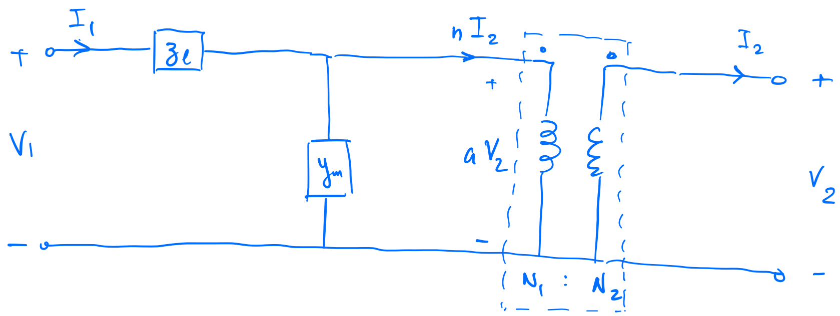
Good approximation when $|y_m| \ll 1/|a^2 z_s|$

$$\frac{\|M - T\|}{\|T\|} < |\epsilon| \ll 1$$

If $y_m = 0$: T equivalent circuit and simplified model are equivalent, $M = T$

Parameter determination

Short & open-circuit tests



Most popular model
(at least for transmission systems)

Parameters (z_l, y_m) can be determined from open and short-circuit tests

- Short-circuit test $(V_2 := 0)$:

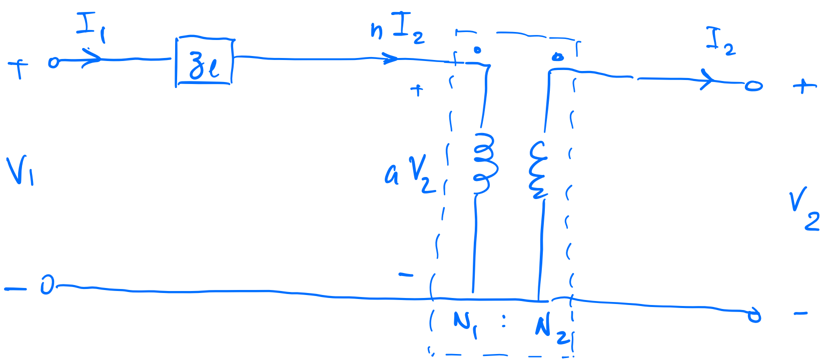
$$z_l = \frac{V_{sc}}{I_{sc}}$$

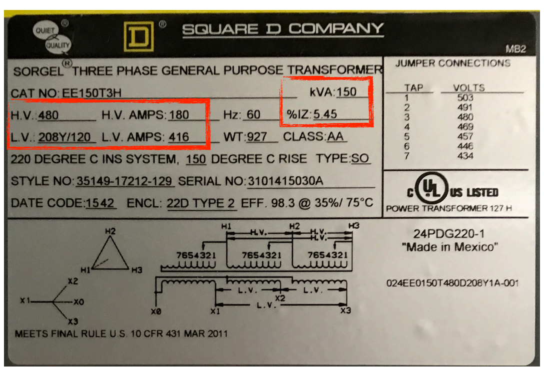
- Open-circuit test $(I_2 := 0)$:

$$\frac{1}{y_m} = \frac{V_{oc}}{I_{oc}} - \frac{V_{sc}}{I_{sc}}$$

Parameter determination

Zero shunt admittance $y_m = 0$





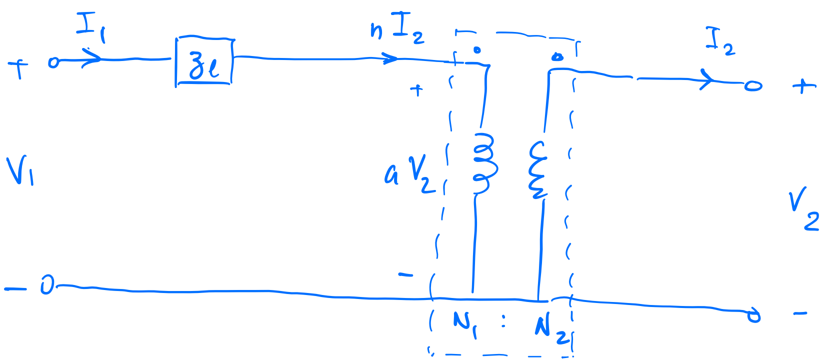
When $y_m = 0$, parameter z_l can be determined from standard 3-phase transformer ratings:

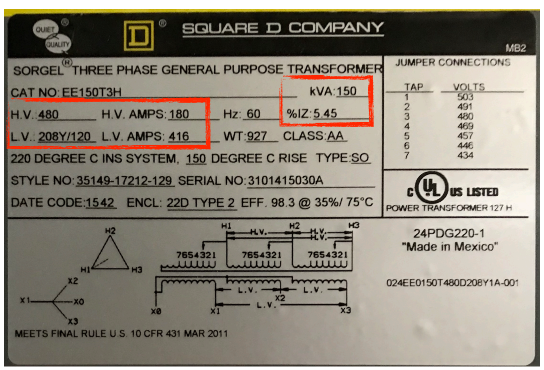
- Rated primary line-to-line voltage $\left| V_{\text{pri}} \right|$
- Rated primary line current $\left| I_{\text{pri}} \right|$
- Impedance voltage β on the primary side, per phase, as % of rated primary voltage

β : voltage needed on the primary side to produce rated primary current across each single-phase transformer is $\beta \times$ rated primary voltage

Parameter determination

Zero shunt admittance $y_m = 0$





For both Y and Δ configurations

$$z_l = \frac{V_{sc}}{I_{sc}}$$

- Δ config:

$$|V_{sc}| = |V_{ab}| = \beta |V_{\text{pri}}|$$

$$|I_{sc}| = |I_{ab}| = \left| \frac{I_{\text{pri}}}{\sqrt{3}} e^{i\pi/6} \right|$$

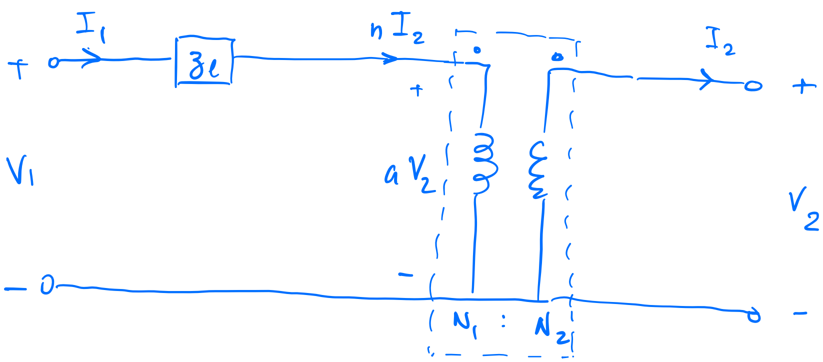
- Y config:

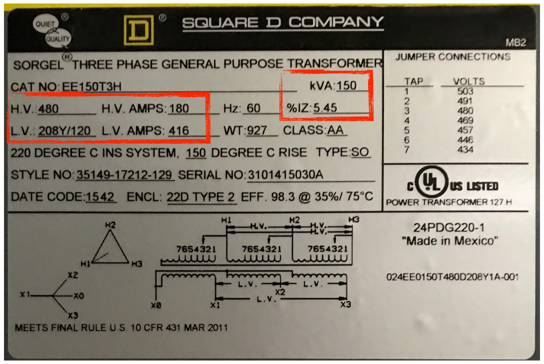
$$|V_{sc}| = |V_{an}| = \beta \left| \frac{V_{\text{pri}}}{\sqrt{3}\, e^{i\pi/6}} \right|$$

$$|I_{sc}| = |I_{an}| = |I_{\text{pri}}|$$

Parameter determination

Zero shunt admittance $y_m = 0$





For both Y and Δ configurations

$$z_l = \frac{V_{sc}}{I_{sc}}$$

- Δ config:

$$|z_l| = \frac{\sqrt{3}\beta |V_{\text{pri}}|}{|I_{\text{pri}}|}$$

- Y config:

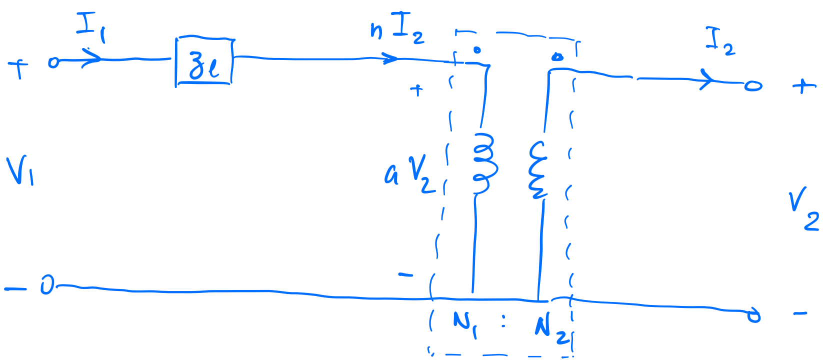
$$|z_l| = \frac{\beta |V_{\text{pri}}|}{\sqrt{3}\,|I_{\text{pri}}|}$$

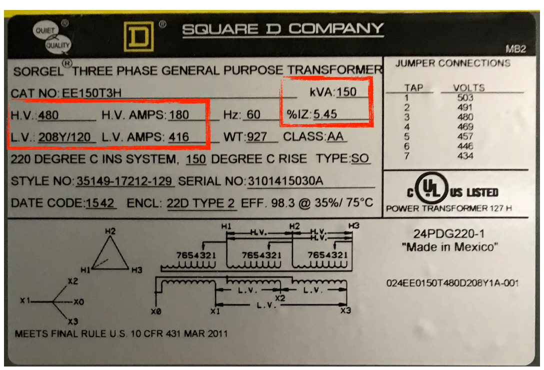
V_{pri} denotes line-to-line voltage even for Y configuration

- Otherwise, $|z_l| = \dfrac{\beta |V_{\text{pri}}|}{|I_{\text{pri}}|}$ for Y configuration if V_{pri} is line-to-neutral

Parameter determination

Zero shunt admittance $y_m = 0$



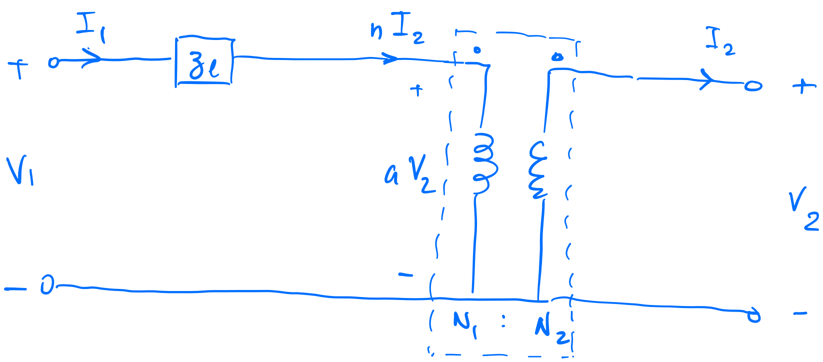


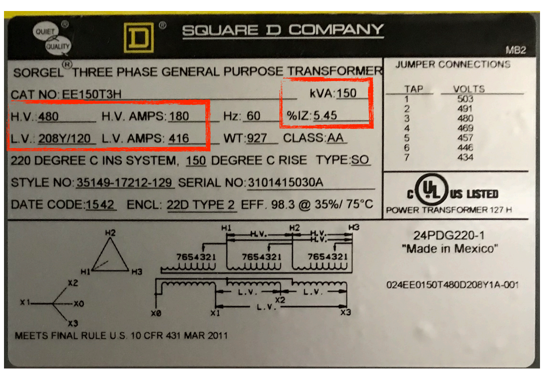
Sometimes $|S_{3\phi}|$ instead of $|I_{\text{pri}}|$ is specified:

- Rated primary line-to-line voltage $|V_{\text{pri}}|$
- Rated 3-phase power $|S_{3\phi}|$
- Impedance voltage β on the primary side, per phase, as % of rated primary voltage

Parameter determination

Zero shunt admittance $y_m = 0$





- Δ config:

$$|S_{3\phi}| = 3|S_{\phi}| = 3|V_{ab}||I_{ab}|$$

$$|V_{sc}| = |V_{ab}| = \beta|V_{\text{pri}}|$$

$$|I_{sc}| = |I_{ab}| = \frac{|S_{3\phi}|}{3|V_{\text{pri}}|}$$

- Y config:

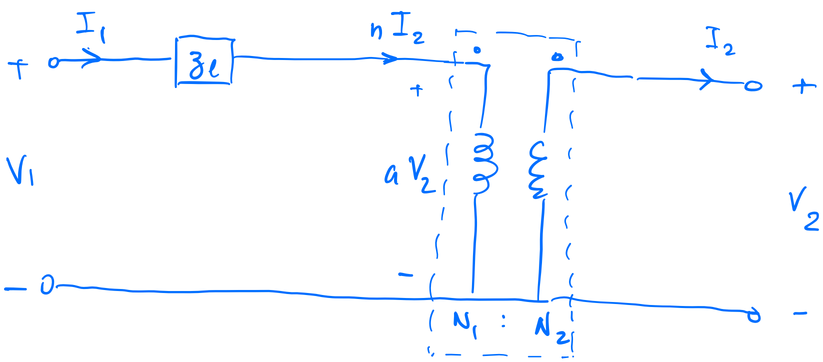
$$|S_{3\phi}| = 3|S_{\phi}| = 3|V_{an}||I_{an}|$$

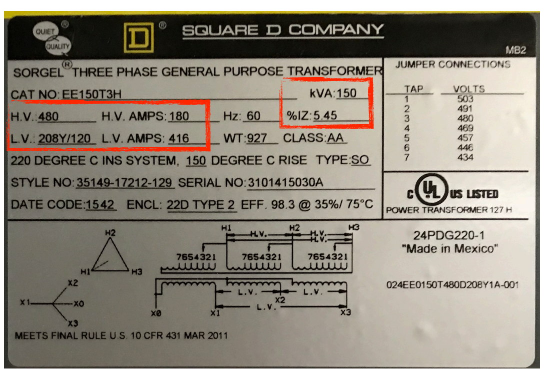
$$|V_{sc}| = |V_{an}| = \beta\left|\frac{V_{\text{pri}}}{\sqrt{3}\,e^{i\pi/6}}\right|$$

$$|I_{sc}| = |I_{an}| = \frac{|S_{3\phi}|}{3\left|\frac{V_{\text{pri}}}{\sqrt{3}\,e^{i\pi/6}}\right|} = \frac{|S_{3\phi}|}{\sqrt{3}|V_{\text{pri}}|}$$

Parameter determination

Zero shunt admittance $y_m = 0$





For both Y and Δ configurations

$$z_l = \frac{V_{sc}}{I_{sc}}$$

- Δ config:

$$|z_l| = \frac{3\beta |V_{\text{pri}}|^2}{|S_{3\phi}|}$$

- Y config:

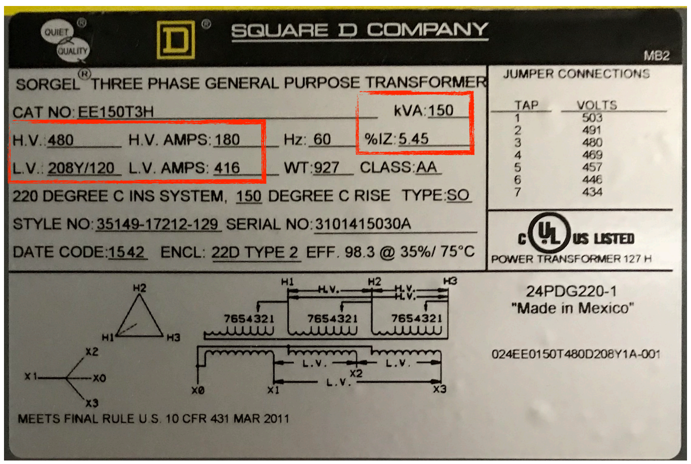
$$|z_l| = \frac{\beta |V_{\text{pri}}|^2}{|S_{3\phi}|}$$

V_{pri} denotes line-to-line voltage even for Y configuration

- Otherwise, $|z_l| = \dfrac{3\beta |V_{\text{pri}}|^2}{|I_{\text{pri}}|}$ for Y configuration if V_{pri} is line-to-neutral

Parameter determination

Example



3-phase transformer ratings (primary):

- Rated 3-phase power $\left|S_{3\phi}\right|$ = 150 kVA
- Rated primary line-to-line voltage $\left|V_{\text{pri}}\right|$ = 480 V
- Rated primary line current $\left|I_{\text{pri}}\right|$ = 180 A
- Impedance voltage β = 5.45% on primary side

Primary in Δ configuration:

$$|S_{3\phi}| = 3\,|S_{ab}| = 3\,|V_{ab}\,\bar{I}_{ab}| = 3\,|V_{\text{pri}}|\,|I_{ab}|$$

Since $I_a = I_{ab} - I_{ca} = I_{ab}\cdot\sqrt{3}\,e^{-i\pi/6}$, we have

$$|I_{\text{pri}}| = \sqrt{3}\,|I_{ab}|$$

Hence

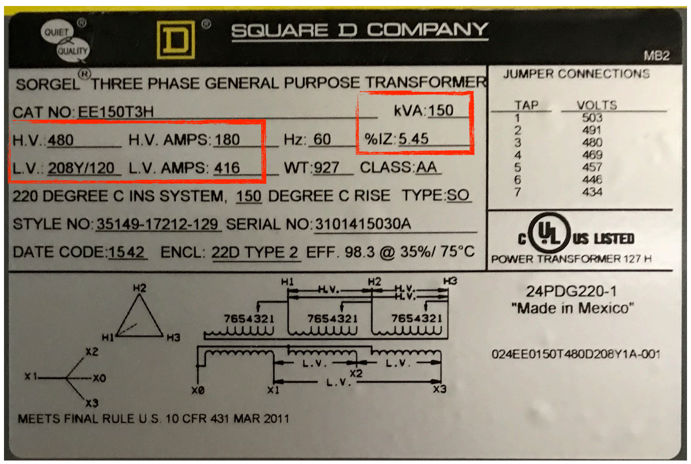
$$|S_{3\phi}| = \sqrt{3}\,|V_{\text{pri}}|\,|I_{\text{pri}}|$$

Verify:

- $\sqrt{3}\,|V_{\text{pri}}|\,|I_{\text{pri}}| = \sqrt{3}\cdot 480\cdot 180 = 149.65\text{ kVA} = |S_{3\phi}|$
- $|z_l| = \dfrac{\sqrt{3}\beta\,|V_{\text{pri}}|}{|I_{\text{pri}}|} = \dfrac{\sqrt{3}\cdot 5.45\%\cdot 480}{180} = 0.2517\Omega$

Parameter determination

Example



3-phase transformer ratings (secondary):

- Rated 3-phase power $\left|S_{3\phi}\right|$ = 150 kVA
- Rated secondary line-to-line voltage $\left|V_{\text{sec}}\right| = 208$ V
- Rated secondary line current $\left|I_{\text{sec}}\right| = 416$ A

Secondary in Y configuration:

$$|S_{3\phi}| = 3\,|S_{an}| = 3\,|V_{an}\,\bar{I}_{an}| = 3\,\left|\frac{V_{\text{sec}}}{\sqrt{3}\,e^{i\pi/6}}\right|\,|I_{\text{sec}}|$$

Hence

$$|S_{3\phi}| = \sqrt{3}\,|V_{\text{sec}}|\,|I_{\text{sec}}|$$

Verify:

- $\sqrt{3}\,|V_{\text{sec}}|\,|I_{\text{sec}}| = \sqrt{3}\cdot 208\cdot 416 = 149.87$ kVA $= |S_{3\phi}|$

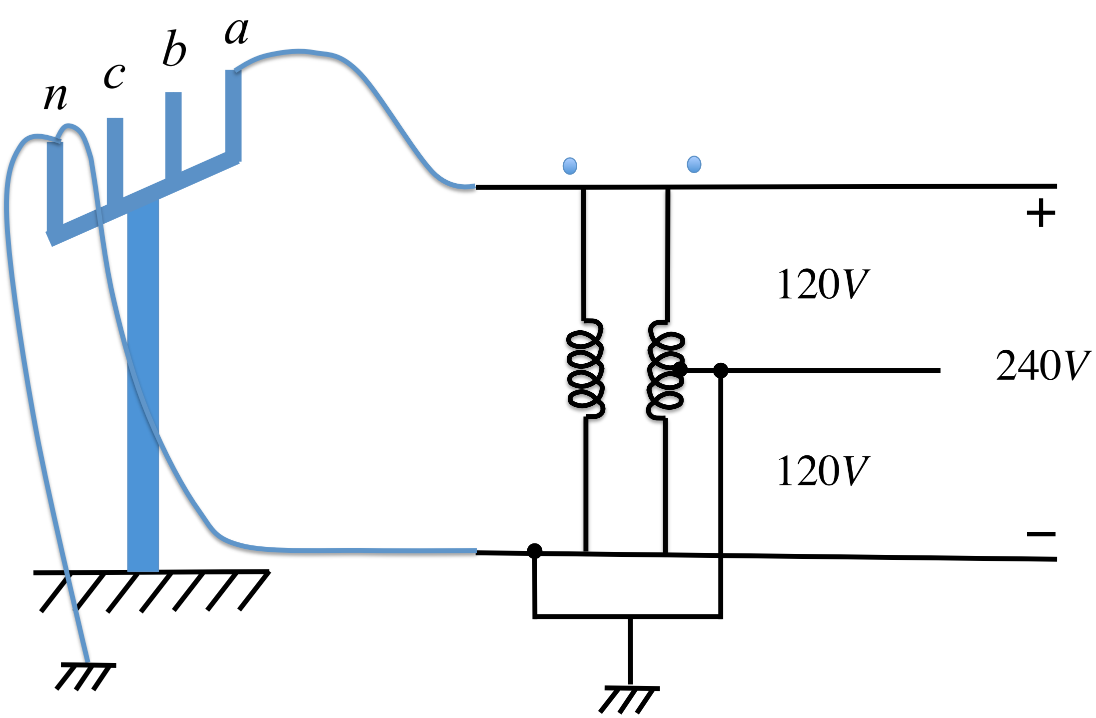
Distribution transformer

Examples

line-to-line voltage (kV) $\lvert V_{ab} \rvert$	phase voltage (kV) $\lvert V_{an} \rvert$	total power (MVA) $\lvert S_{3\phi} \rvert$
4.8	2.8	3.3
12.47	7.2	8.6
22.9	13.2	15.9
34.5	19.9	23.9

Distribution transformer

Examples

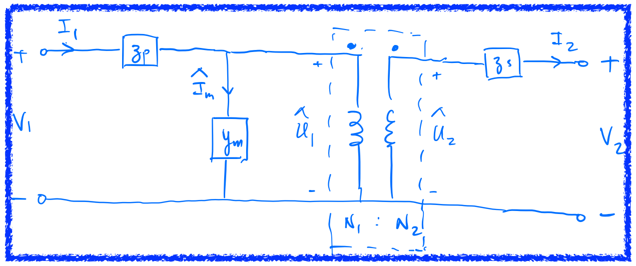


Common deployment in US

- Single phase
- Split-phase 120/240 V

Nonideal transformer

Circuit models

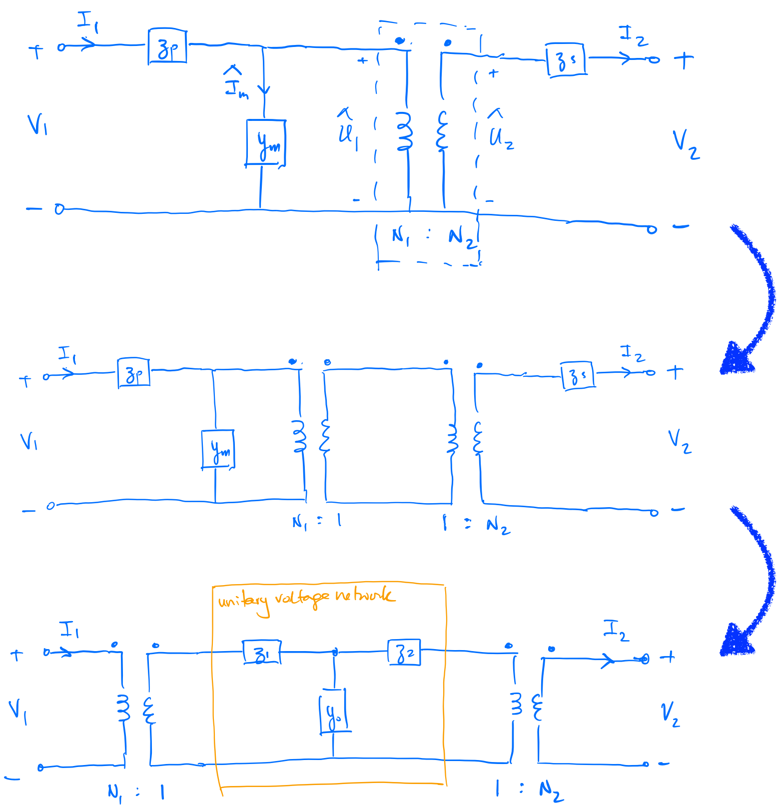


T eq. circuit $\equiv$ Unitary voltage ntk

Simplified model

Unitary voltage network

Single-phase 2-winding transformer



equivalent to 2 ideal transformers in series

reference imp & adm across ideal transformers

UVN-based model

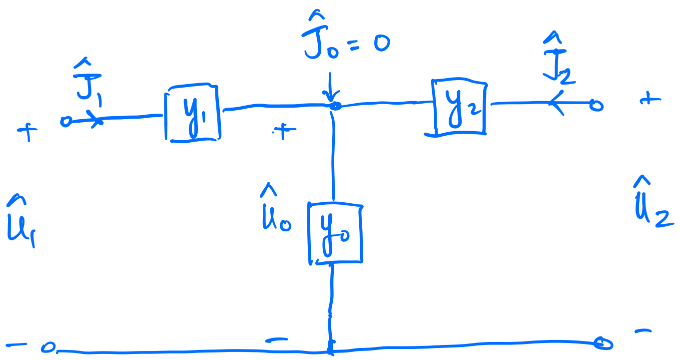
- Unitary voltage network (UVN) connecting 2 ideal transformers
- Equivalent to T equivalent circuit
- Simplified model is an approximation

Advantages

- UVN can be generalized to incorporate multiple windings, e.g., split-phase transformers
- Ideal transformers on both ends can be connected in various ways, e.g., 3-phase transformers in Y/Δ configurations, non-standard transformers

Single-phase transformer

Unitary voltage network



$$\hat{J}_1 = y_1(\hat{U}_1 - \hat{U}_0), \qquad \hat{J}_2 = y_2(\hat{U}_2 - \hat{U}_0)$$

$$y_0\hat{U}_0 = \hat{J}_0 + \hat{J}_1 + \hat{J}_2$$

Admittance matrix

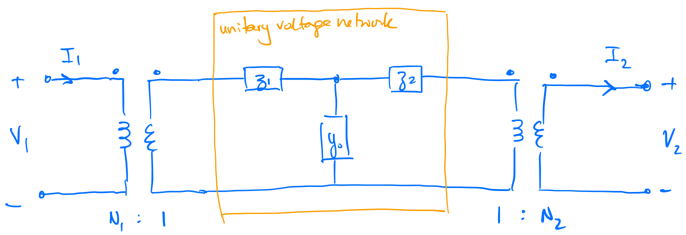
$$\begin{bmatrix} \hat{J}_0 \\ \hat{J}_1 \\ \hat{J}_2 \end{bmatrix} = \begin{bmatrix} y_0 + y_1 + y_2 & -y_1 & -y_2 \\ -y_1 & y_1 & 0 \\ -y_2 & 0 & y_2 \end{bmatrix} \begin{bmatrix} \hat{U}_0 \\ \hat{U}_1 \\ \hat{U}_2 \end{bmatrix}$$

Since $\hat{J}_0 = 0$, can eliminate $\hat{U}_0$ to obtain Kron reduced admittance matrix

$$\begin{bmatrix} \hat{J}_1 \\ \hat{J}_2 \end{bmatrix} = \underbrace{\frac{1}{\sum_i y_i} \begin{bmatrix} y_1(y_0 + y_2) & -y_1y_2 \\ -y_1y_2 & y_2(y_0 + y_1) \end{bmatrix}}_{Y_{\text{uvn}}} \begin{bmatrix} \hat{U}_1 \\ \hat{U}_2 \end{bmatrix}$$

Single-phase transformer

External model: admittance matrix



$$\begin{bmatrix} \hat{J}_1 \\ \hat{J}_2 \end{bmatrix} = \underbrace{\frac{1}{\sum_i y_i} \begin{bmatrix} y_1(y_0 + y_2) & -y_1 y_2 \\ -y_1 y_2 & y_2(y_0 + y_1) \end{bmatrix}}_{Y_{\text{uvn}}} \begin{bmatrix} \hat{U}_1 \\ \hat{U}_2 \end{bmatrix}$$

Let

$$I := \begin{bmatrix} I_1 \\ -I_2 \end{bmatrix}, \quad V := \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$M := \begin{bmatrix} 1/N_1 & 0 \\ 0 & 1/N_2 \end{bmatrix}$$

Conversion between internal vars & terminal vars across ideal transformers

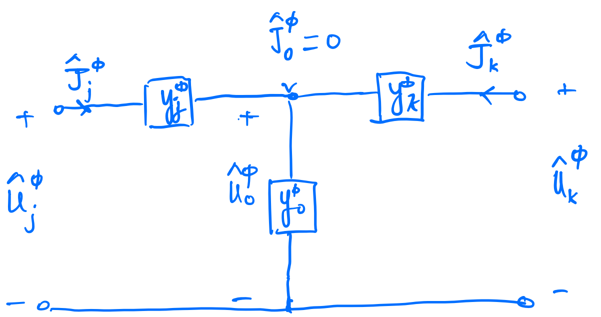
$$\hat{U} = MV, \quad \hat{J} = M^{-1} I$$

Hence, external model:

$$I = \left(M Y_{\text{uvn}} M \right) V$$

Three-phase transformers

Standard configurations



$$Y_{\mathsf{uvn}} \;:=\; \left(\mathbb{I}_2 \otimes \left(\sum_{i=0}^{2} y_i \right)^{-1} \right) \begin{bmatrix} y_j(y_0+y_k) & -y_jy_k \\ -y_jy_k & y_k(y_0+y_j) \end{bmatrix}$$

Let

$$I \;:=\; \begin{bmatrix} I_1^{abc} \\ -I_2^{abc} \end{bmatrix} \in \mathbb{C}^6, \quad V \;:=\; \begin{bmatrix} V_1^{abc} \\ V_2^{abc} \end{bmatrix} \in \mathbb{C}^6$$

$$M \;:=\; \begin{bmatrix} 1/N_1^{abc} & 0 \\ 0 & 1/N_2^{abc} \end{bmatrix} \in \mathbb{C}^{6\times 6}$$

External model:

$$I \;=\; D^{\mathsf{T}}(MY_{\mathsf{uvn}}M)D\left(V-\gamma\right)$$

where $\gamma := \left(V_1^n 1, V_2^n 1\right) \in \mathbb{C}^6$ are neutral voltages in YY configuration, and

YY config: $\quad D := \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{bmatrix}$

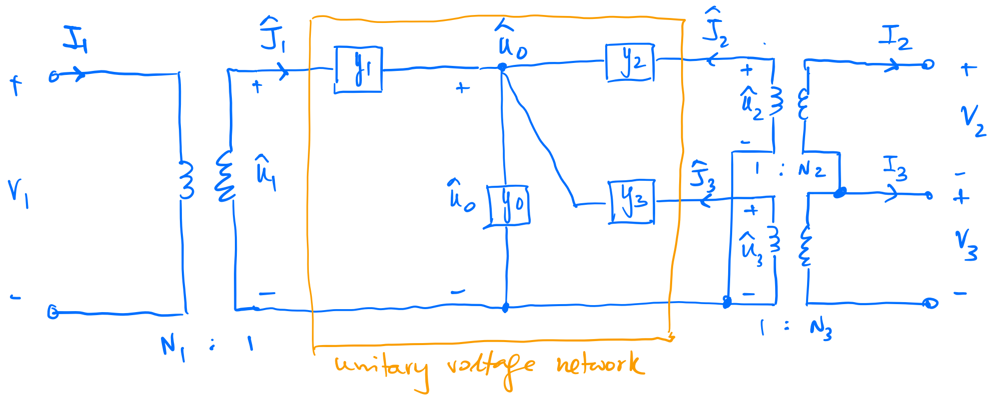
$\Delta\Delta$ config: $\quad D := \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix}$

ΔY config: $\quad D := \begin{bmatrix} \Gamma & 0 \\ 0 & \mathbb{I} \end{bmatrix}$

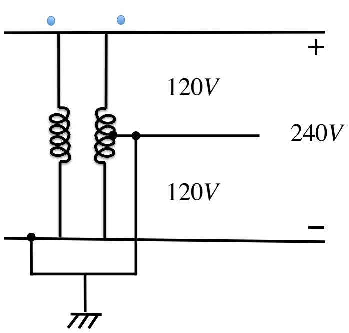
$Y\Delta$ config: $\quad D := \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \Gamma \end{bmatrix}$

Multi-winding transformers

Example: split-phase transformer



$$\begin{bmatrix} \hat{J}_0 \\ \hat{J}_1 \\ \hat{J}_2 \\ \hat{J}_3 \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{3} & -y_1 & -y_2 & -y_3 \\ -y_1 & y_1 & 0 & 0 \\ -y_2 & 0 & y_2 & 0 \\ -y_3 & 0 & 0 & y_3 \end{bmatrix} \begin{bmatrix} \hat{U}_0 \\ \hat{U}_1 \\ \hat{U}_2 \\ \hat{U}_3 \end{bmatrix}$$

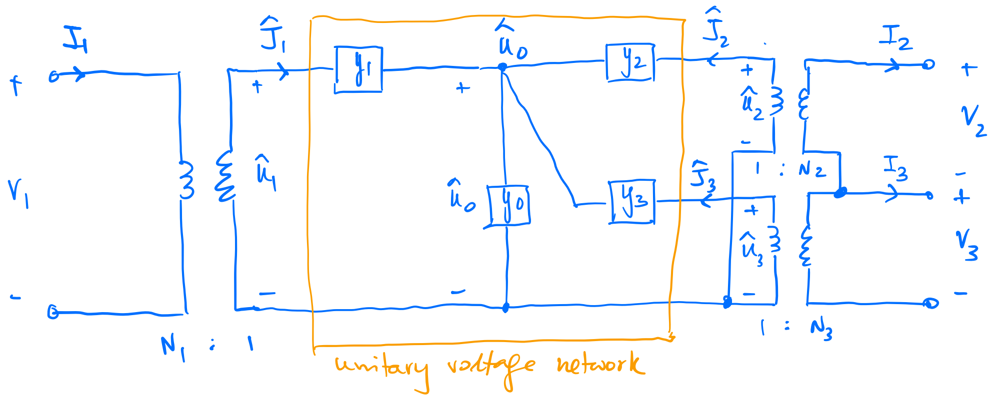


UVN: Kron-reduced admittance matrix

$$\begin{bmatrix} \hat{J}_1 \\ \hat{J}_2 \\ \hat{J}_3 \end{bmatrix} = \underbrace{\frac{1}{\sum_i y_i} \begin{bmatrix} y_1(y_0+y_2+y_3) & -y_1y_2 & -y_1y_3 \\ -y_2y_1 & y_2(y_0+y_1+y_3) & -y_2y_3 \\ -y_3y_1 & -y_3y_2 & y_3(y_0+y_1+y_2) \end{bmatrix}}_{Y_{\text{uvn}}} \begin{bmatrix} \hat{U}_1 \\ \hat{U}_2 \\ \hat{U}_3 \end{bmatrix}$$

Multi-winding transformers

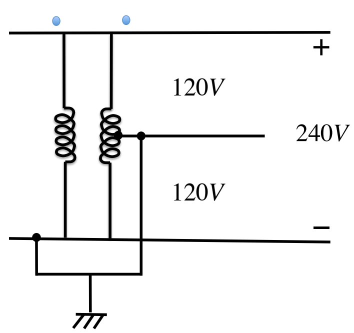
Example: split-phase transformer



Let

$$I \;:=\; \begin{bmatrix} I_1 \\ -I_2 \\ -I_3 \end{bmatrix}, \quad V \;:=\; \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$M \;:=\; \begin{bmatrix} 1/N_1 & 0 & 0 \\ 0 & 1/N_2 & 0 \\ 0 & 0 & 1/N_3 \end{bmatrix}$$

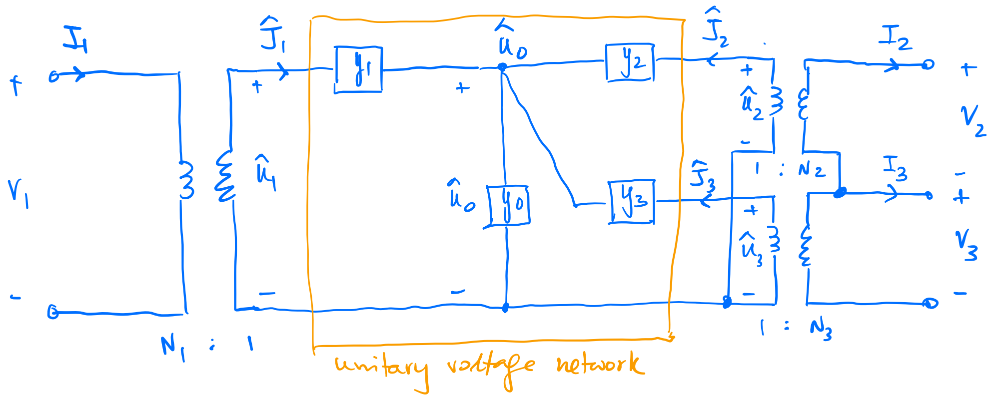


Conversion between internal vars & terminal vars across ideal transformers: $\hat{U} = MV$ and

$$\hat{J} \;=\; M^{-1} \begin{bmatrix} I_1 \\ -I_2 \\ -I_2 - I_3 \end{bmatrix} =: \; M^{-1}AI \quad \text{where } A := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Multi-winding transformers

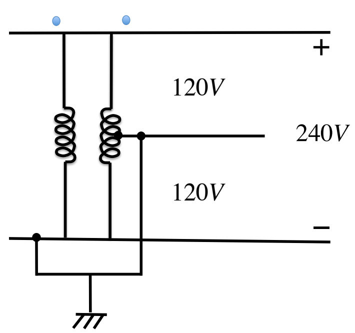
Example: split-phase transformer



Let

$$I := \begin{bmatrix} I_1 \\ -I_2 \\ -I_3 \end{bmatrix}, \quad V := \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$M := \begin{bmatrix} 1/N_1 & 0 & 0 \\ 0 & 1/N_2 & 0 \\ 0 & 0 & 1/N_3 \end{bmatrix}$$



Eliminate internal vars $(\hat{J}, \hat{U})$ from

$$\hat{U} = Y_{\text{uvn}} \hat{J}, \quad \hat{U} = MV, \quad \hat{J} = M^{-1} A I$$

External model:

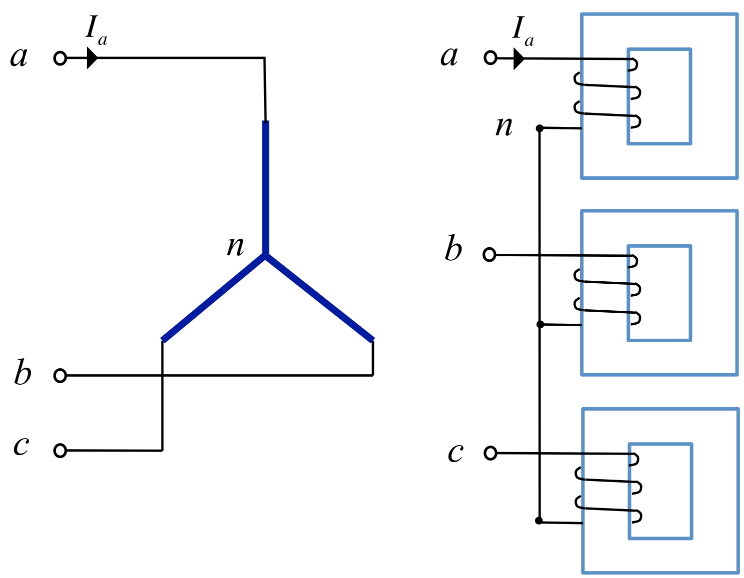
$$I = A^{-1} \left(M Y_{\text{uvn}} M \right) V$$

Outline

1. Single-phase transformer
2. Three-phase transformer
    - Ideal transformer
    - Equivalent circuit
3. Equivalent impedance
4. Per-phase analysis
5. Per-unit normalization

Ideal transformer

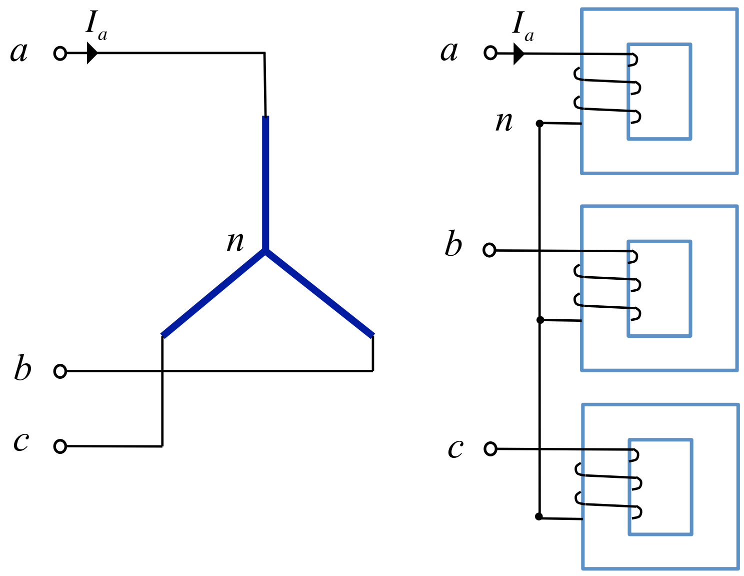
Connectivity



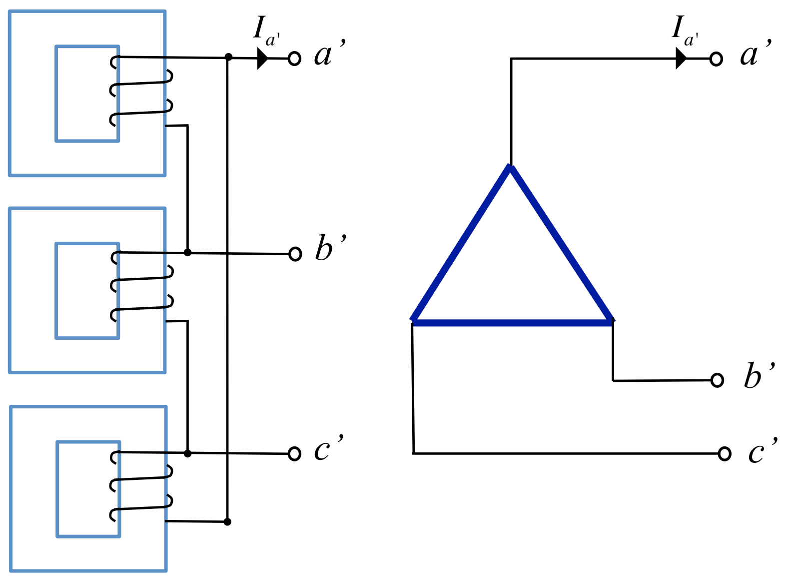
(a) Primary winding in Y configuration

Ideal transformer

Connectivity



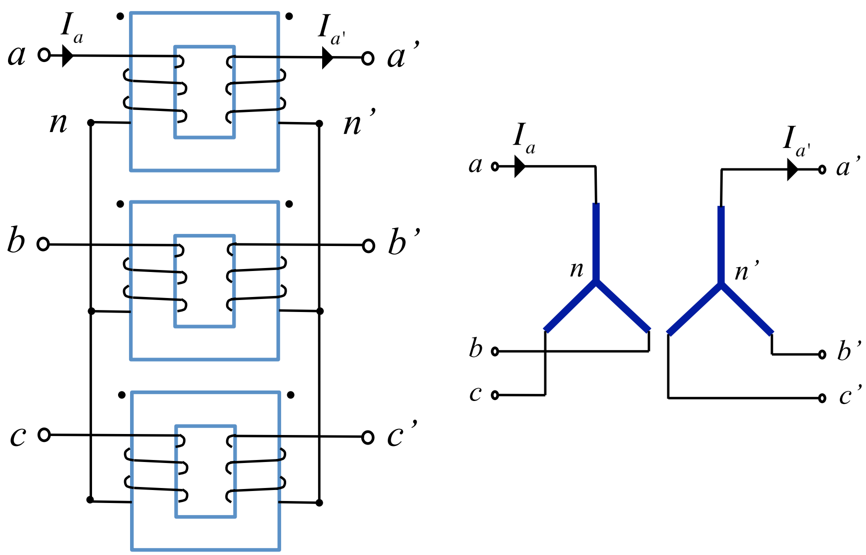
(a) Primary winding in Y configuration



(b) Secondary winding in Δ configuration

Ideal transformer

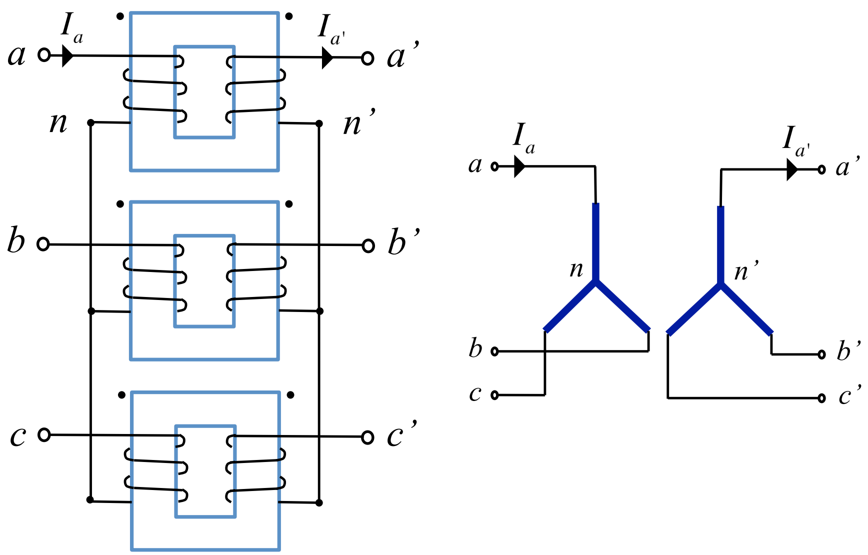
Configurations



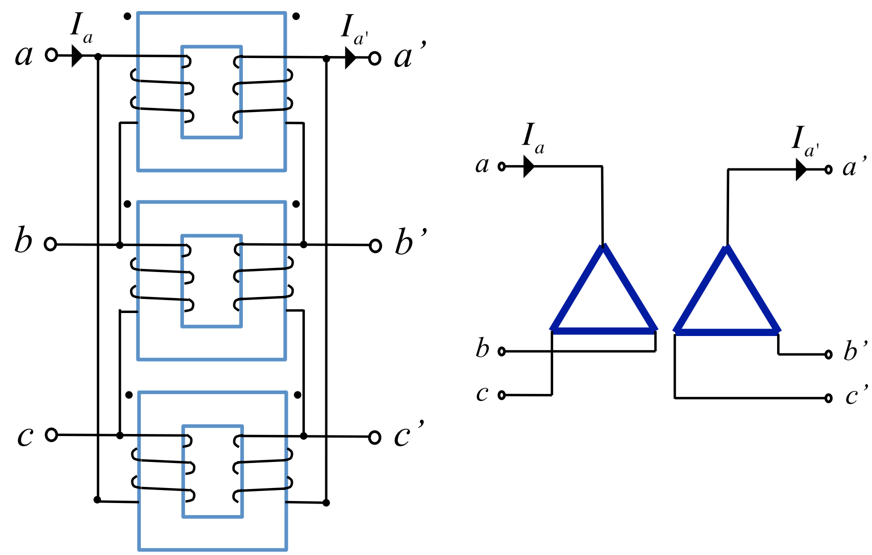
YY

Ideal transformer

Configurations



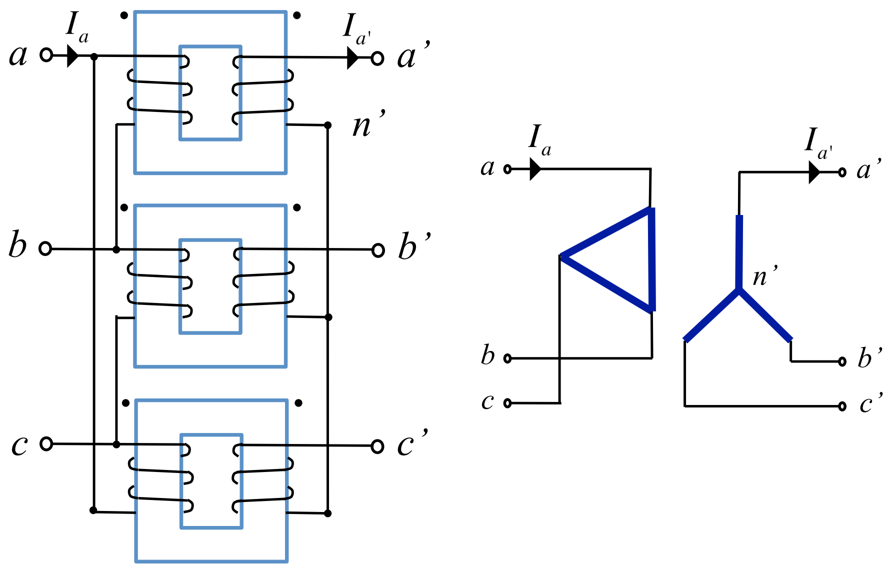
YY



$\Delta\Delta$

Ideal transformer

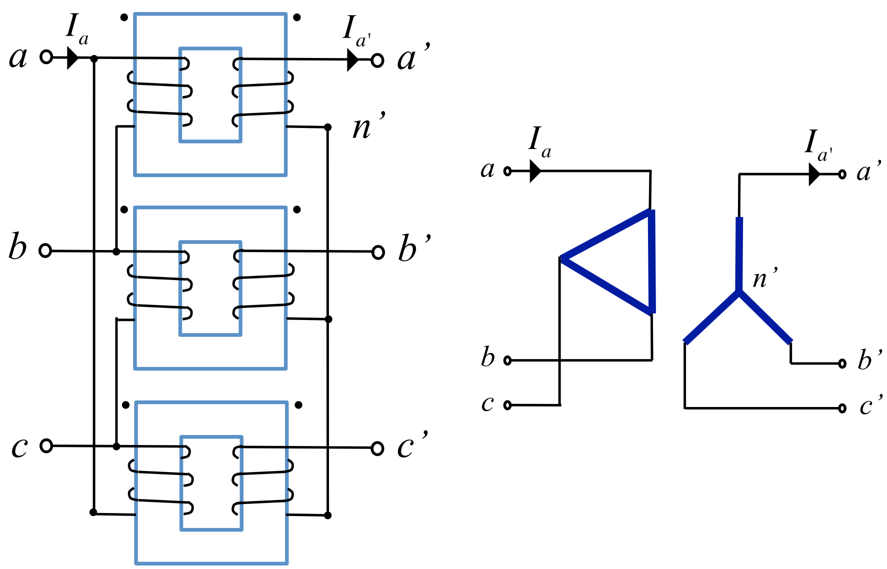
Configurations



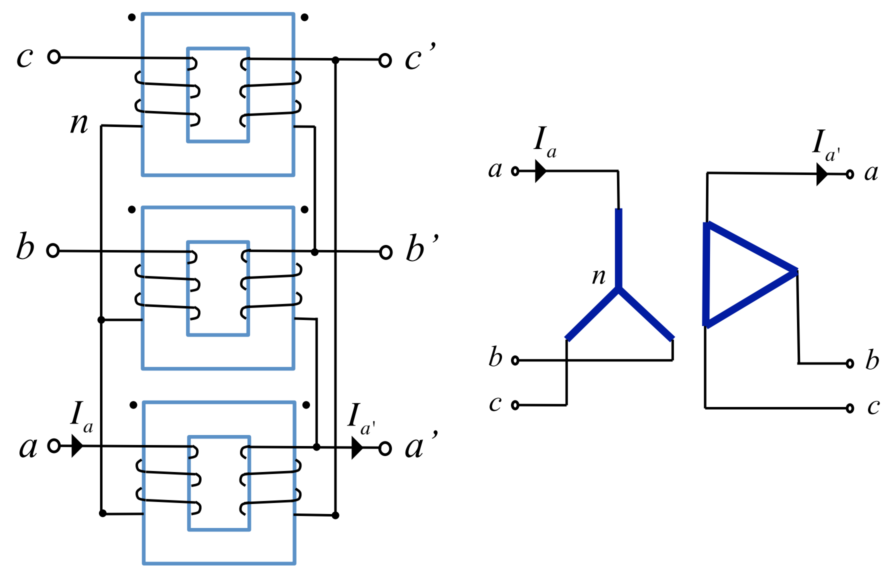
ΔY

Ideal transformer

Configurations



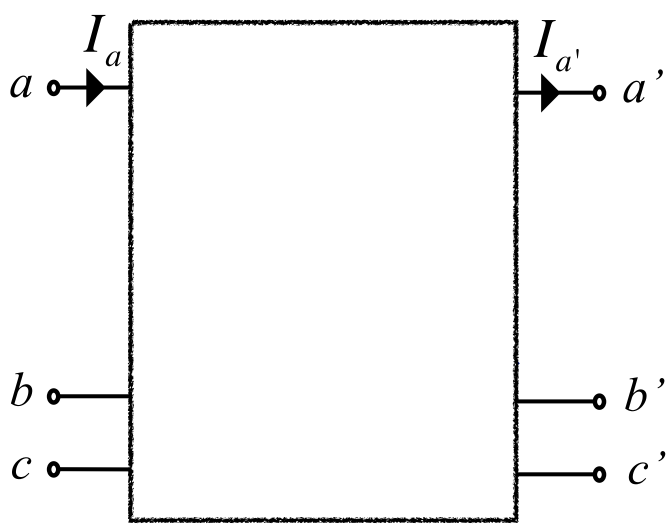
ΔY



$Y\Delta$

Ideal transformer

Configurations



What is external model?

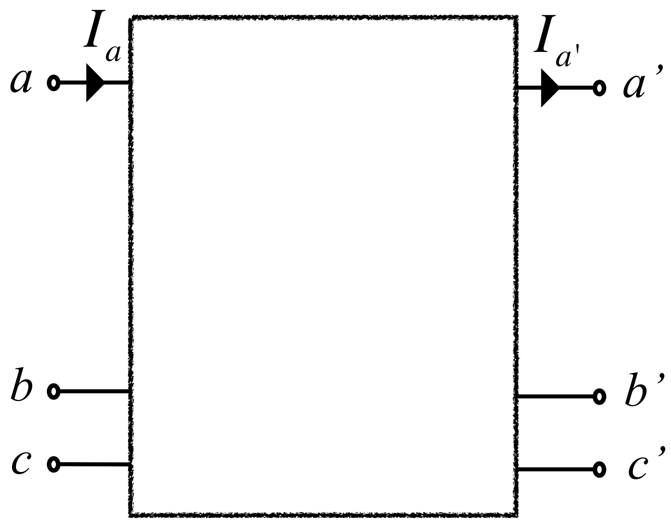
- Line-to-line voltage gain $\dfrac{V_{a'b'}}{V_{ab}}$
- Line current gain $\dfrac{I_{a'}}{I_a}$

What is YY equivalent circuit?

- Yields per-phase circuit

Ideal transformer

Configurations



Derivation

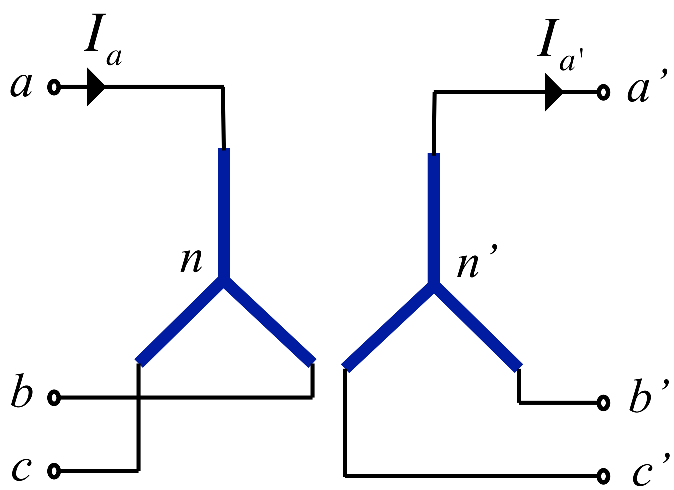
- Single-phase voltage & current gains
- Derive external model
- Derive YY equivalent circuit

Use conversion between phase & line vars

- $V_{ab} = \sqrt{3}\, e^{i\pi/6}\, V_{an}, \quad V_{a'b'} = \sqrt{3}\, e^{i\pi/6}\, V_{a'n'}$
- $I_a = \sqrt{3}\, e^{-i\pi/6}\, I_{ab}, \quad I_{a'} = -\sqrt{3}\, e^{-i\pi/6}\, I_{a'b'}$

Ideal transformer

YY configuration



- Single-phase gains

$$\frac{V_{a'n'}}{V_{an}} = n, \qquad \frac{I_{a'}}{I_a} = \frac{1}{n}$$

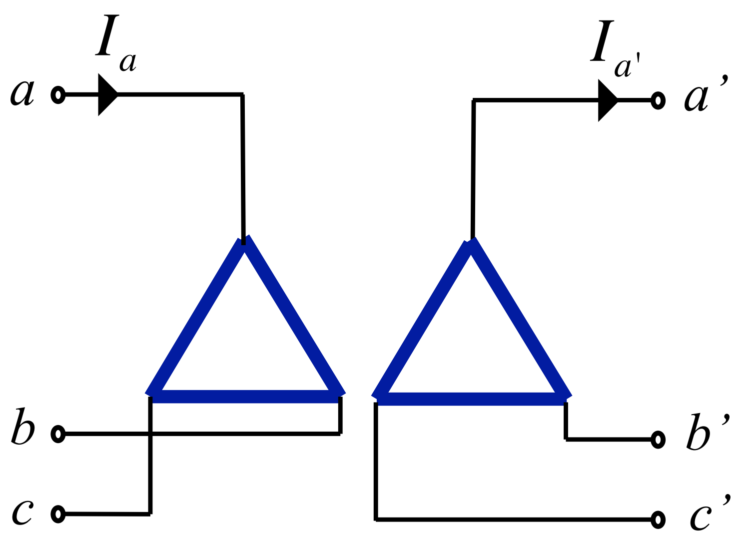
- External model

$$\frac{V_{a'b'}}{V_{ab}} = n, \qquad \frac{I_{a'}}{I_a} = \frac{1}{n}$$

external model = internal model

Ideal transformer

ΔΔ configuration



- Single-phase gains

$$\frac{V_{a'b'}}{V_{ab}} = n, \qquad \frac{-I_{a'b'}}{I_{ab}} = \frac{1}{n}$$

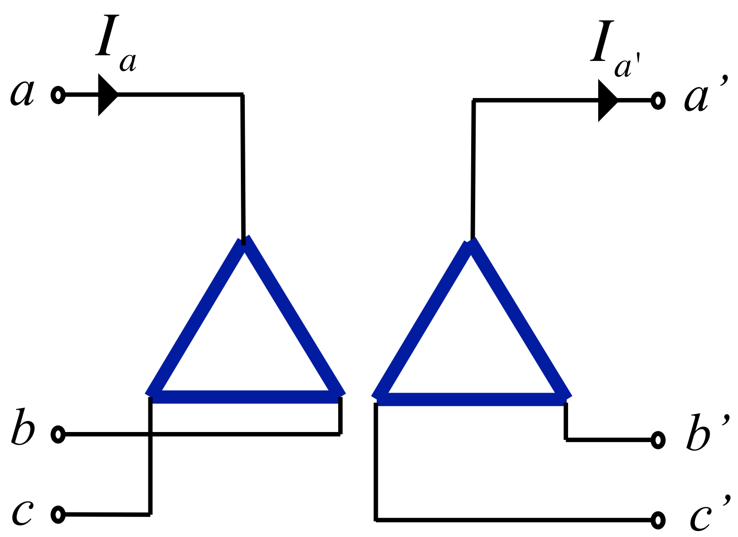
- External model

$$\frac{I_{a'}}{I_a} = \frac{\sqrt{3}\, e^{-i\pi/6}\, I_{a'b'}}{\sqrt{3}\, e^{-i\pi/6}\, I_{ab}} = \frac{1}{n}$$

external model = internal model

Ideal transformer

ΔΔ configuration



- Single-phase gains

$$\frac{V_{a'b'}}{V_{ab}} = n, \qquad \frac{-I_{a'b'}}{I_{ab}} = \frac{1}{n}$$

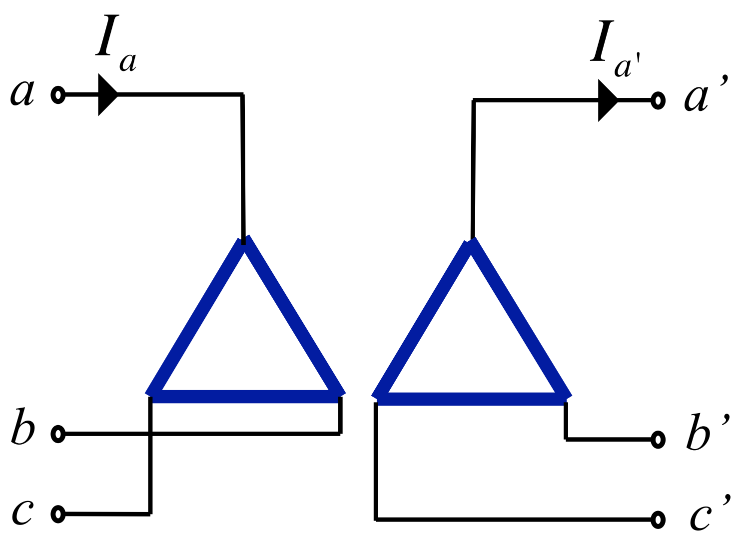
- Equivalent YY circuit

$$\frac{V^Y_{a'n'}}{V^Y_{an}} = \frac{\left(\sqrt{3}\, e^{i\pi/6}\right)^{-1} V^Y_{a'b'}}{\left(\sqrt{3}\, e^{i\pi/6}\right)^{-1} V^Y_{ab}} = n$$

$$\frac{-I^Y_{a'n'}}{I^Y_{an}} = \frac{I_{a'}}{I_a} = \frac{1}{n}$$

Ideal transformer

ΔΔ configuration



- Single-phase gains

$$\frac{V_{a'b'}}{V_{ab}} = n, \qquad \frac{-I_{a'b'}}{I_{ab}} = \frac{1}{n}$$

- Equivalent YY circuit

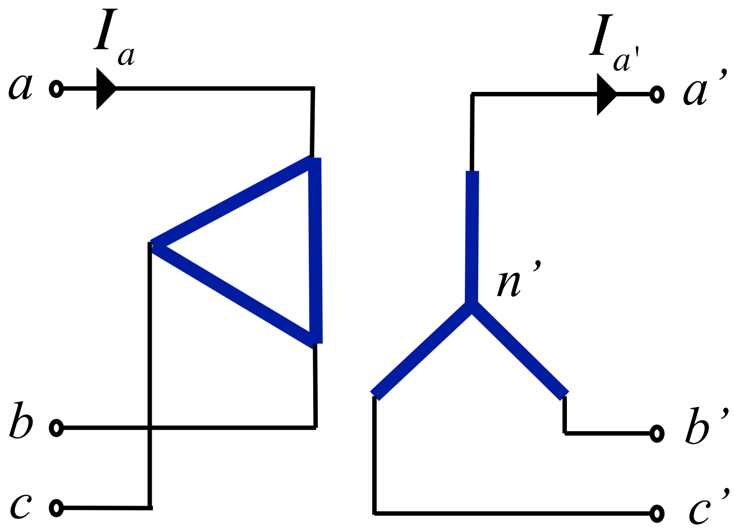
$$\frac{V_{a'b'}}{V_{ab}} = \frac{V^Y_{a'n'}}{V^Y_{an}} = n$$

$$\frac{I_{a'}}{I_a} = \frac{-I^Y_{a'n'}}{I^Y_{an}} = \frac{1}{n}$$

external model = YY equivalent = internal model

Ideal transformer

ΔY configuration



- Single-phase gains

$$\frac{V_{a'n'}}{V_{ab}} = n, \qquad \frac{-I_{a'n'}}{I_{ab}} = \frac{1}{n}$$

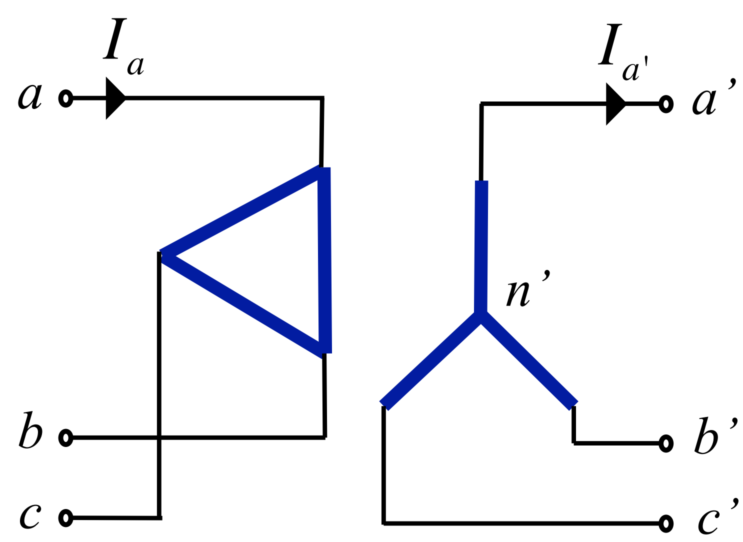
- External model

$$\frac{V_{a'b'}}{V_{ab}} = \frac{\sqrt{3}e^{i\pi/6}V_{a'n'}}{V_{ab}} = \sqrt{3}e^{i\pi/6}n$$

$$\frac{I_{a'}}{I_a} = \frac{-I_{a'n'}}{\sqrt{3}e^{-i\pi/6}\,I_{ab}} = \frac{1}{\sqrt{3}e^{-i\pi/6}\,n}$$

Ideal transformer

ΔY configuration

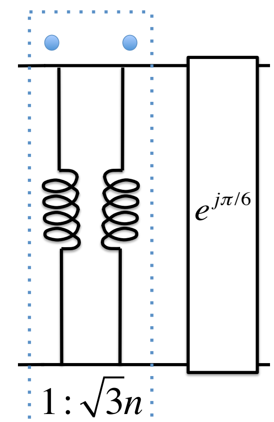


- Single-phase gains

$$\frac{V_{a'n'}}{V_{ab}} = n, \qquad \frac{-I_{a'n'}}{I_{ab}} = \frac{1}{n}$$

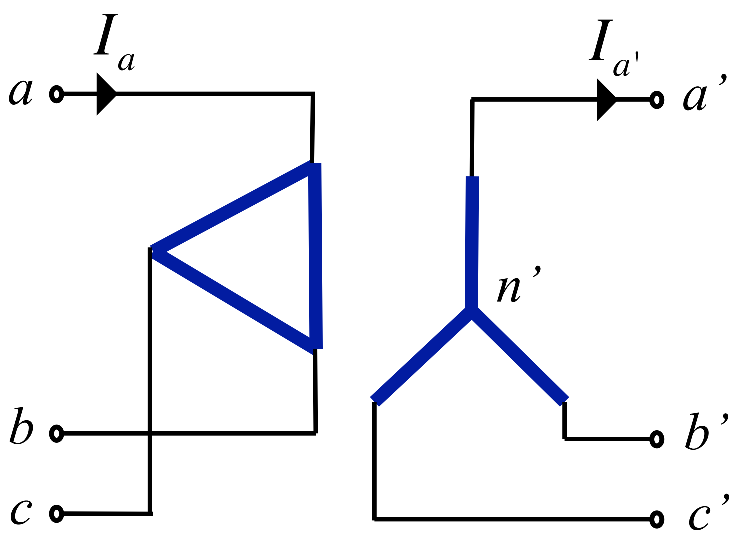
- Complex voltage gain

$$K_{\Delta Y}(n) := \sqrt{3}e^{i\pi/6}n$$



Ideal transformer

ΔY configuration



- Single-phase gains

$$\frac{V_{a'n'}}{V_{ab}} = n, \qquad \frac{-I_{a'n'}}{I_{ab}} = \frac{1}{n}$$

- Complex voltage gain

$$K_{\Delta Y}(n) := \sqrt{3}e^{i\pi/6}n$$

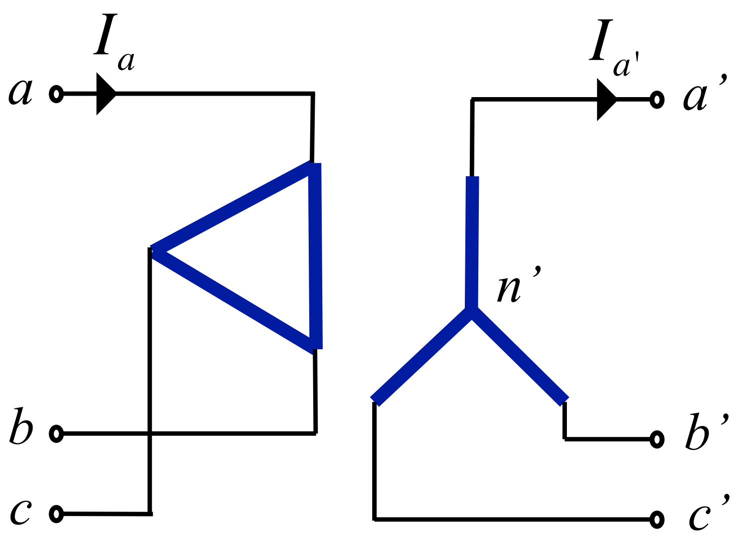
- External model

$$V_{a'b'} = K_{\Delta Y}(n)\, V_{ab}$$

$$I_{a'} = \frac{I_a}{K_{\Delta Y}^*(n)}$$

Ideal transformer

ΔY configuration



terminal behavior = YY equivalent

- Single-phase gains

$$\frac{V_{a'n'}}{V_{ab}} = n, \qquad \frac{-I_{a'n'}}{I_{ab}} = \frac{1}{n}$$

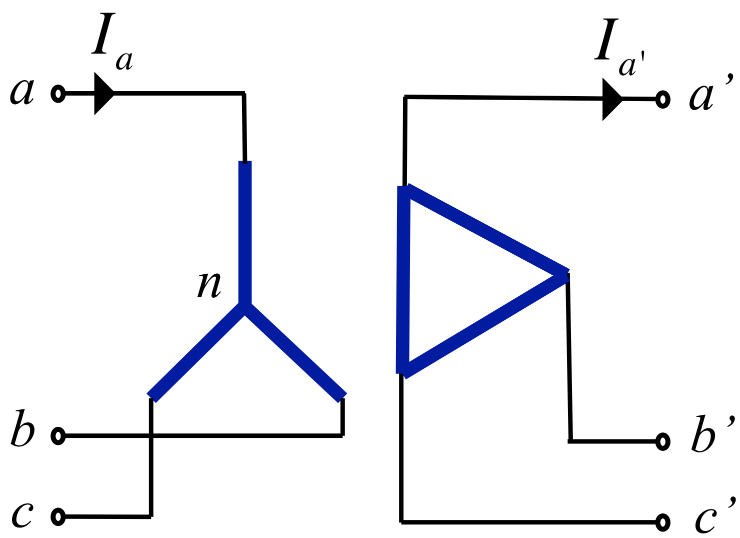
- Equivalent YY circuit

$$\frac{V_{a'n'}}{V_{an}^{Y}} = \frac{V_{a'n'}}{\left(\sqrt{3}e^{i\pi/6}\right)^{-1} V_{ab}} = K_{\Delta Y}(n)$$

$$\frac{I_{a'}}{I_a} = \frac{-I_{a'n'}}{\sqrt{3}e^{-i\pi/6}\, I_{ab}} = \frac{1}{K_{\Delta Y}^{*}(n)}$$

Ideal transformer

$Y\Delta$ configuration



- Single-phase gains

$$\frac{V_{a'c'}}{V_{an}} = n, \qquad \frac{I_{c'a'}}{I_{an}} = \frac{1}{n}$$

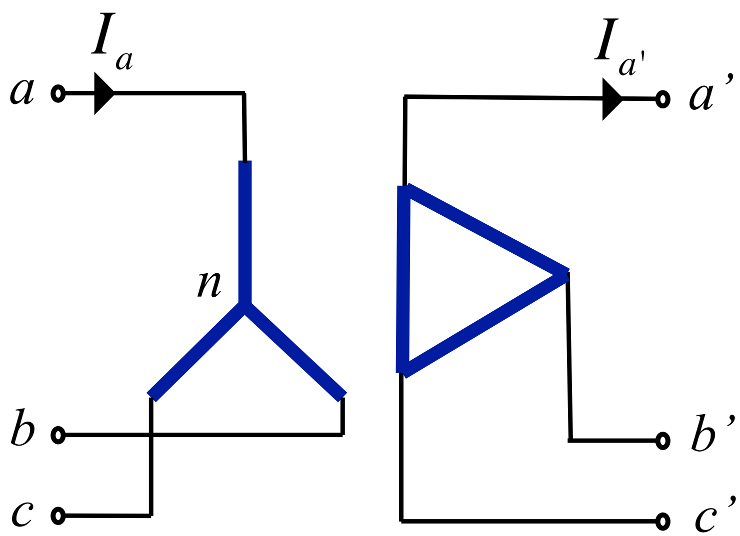
- External model

$$\frac{V_{a'c'}}{V_{ac}} = \frac{V_{a'c'}}{\sqrt{3}e^{-i\pi/6}V_{an}} = \frac{n}{\sqrt{3}}\,e^{i\pi/6}$$

$$\frac{I_{a'}}{I_a} = \frac{\sqrt{3}\,e^{i\pi/6}\,I_{c'a'}}{I_{an}} = \frac{\sqrt{3}\,e^{i\pi/6}}{n}$$

Ideal transformer

$Y\Delta$ configuration



- Single-phase gains

$$\frac{V_{a'c'}}{V_{an}} = n, \qquad \frac{I_{c'a'}}{I_{an}} = \frac{1}{n}$$

- Complex voltage gain

$$K_{Y\Delta}(n) := \frac{n}{\sqrt{3}} e^{i\pi/6}$$

- External mdoel

$$V_{a'b'} = K_{Y\Delta}(n)\, V_{ab}$$
$$I_{a'} = K^*_{Y\Delta}(n)\, I_a$$

Ideal transformer

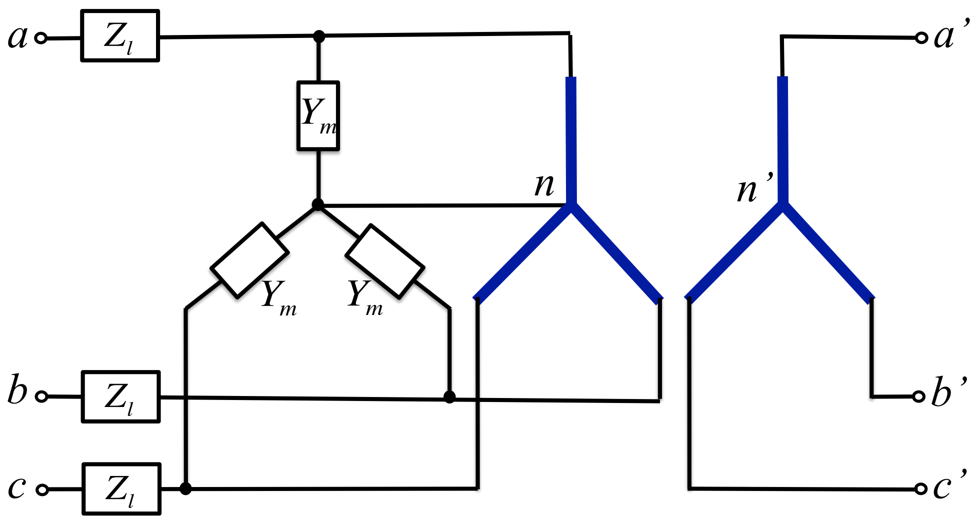
Summary

Property	Gain
Voltage gain	$K(n)$
Current gain	$\frac{1}{K^*(n)}$
Power gain	1
Sec Z_l referred to pri	$\frac{Z_l}{\lvert K(n)\rvert^2}$

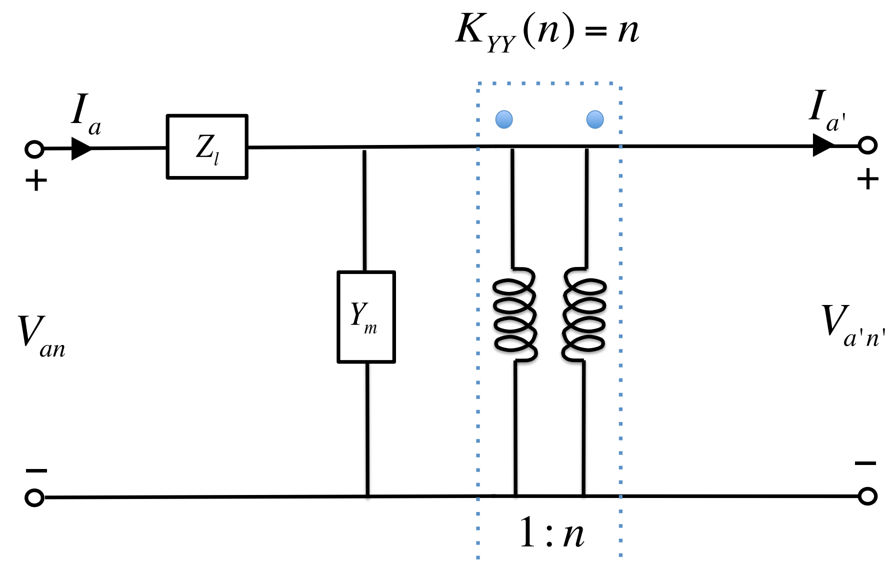
Configuration	Gain
YY	$K_{YY}(n) := n$
$\Delta\Delta$	$K_{\Delta\Delta}(n) := n$
ΔY	$K_{\Delta Y}(n) := \sqrt{3}n\ e^{\mathbf{i}\pi/6}$
$Y\Delta$	$K_{Y\Delta}(n) := \frac{n}{\sqrt{3}}\ e^{\mathbf{i}\pi/6}$

Equivalent circuit

YY configuration



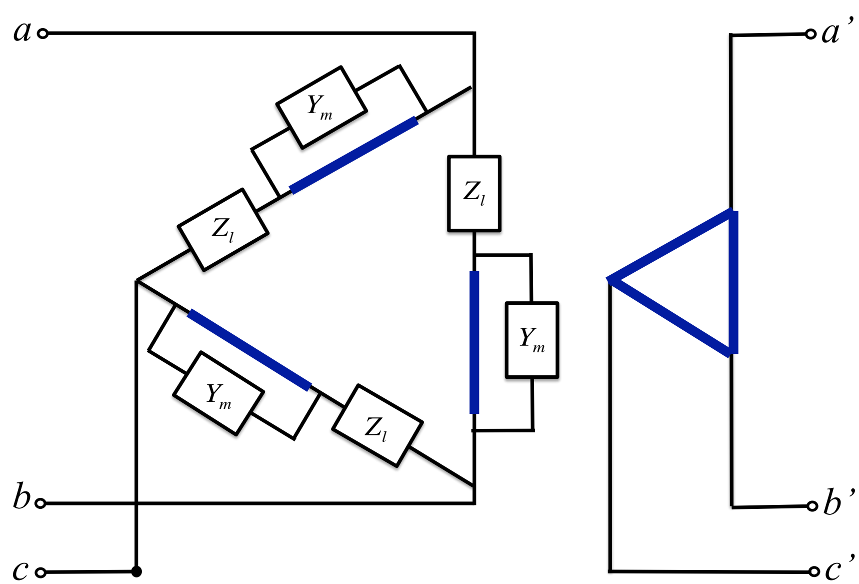
equivalent circuit



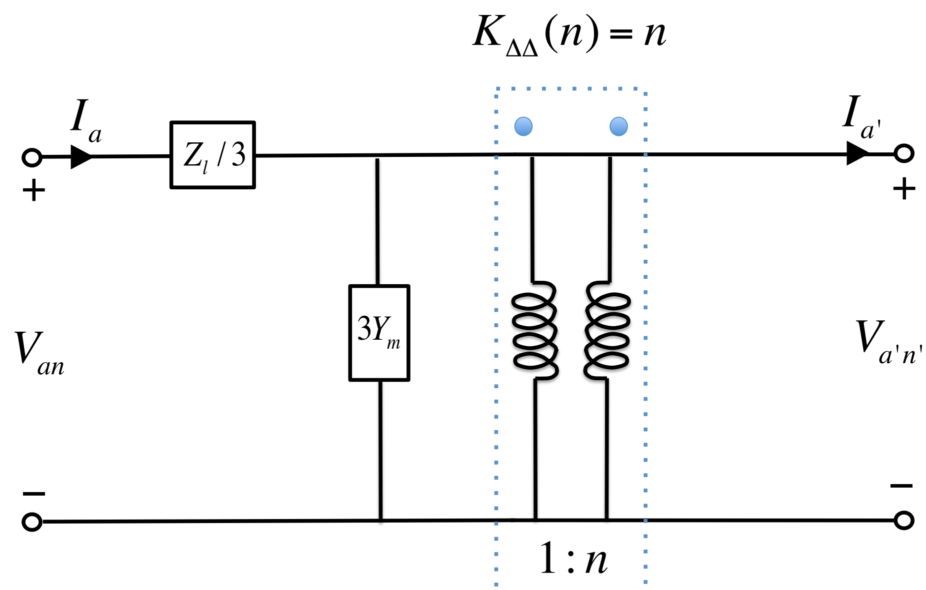
per-phase circuit

Equivalent circuit

ΔΔ configuration



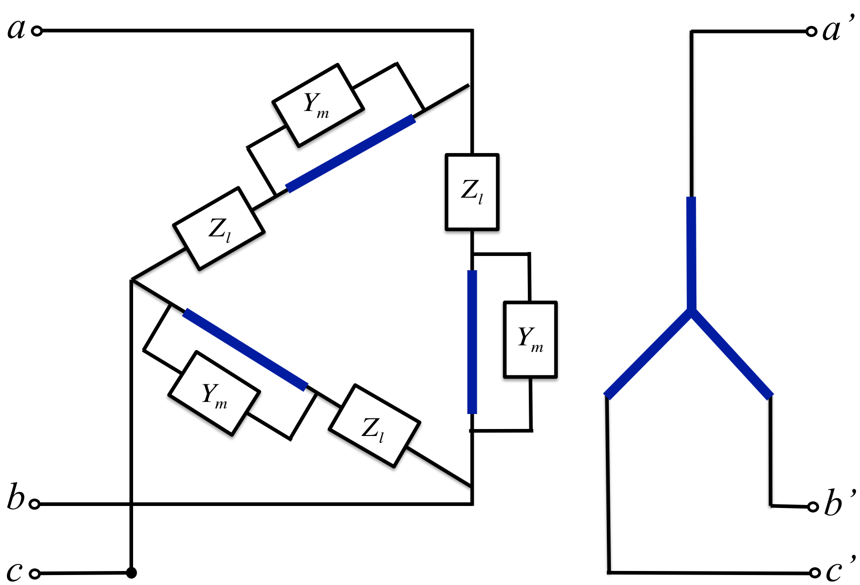
equivalent circuit



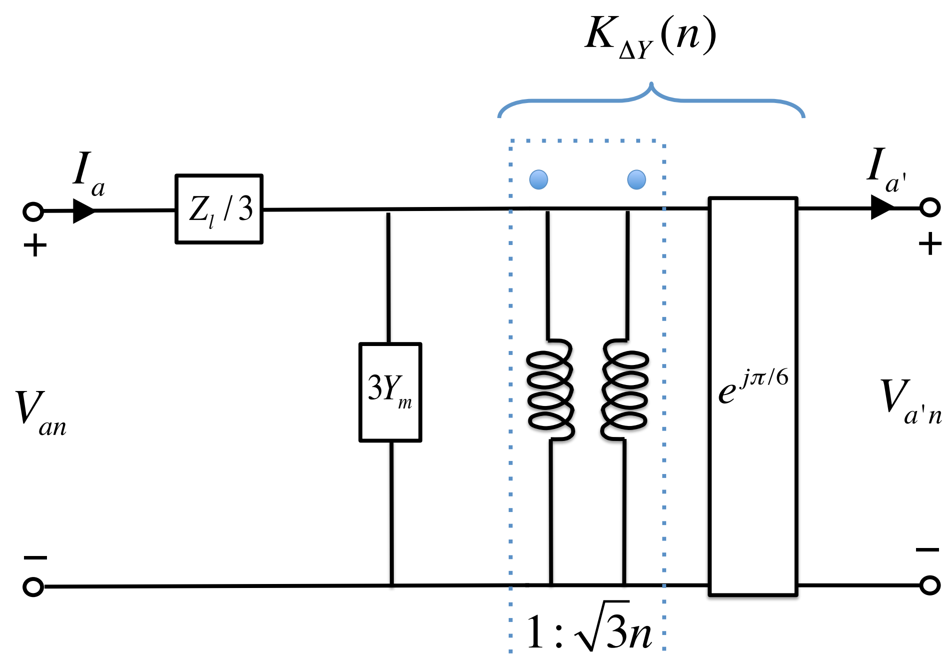
per-phase circuit

Equivalent circuit

ΔY configuration



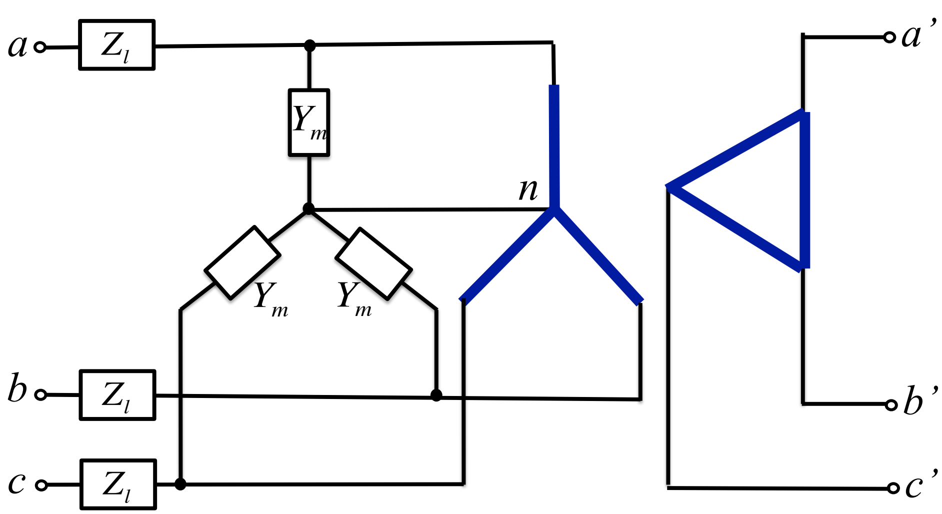
equivalent circuit



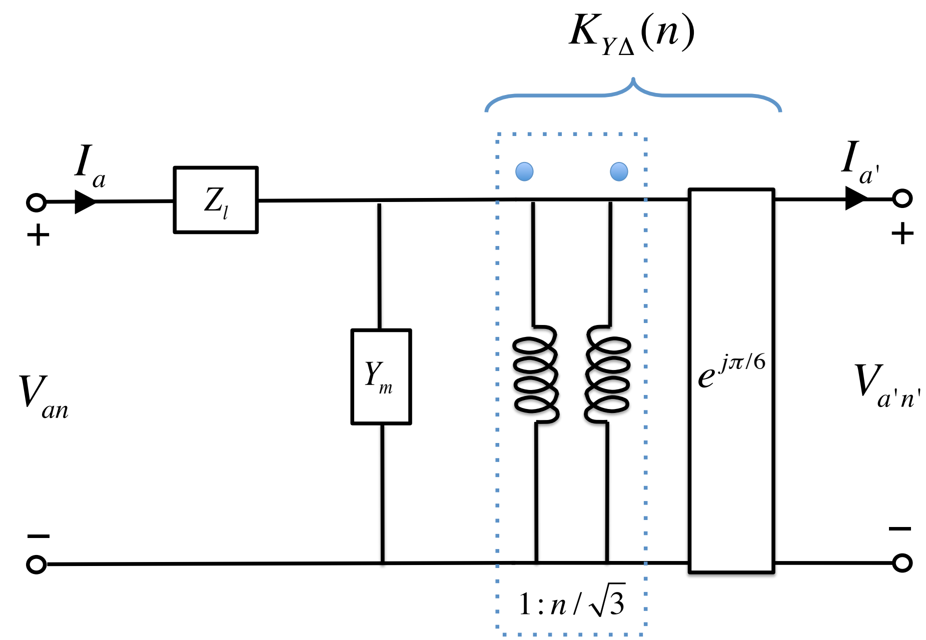
per-phase circuit

Equivalent circuit

$Y\Delta$ configuration



equivalent circuit



per-phase circuit

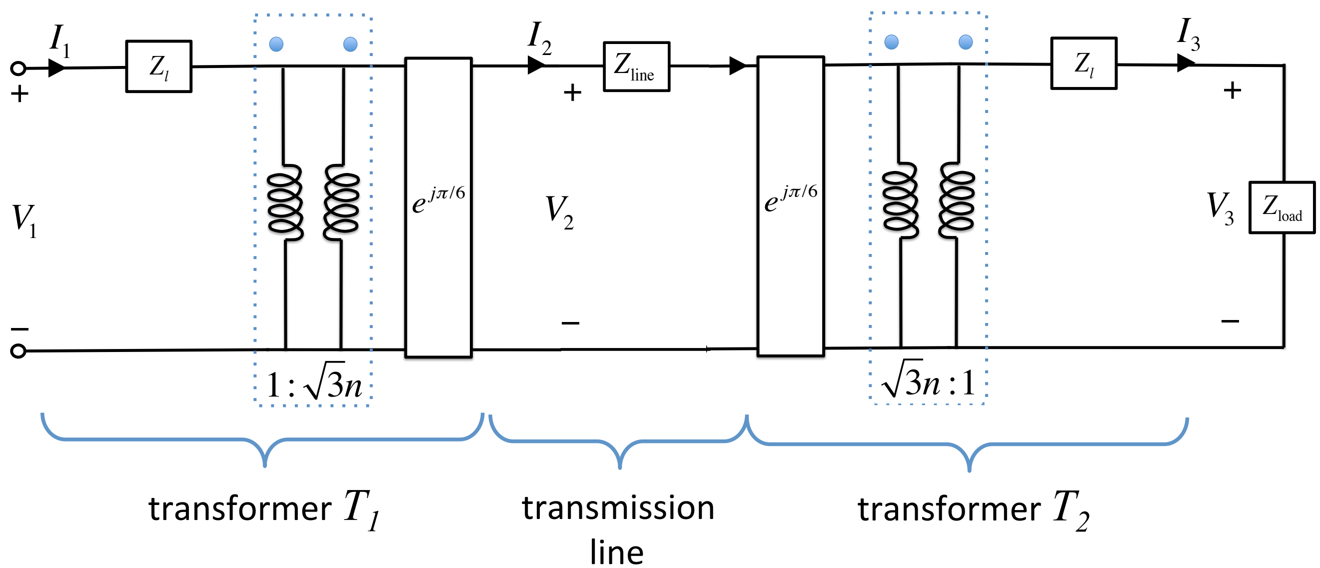
Outline

1. Single-phase transformer
2. Three-phase transformer
3. Equivalent impedance
    - Equivalence
    - Transmission matrix
    - Driving-point impedance
4. Per-phase analysis
5. Per-unit normalization

Motivation

Short cut in analyzing circuits containing transformers

- Thevenin equivalent of impedances in series and in parallel
- Equivalent impedances in primary or secondary circuits



Equivalent impedances

- referring Z_s in secondary to primary

$$Z_p = \frac{Z_s}{|K(n)|^2}$$

"It is equivalent to replace Z_s in the secondary circuit by Z_p in the primary circuit"

- referring Z_p in primary to secondary

$$Z_s = |K(n)|^2 Z_p$$

"It is equivalent to replace Z_p in the primary circuit by Z_s in the secondary circuit"

Equivalent admittances

- referring Y_s in secondary to primary

$$Y_p = |K(n)|^2 \; Y_s$$

"It is equivalent to replace Y_s in the secondary circuit by Y_p in the primary circuit"

- referring Y_p in primary to secondary

$$Y_s = \frac{Y_p}{|K(n)|^2}$$

"It is equivalent to replace Y_p in the primary circuit by Y_s in the secondary circuit"

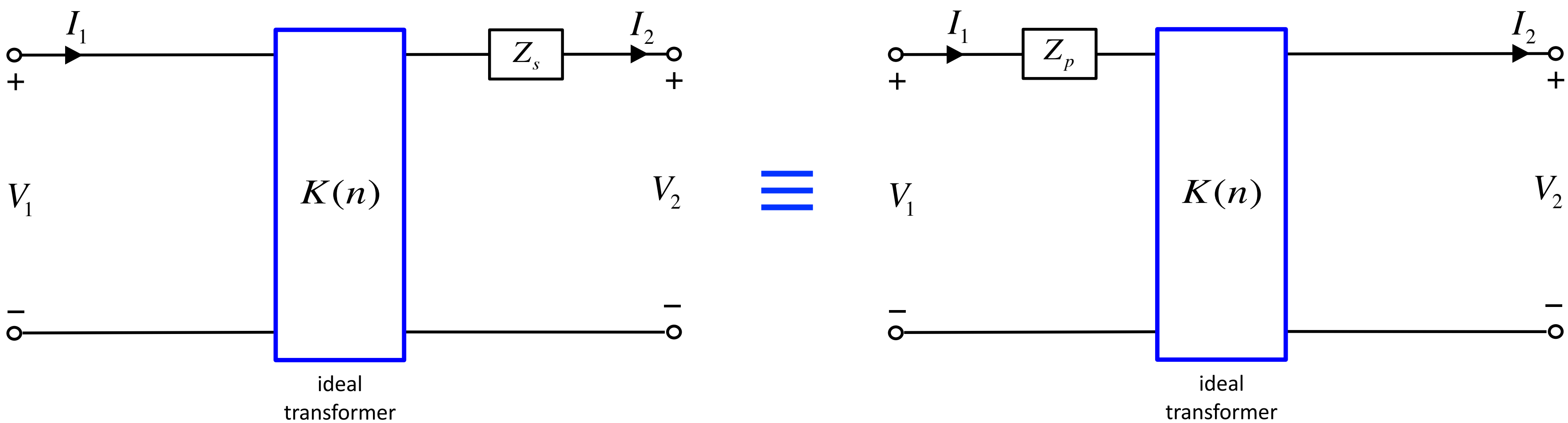
Equivalent impedances

What is equivalence ?

- Same transmission matrices
- Same driving-point impedance

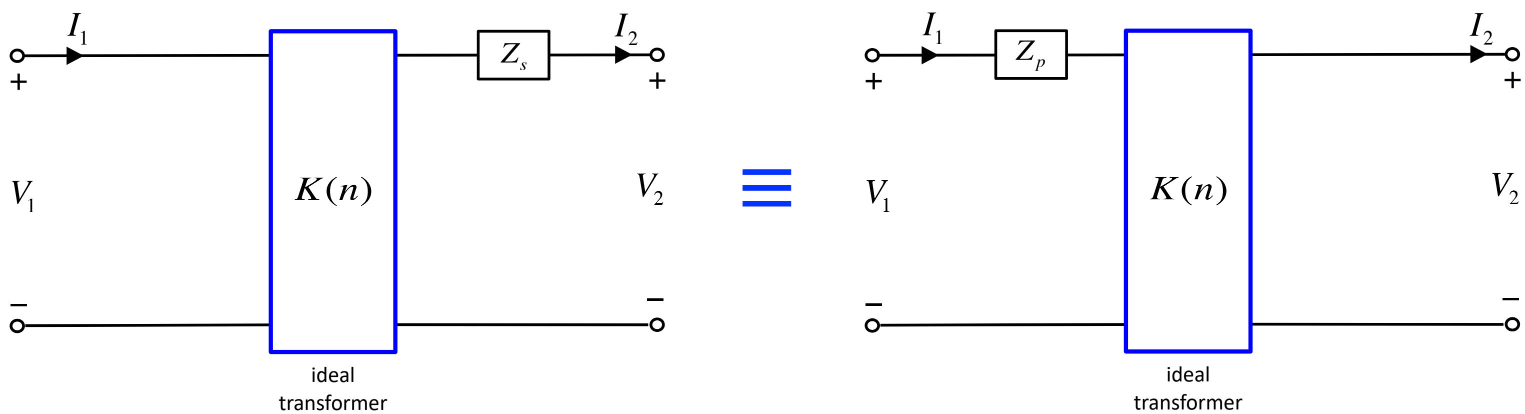
This is a simple consequence of Kirchhoff's and Ohm's laws

Transmission matrix



External models (transmission matrices) of 2 circuits are equal if and only if $Z_p = \dfrac{Z_s}{|K(n)|^2}$

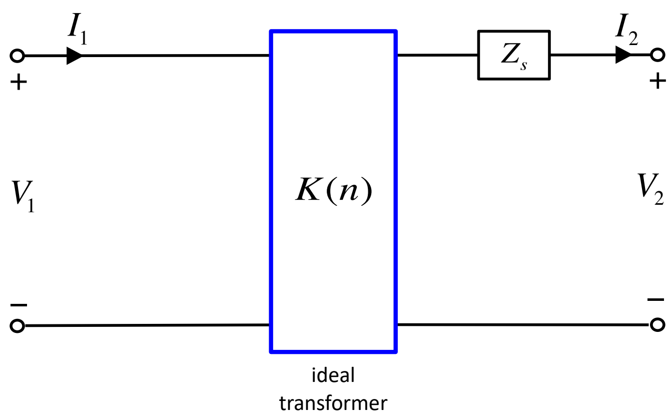
Transmission matrix

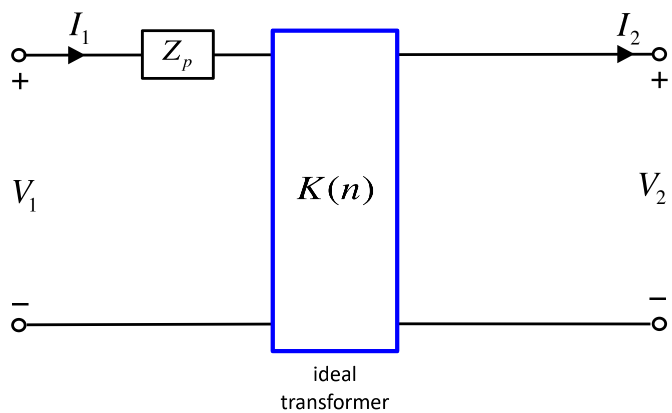


$$\begin{bmatrix} V \\ I \end{bmatrix} = \begin{bmatrix} 1 & Z_s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} K^{-1}(n) & 0 \\ 0 & K^*(n) \end{bmatrix} \begin{bmatrix} V \\ I \end{bmatrix}$$

Transmission matrix





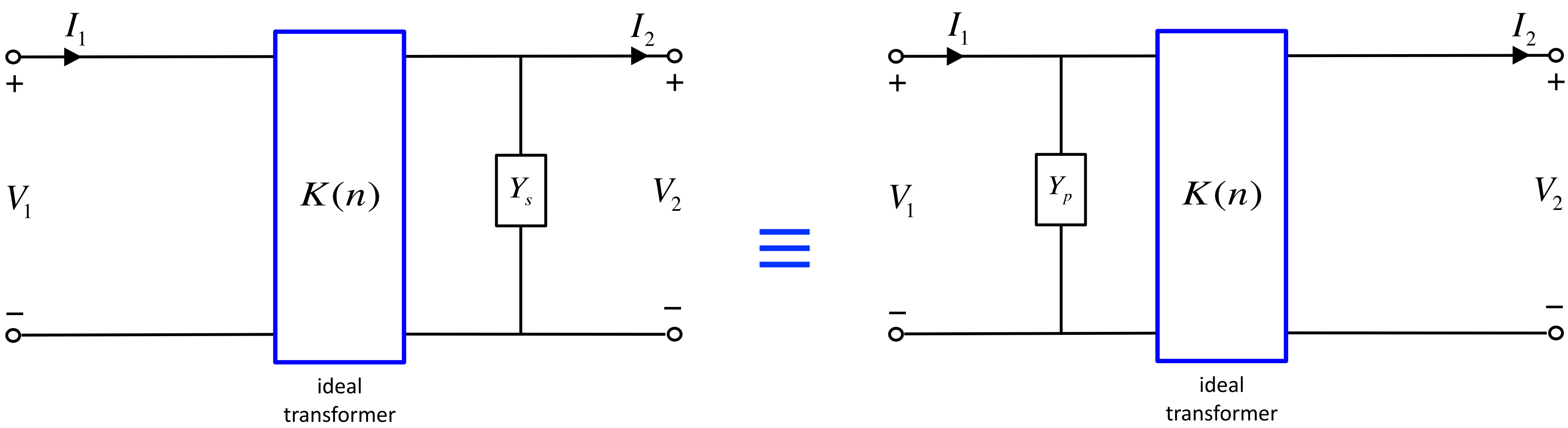
$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} K^{-1}(n) & K^{-1}(n)Z_s \\ 0 & K^*(n) \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} K^{-1}(n) & K^*(n)Z_p \\ 0 & K^*(n) \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

External models (transmission matrices) of 2 circuits are equal

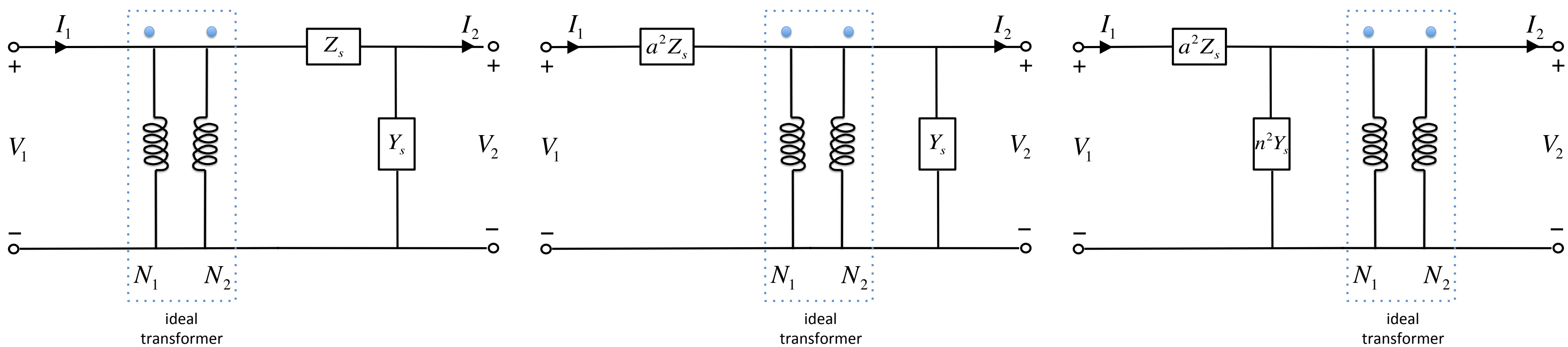
if and only if $Z_p = \dfrac{Z_s}{|K(n)|^2}$

Transmission matrix



External models (transmission matrices) of 2 circuits are equal
if and only if $Y_p \;=\; |K(n)|^2\, Y_s$

Transmission matrix
Example



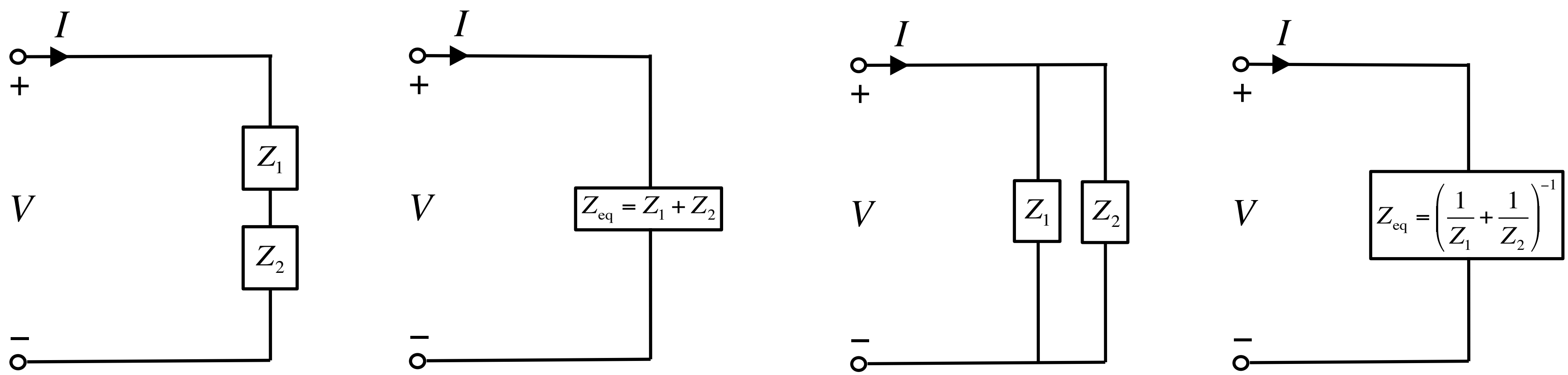
(a) (Z_s, Y_s) in the secondary circuit.

(b) Refer Z_s to the primary.

(c) Refer Y_s to the primary.

Driving-point impedance

Thevenin equivalent

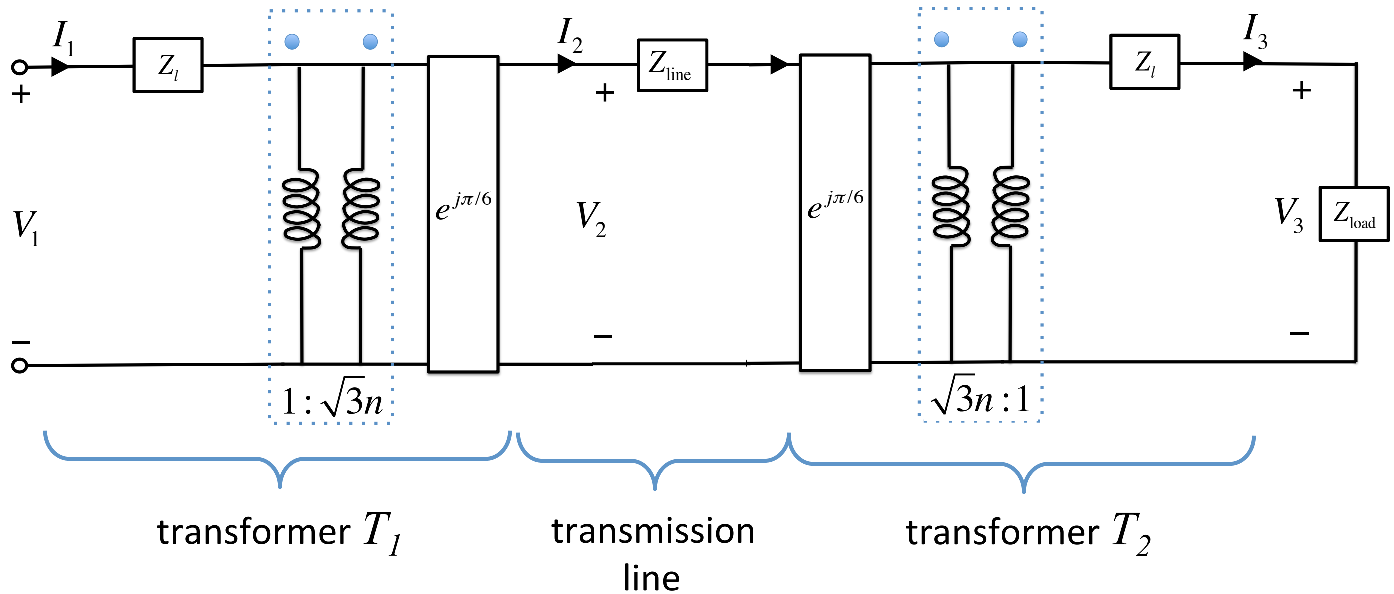


(a) Impedances in series (b) Impedances in parallel

Thevenin equivalent is a short cut in analyzing circuits with impedances only

Driving-point impedance

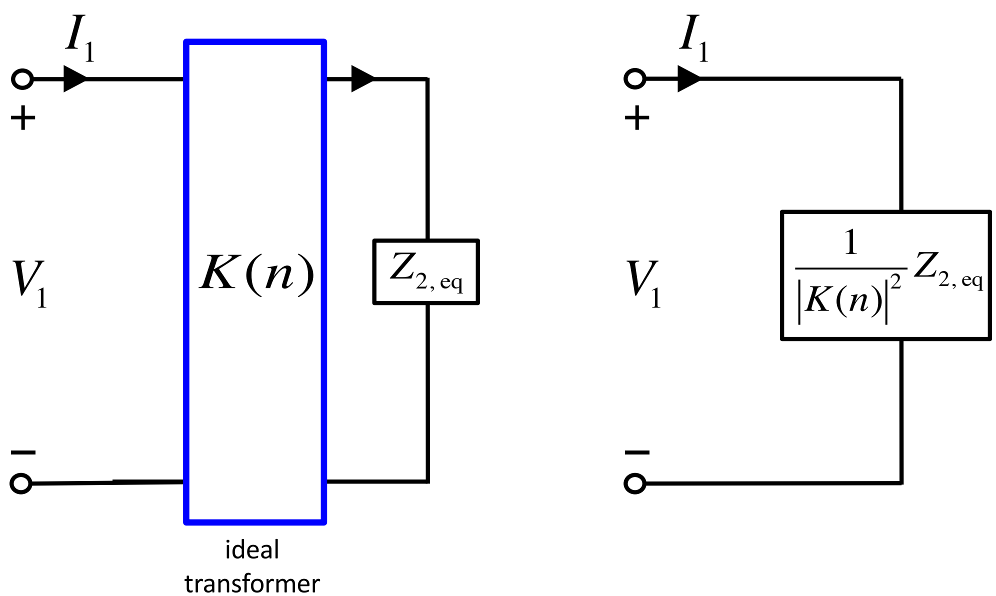
Thevenin equivalent



What if circuits contain both impedance and transformers ?

Driving-point impedance

Referring impedance from secondary to primary

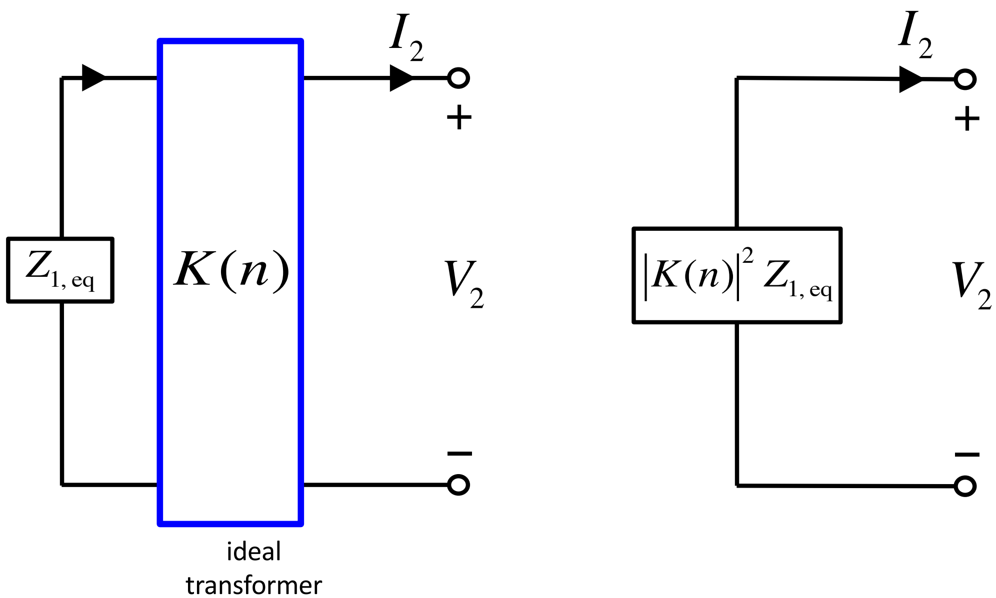


Both circuits have same driving-point impedance V_1 / I_1 on primary side

- Can verify using Kirchhoff's and Ohm's laws

Driving-point impedance

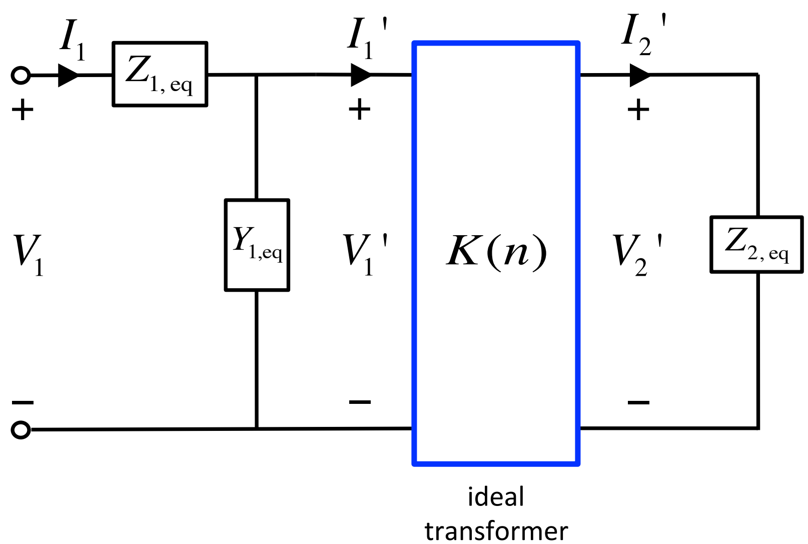
Referring impedance from primary to secondary



Both circuits have same driving-point impedance V_2/I_2 on secondary side

- Can verify using Kirchhoff's and Ohm's laws

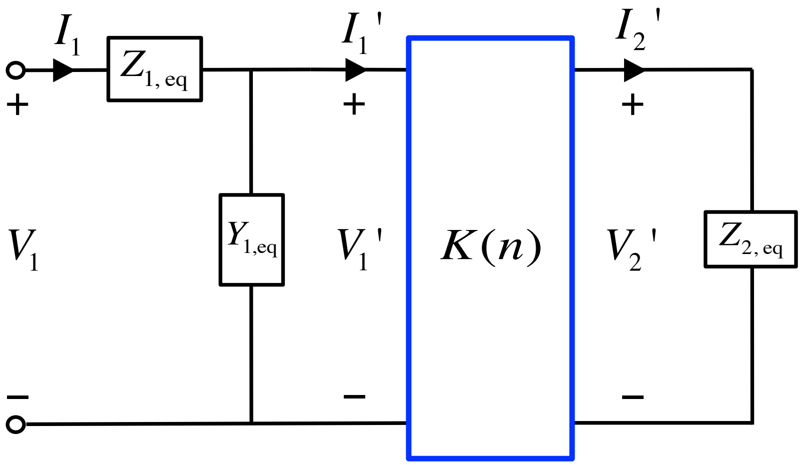
Driving-point impedance
Example

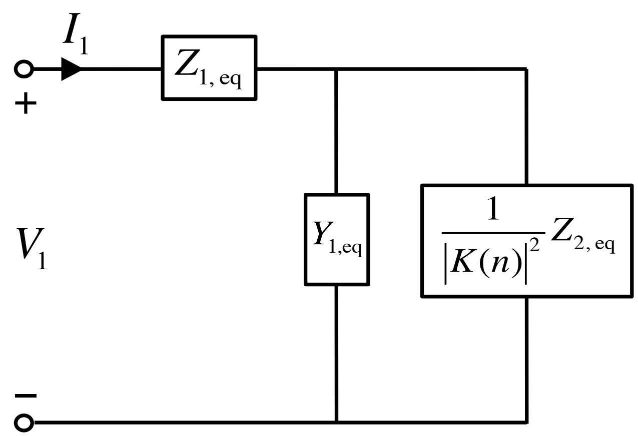


To find V_1/I_1, can analyze using Kirchhoff's and Ohm's laws

Driving-point impedance

Example

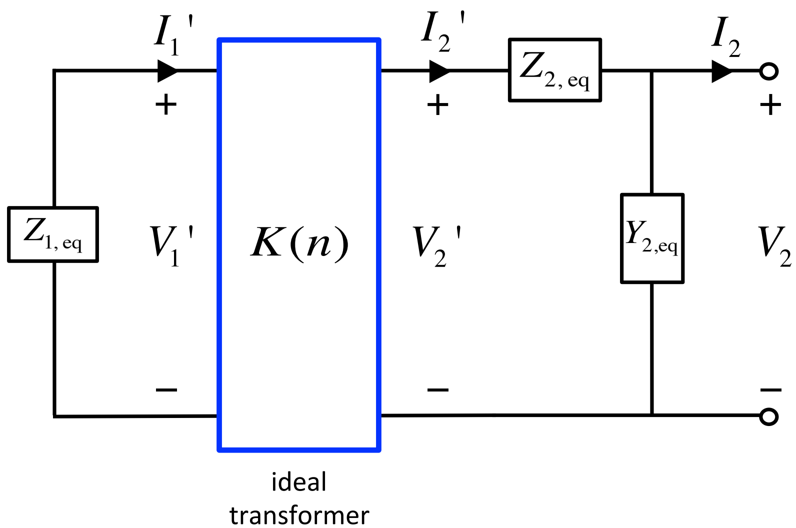




$$\frac{V_1}{I_1} = Z_{1,\text{eq}} + \left(Y_{1,\text{eq}} + \frac{1}{Z_{2,\text{eq}}/|K(n)|^2} \right)^{-1}$$

Driving-point impedance

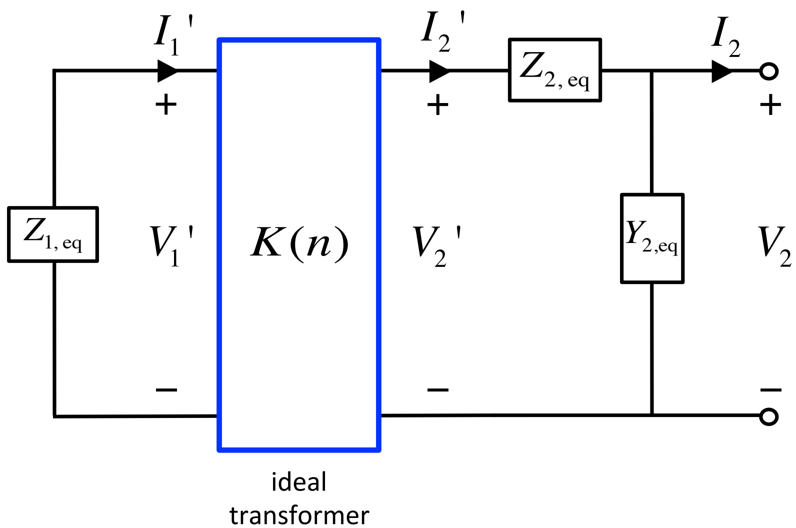
Example

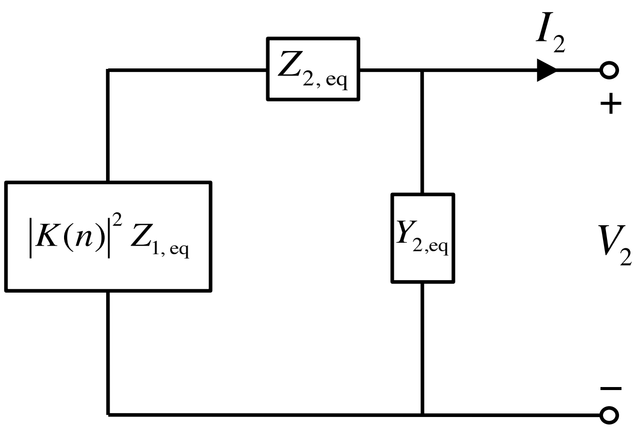


To find V_2/I_2, can analyze using Kirchhoff's and Ohm's laws

Driving-point impedance

Example





$$\frac{V_2}{I_2} = \left(Y_{2,\text{eq}} + \frac{1}{Z_{2,\text{eq}} + |K(n)|^2 \cdot Z_{1,\text{eq}}} \right)^{-1}$$

Driving-point impedance

Reference from one circuit to the other is not always applicable

- Example: circuits containing parallel paths (see example later)
- Generally applicable in a radial network without parallel paths
- Can always analyze original circuit using Kirchhoff's and Ohm's laws

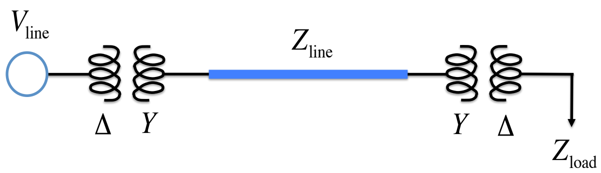
Outline

Per-phase analysis
Procedure

1. Convert all sources and loads in Δ configurations into their Y equivalents
2. Convert all ideal transformers in Δ configurations into their Y equivalents
3. Obtain phase a equivalent circuit by connecting all neutrals
4. Solve for desired phase-a variables
   - Use Thevenin equivalent of series impedances and shunt admittances in networks containing transformers whenever applicable, e.g., for a radial network
5. Obtain variables for phases b and c by subtracting $120°$ and $240°$ from phase a variables (positive sequence sources)
   - If variables in the internal of Δ configurations are desired, derive them from original circuits

Per-phase analysis

Example

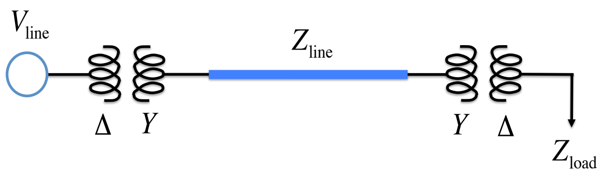


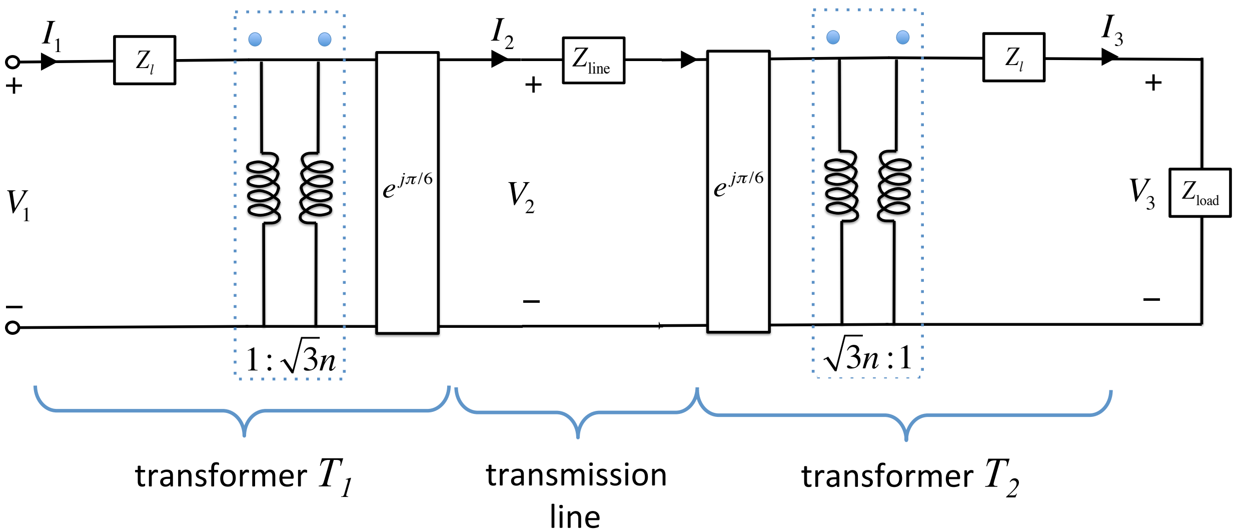
Balanced 3ϕ system

- Generator with line voltage V_{line}
- Step-up ΔY transformer
- Transmission line with series impedance Z_{line}
- Step-down ΔY transformer (primary on right)
- Load with impedance Z_{load}
- Single-phase transformer with voltage gain n and series impedance $3Z_l$ on primary side

Per-phase analysis

Example





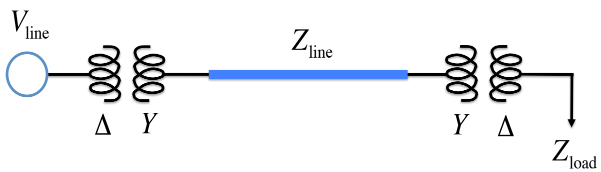
Balanced 3ϕ system

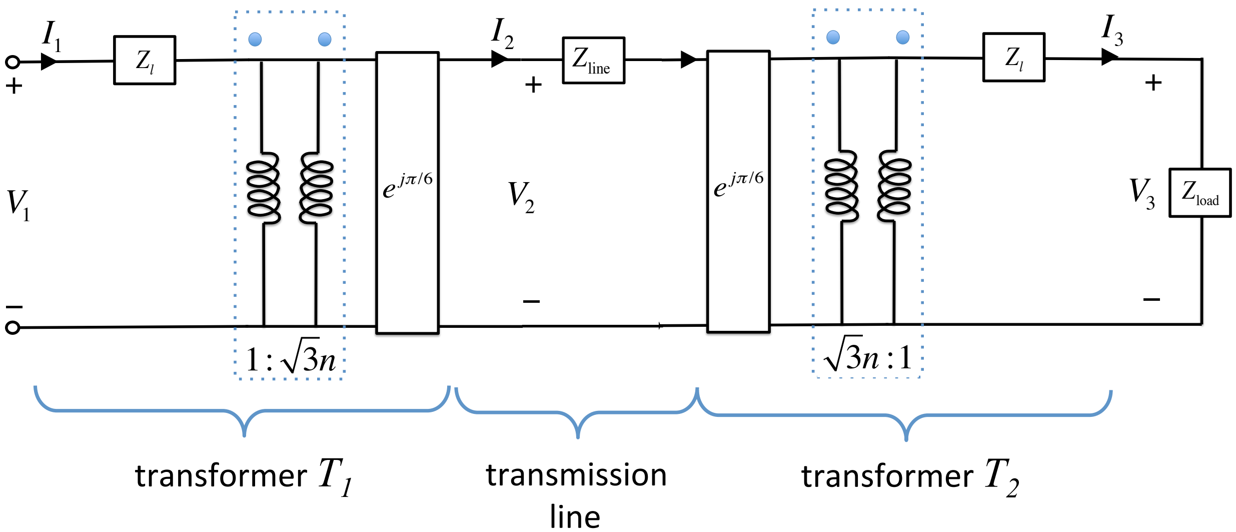
- Generator with line voltage V_{line}
- Step-up ΔY transformer
- Transmission line with series impedance Z_{line}
- Step-down ΔY transformer (primary on right)
- Load with impedance Z_{load}
- Single-phase transformer with turns ratio n and series impedance $3Z_l$ on primary side

$$V_1 = \frac{V_{\text{line}}}{\sqrt{3}\, e^{i\pi/6}} \qquad Z^Y = Z_l$$

Per-phase analysis

Example





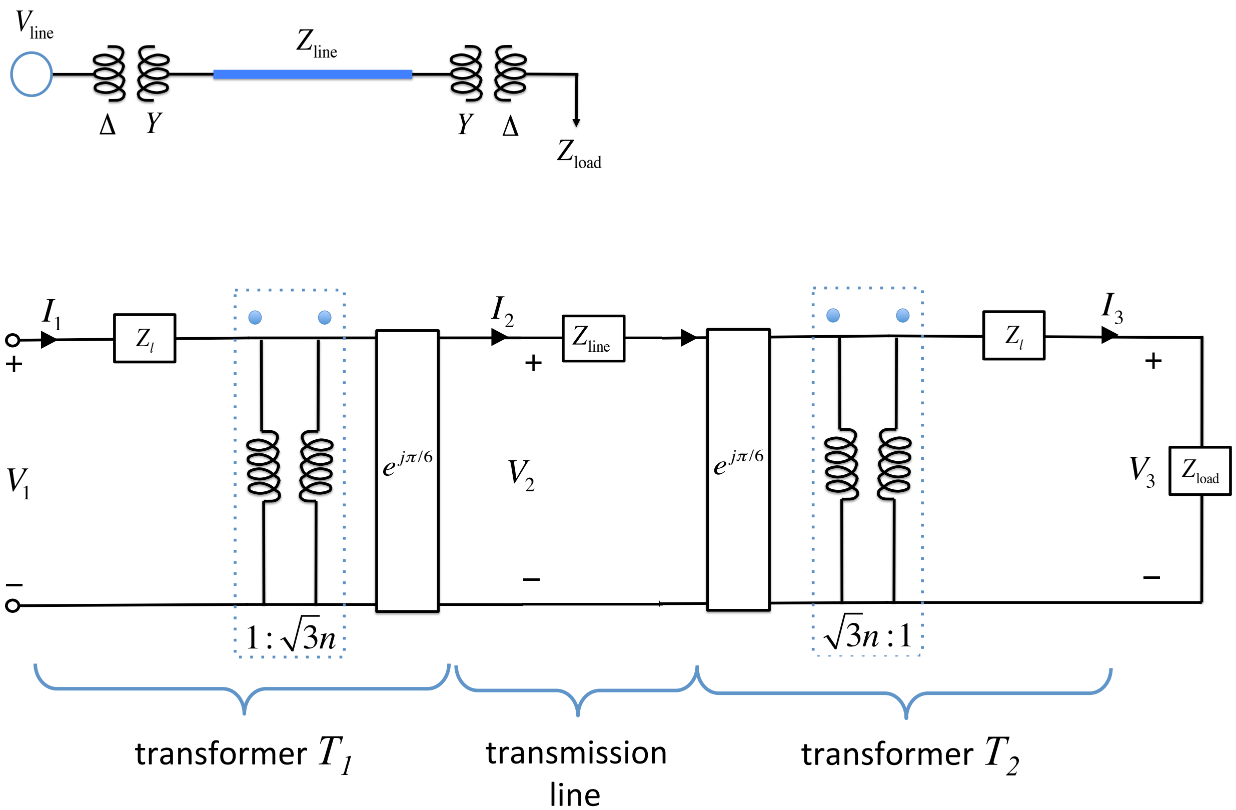
Calculate

- Generator current I_1
- Transmission line current I_2
- Load current I_3
- Load voltage V_3
- Power delivered to load: $V_3 I_3^*$

$$V_1 = \frac{V_{\text{line}}}{\sqrt{3}\, e^{i\pi/6}} \qquad Z^Y = Z_l$$

Per-phase analysis

Example



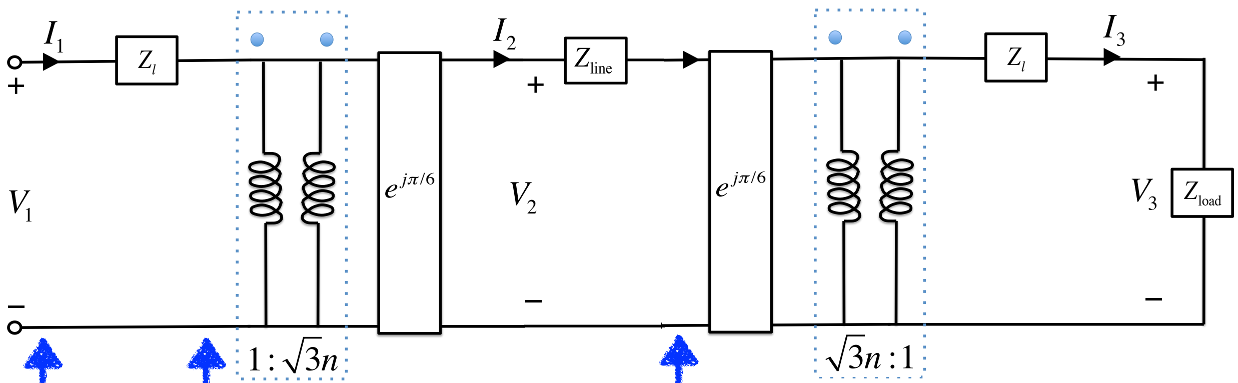
Solution strategy

- Refer all impedances to primary side of step-up transformer
- Derive driving-point impedance V_1/I_1
- Derive generator current I_1
- Propagate calculation towards load

$$V_1 = \frac{V_{\text{line}}}{\sqrt{3}\, e^{i\pi/6}} \qquad Z^Y = Z_l$$

Per-phase analysis

Example



$$|K(n)|^2 \, (Z_{\text{load}} + Z_l)$$

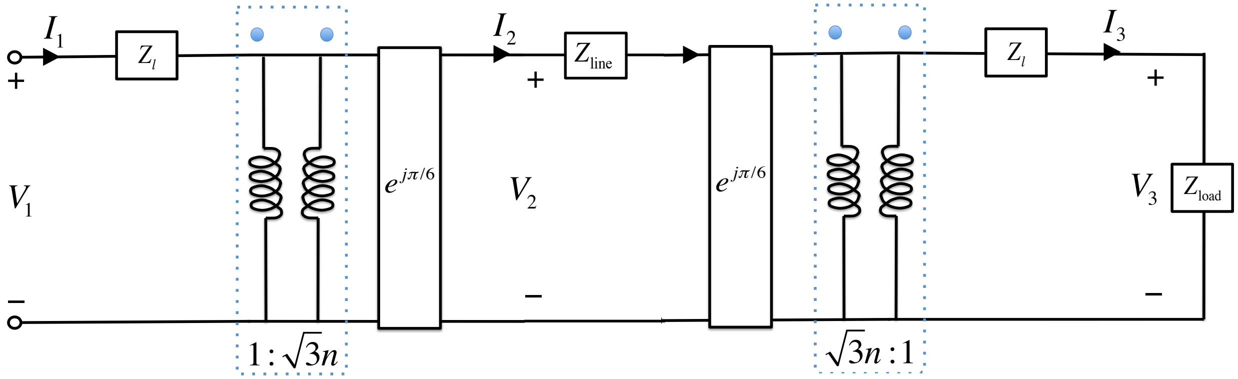
$$\frac{1}{|K(n)|^2} \cdot \left(Z_{\text{line}} + |K(n)|^2 \, (Z_{\text{load}} + Z_l) \right)$$

$$\frac{V_1}{I_1} = 2Z_l + \frac{Z_{\text{line}}}{|K(n)|^2} + Z_{\text{load}}$$

transformer gains on Z_{load} is canceled

Per-phase analysis

Example



$$I_1 = \frac{V_{\text{line}} \Big/ \left(\sqrt{3} e^{i\pi/6}\right)}{2 Z_l + \frac{Z_{\text{line}}}{|K(n)|^2} + Z_{\text{load}}}$$

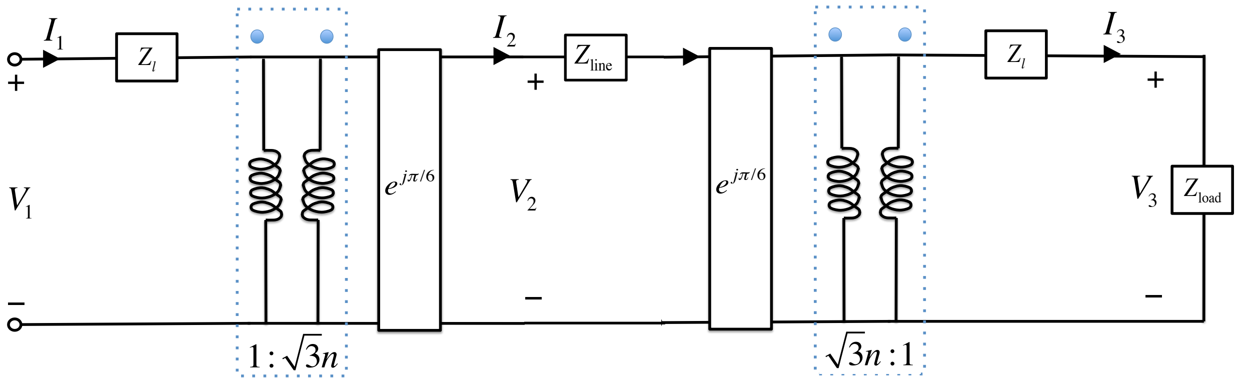
$$I_3 = K^*(n)\, I_2 = I_1$$

$$V_3 = Z_{\text{load}}\, I_3 = Z_{\text{load}}\, I_1$$

$$I_2 = \frac{I_1}{K^*(n)}$$

Per-phase analysis

Example



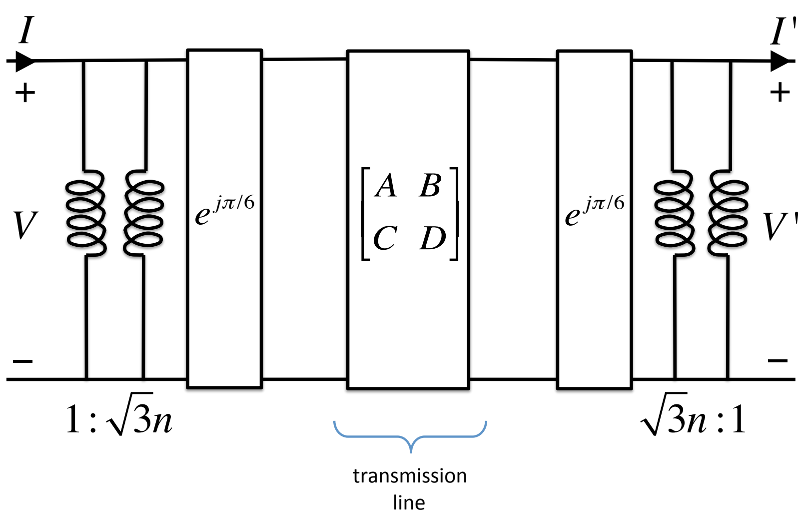
$$I_1 \;=\; \frac{V_{\text{line}} \Big/ \left(\sqrt{3}e^{i\pi/6}\right)}{2\,Z_l + \frac{Z_{\text{line}}}{|K(n)|^2} + Z_{\text{load}}}$$

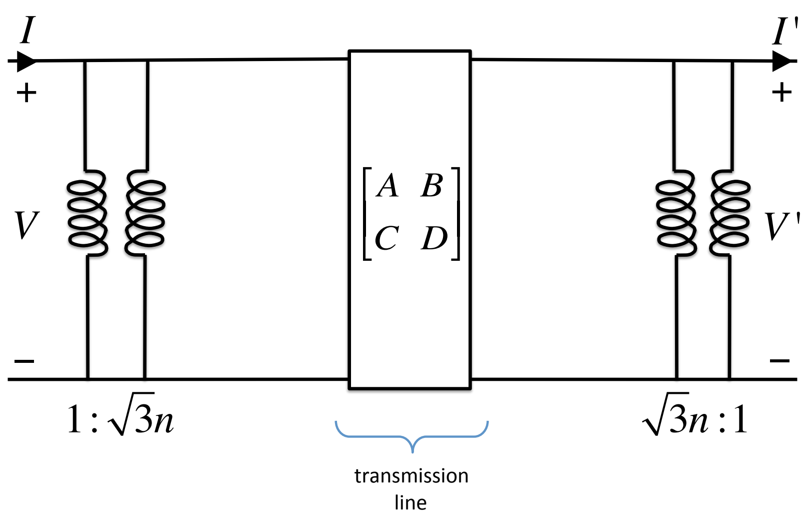
$$I_3 \;=\; I_1$$

$$V_3 \;=\; Z_{\text{load}}\, I_1$$

- External behavior does not depend on connection-induced phase shift $e^{i\pi/6}$
- Only internal variables I_{line} does

Simplified model for terminal behavior





Terminal behavior does not depend on $e^{i\pi/6}$

- The simplified model has the same transmission matrix

Normal systems

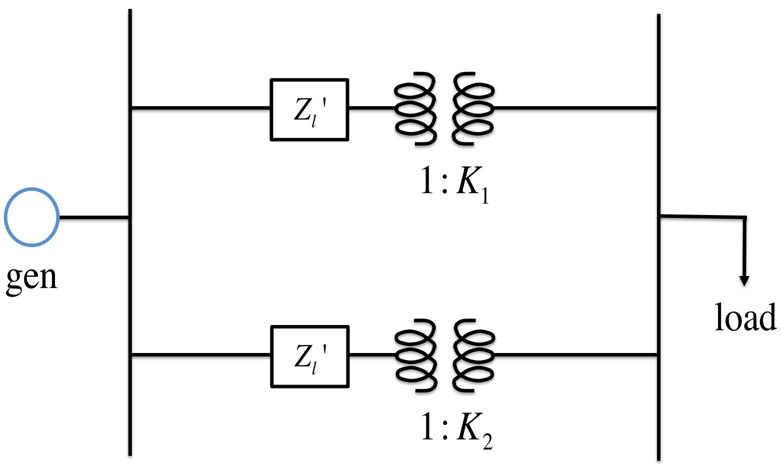
A system is normal if, in its per-phase circuit, the product of complex ideal transformer gains around every loop is 1

Equivalently, on each parallel path,

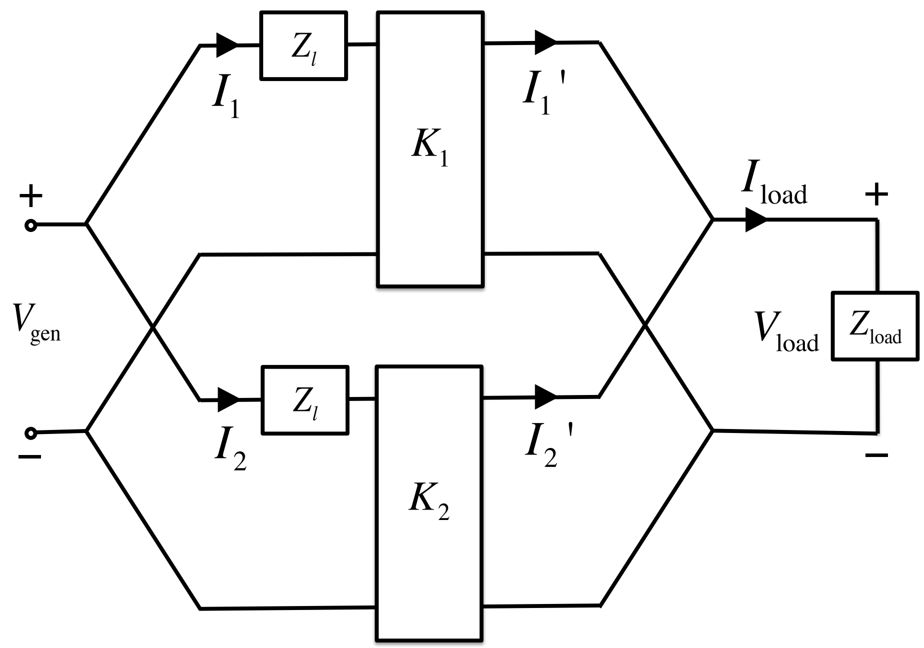
1. Product of ideal transformer gain magnitudes is the same, and
2. Sum of ideal transformer phase shifts is the same

Normal systems

Example



Generator & load connected by two 3ϕ transformers in parallel (forming a loop)



Per-phase circuit

Normal systems

Example

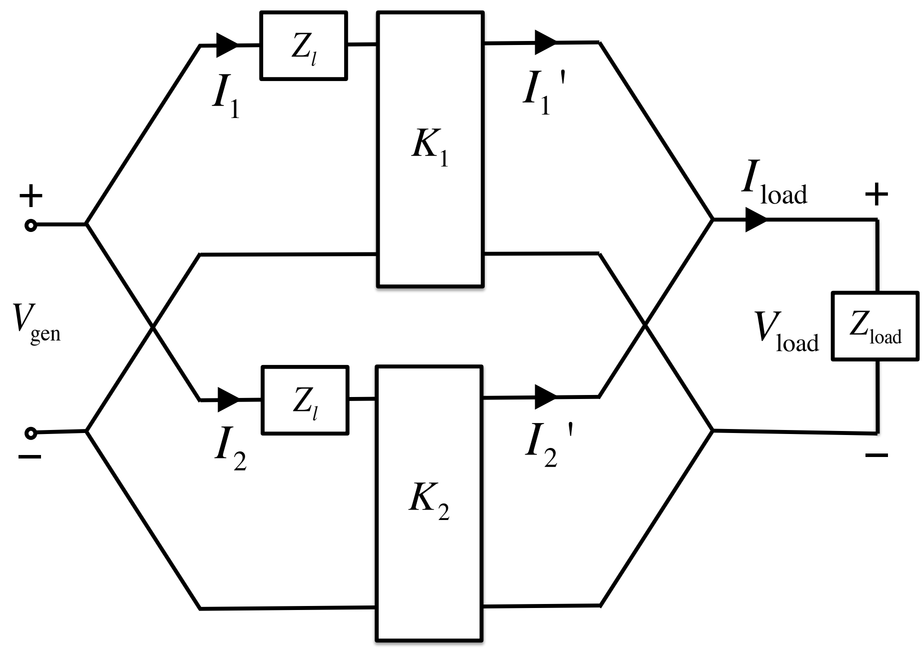
Calculate

- Load current I_{load}
- Line currents I_1', I_2'

in terms of $V_{\text{gen}}, Z_l, Z_{\text{load}}$

Implications when

- $K_2 = K_1$ (normal system)
- $K_2 = K_1 \, e^{i\theta}$
- $K_2 = k \cdot K_1$



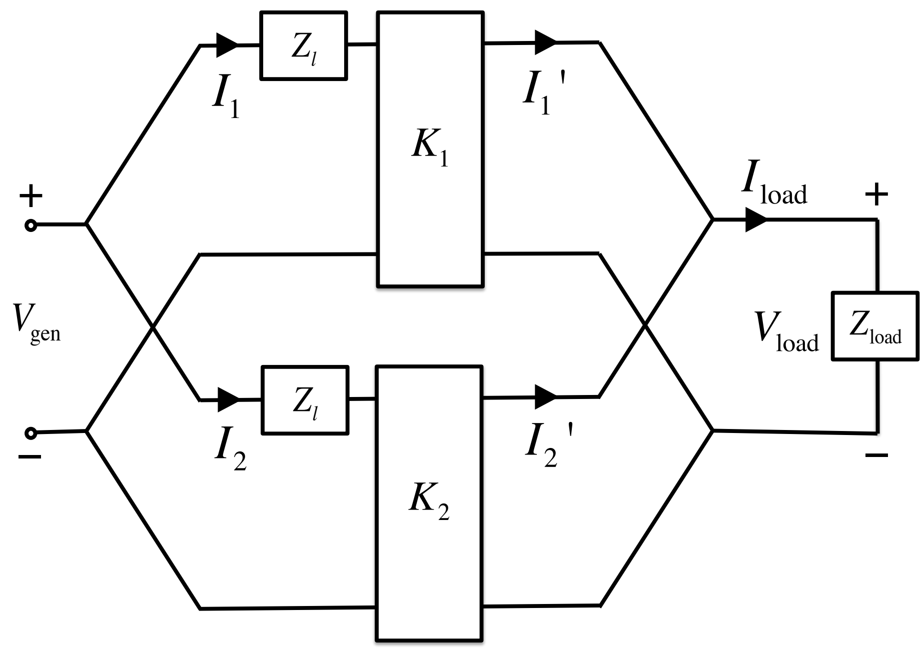
Per-phase circuit

Normal systems

Example

$K_2 = K_1$ (normal system):

- $I_1' = I_2'$
- $\frac{I_{\text{load}}}{I_1'} = \frac{I_{\text{load}}}{I_2'} = 2$



Per-phase circuit

Normal systems

Example

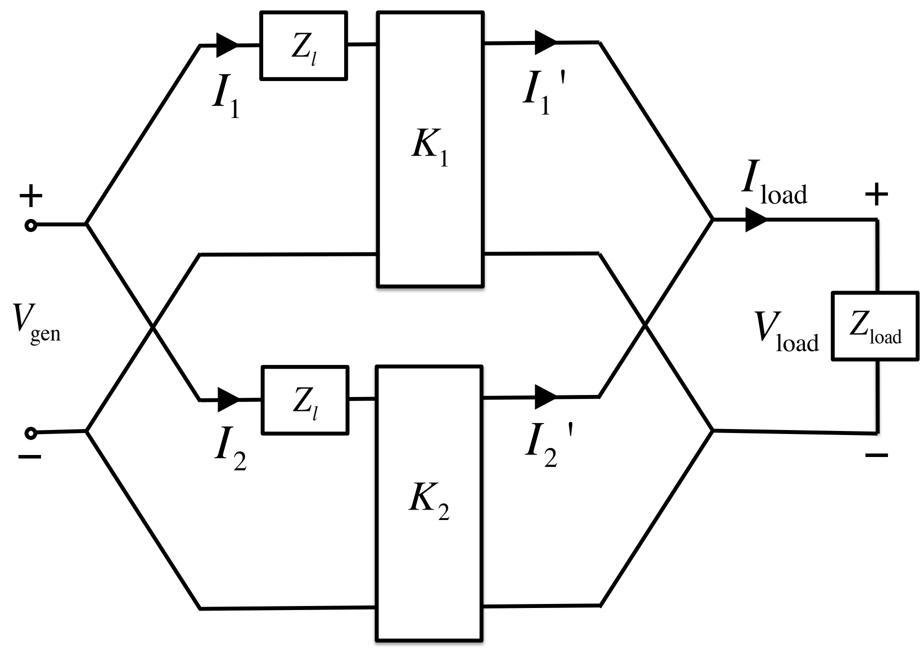
$K_2 = K_1 e^{i\theta}$:

- $I_1' \neq I_2'$
- $\dfrac{\left|I_{\text{load}}\right|}{|I_1'|} = \dfrac{\left|1+e^{i\theta}\right|}{|\alpha_1|}, \quad \dfrac{\left|I_{\text{load}}\right|}{|I_2'|} = \dfrac{\left|1+e^{i\theta}\right|}{|\alpha_2|}$

Example: $K_2 = K_1 e^{i\pi/6}$:

- $\dfrac{\left|I_{\text{load}}\right|}{|I_1'|} = 20.6\,\%, \quad \dfrac{\left|I_{\text{load}}\right|}{|I_2'|} = 17.1\,\%$

Most current loops between transformers without entering load



Per-phase circuit

Normal systems

Example

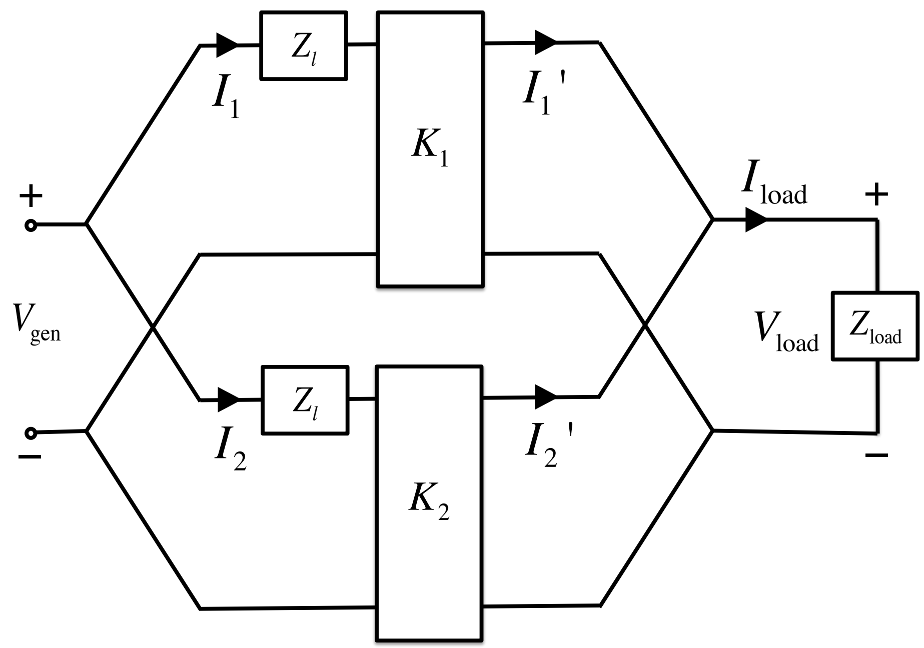
$K_2 = K_1\, e^{i\theta}$:

- $I_1' \neq I_2'$
- $\dfrac{\left| I_{\text{load}} \right|}{|\,I_1'\,|} = \dfrac{\left| 1 + e^{i\theta} \right|}{|\,\alpha_1\,|}, \quad \dfrac{\left| I_{\text{load}} \right|}{|\,I_2'\,|} = \dfrac{\left| 1 + e^{i\theta} \right|}{|\,\alpha_2\,|}$

Example: $K_2 = K_1\, e^{i\pi/6}$:

- $S_{\text{gen}} = 183\,\angle 71°, \quad S_{\text{load}} = 60\,\angle 0°$ MVA

Most current loops between transformers without entering load



Per-phase circuit

Normal systems

Example

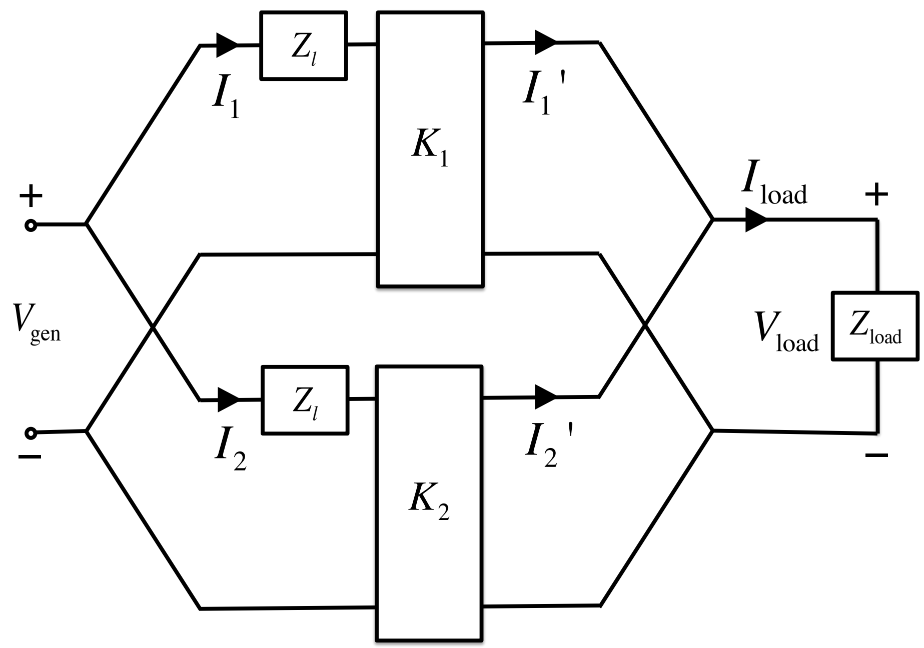
$K_2 = k \cdot K_1$:

- $I_1' \neq I_2'$
- $\dfrac{|I_{\text{load}}|}{|I_1'|} = \dfrac{1+k^{-1}}{|\alpha_1|}, \quad \dfrac{|I_{\text{load}}|}{|I_2'|} = \dfrac{1+k}{|\alpha_2|}$

Example: $K_2 = 2K_1$:

- $\dfrac{|I_{\text{load}}|}{|I_1'|} = 29.4\,\%, \quad \dfrac{|I_{\text{load}}|}{|I_2'|} = 29.9\,\%$

Most current loops between transformers without entering load



Per-phase circuit

Outline

1. Single-phase transformer
2. Three-phase transformer
3. Equivalent impedance
4. Per-phase analysis
5. Per-unit normalization
    - Kirchhoff's and Ohm's laws
    - Across ideal transformer
    - 3ϕ quantities
    - Per-unit per-phase analysis

Per-unit normalization

- Quantities of interest: voltages V, currents I, power S, impedances Z
- $$\text{quantity in p.u.} = \frac{\text{actual quantity}}{\text{base value of quantity}}$$
- Base values
  - Real positive values
  - Same units as actual quantities
- Choose base values to satisfy same physical laws
  - Kirchhoff's and Ohm's laws
  - Across ideal transformer
  - Relationship between 3ϕ and 1ϕ quantities

Per-unit normalization

General procedure

1. Choose voltage base value V_{1B} for (say) area 1
2. Choose power base value S_B for entire network
3. Calculate all other base values from physical laws

Example: Choose

1. V_{1B} = nominal voltage magnitude of area 1
2. S_B = rated apparent power of a transformer in area 1

How to calculate the other base values $\left(V_{iB}, I_{iB}, Z_{iB}\right)$?

- Consider single-phase or per-phase circuit of balanced 3ϕ system

Kirchhoff's and Ohm's laws

Given base values $\left(V_{1B}, S_B\right)$, within area 1:

$$I_{1B} := \frac{S_B}{V_{1B}} \quad A, \qquad Z_{1B} := \frac{V_{1B}^2}{S_B} \quad \Omega$$

Then: physical laws are satisfied by both the base values and p.u. quantities

$$\begin{aligned} V_{1B} &= Z_{1B} I_{1B}, & V_{1\text{pu}} &= Z_{1\text{pu}} I_{1\text{pu}} \\ S_B &= V_{1B} I_{1B}, & S_{1\text{pu}} &= V_{1\text{pu}} I_{1\text{pu}} \end{aligned}$$

Can perform circuit analysis using pu quantities instead of actual quantities

Kirchhoff's and Ohm's laws

Other quantities

These quantities $\left(V_{1B}, S_B, I_{1B}, Z_{1B}\right)$ serve as base values for other quantities within area 1, with appropriate units

- S_B is base value for real power in W, reactive power in var

$$P_{1\text{pu}} \ := \ \frac{P_1}{S_B}, \qquad Q_{1\text{pu}} \ := \ \frac{Q_1}{S_B}, \qquad S_{1\text{pu}} = P_{1\text{pu}} + iQ_{1\text{pu}}$$

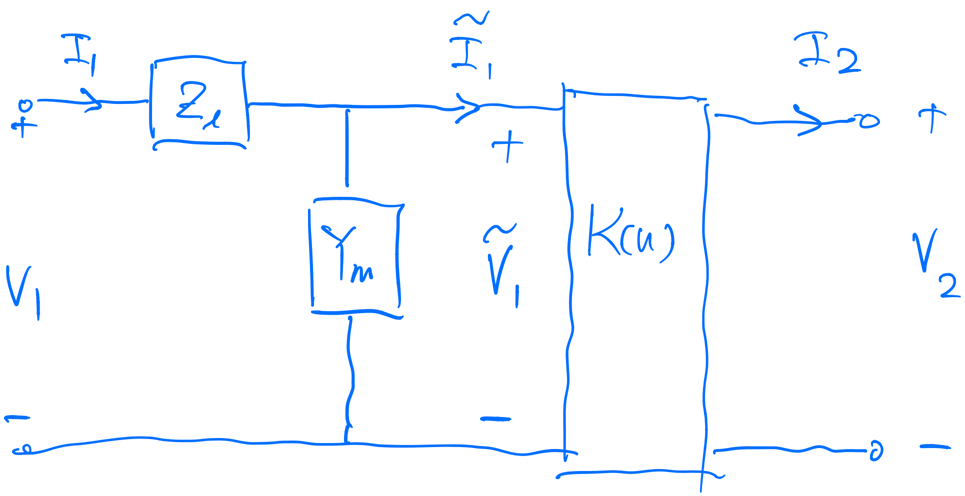
- Z_{1B} is base value for resistances & reactances in Ω

$$R_{1\text{pu}} \ := \ \frac{R_1}{Z_{1B}}, \qquad X_{1\text{pu}} \ := \ \frac{X_1}{Z_{1B}}, \qquad Z_{1\text{pu}} = R_{1\text{pu}} + iX_{1\text{pu}}$$

- $Y_{1B} := 1/Z_{1B}$ in $\Omega^{-1}$is base value for conductances, susceptances, & admittances

$$G_{1\text{pu}} \ := \ \frac{G_1}{Y_{1B}}, \qquad B_{1\text{pu}} \ := \ \frac{B_1}{Y_{1B}}, \qquad Y_{1\text{pu}} = G_{1\text{pu}} + iB_{1\text{pu}} \ = \ \frac{1}{Z_{1\text{pu}}}$$

Across ideal transformer



area 1 area 2

Choose $\left(V_{2B}, I_{2B}, Z_{2B}\right)$ according to

$$V_{2B} := \ |K(n)|\, V_{1B} \quad V$$

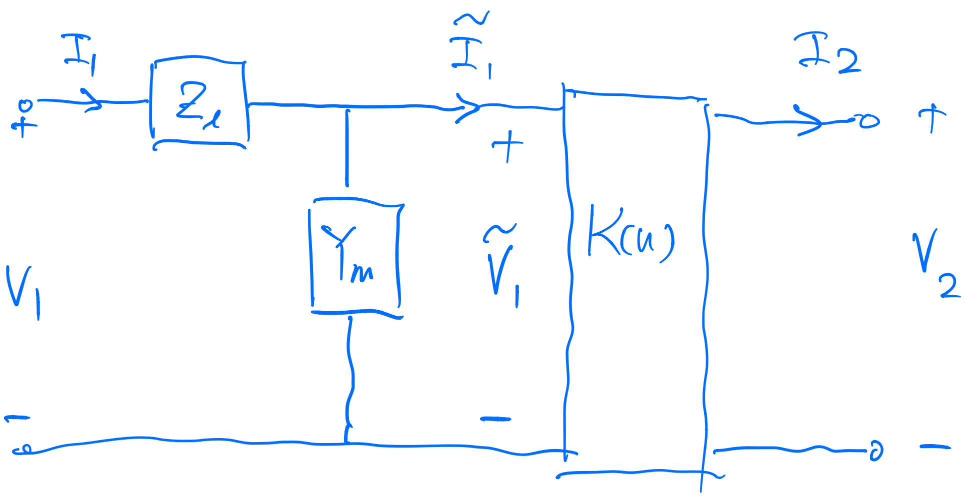
$$I_{2B} := \ \frac{I_{1B}}{|K(n)|} \quad A$$

$$Z_{2B} := \ |K(n)|^2\, Z_{1B} \quad \Omega$$

Base values remain real positive

S_B remains base value for power

Across ideal transformer



External behavior

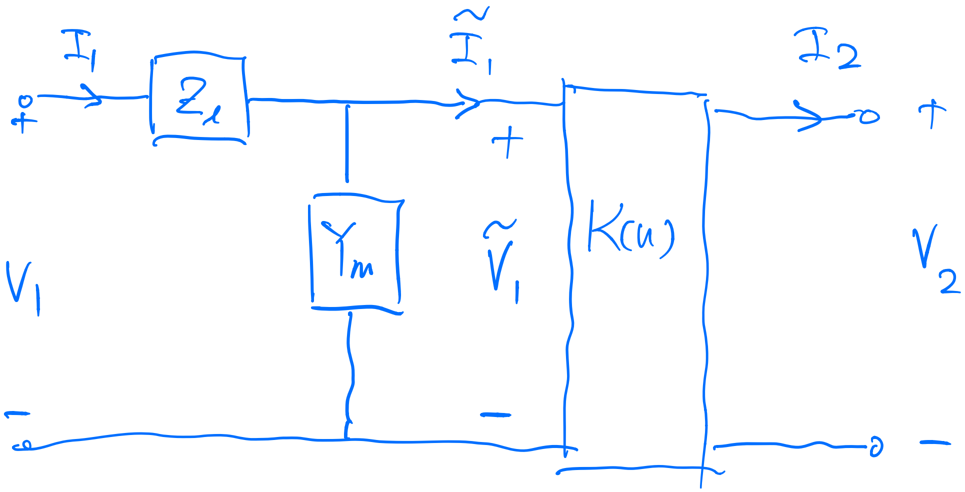
$$\tilde{V}_{1\text{pu}} = \frac{\tilde{V}_1}{V_{1B}} = \frac{V_2}{K(n)} \frac{|K(n)|}{V_{2B}} = V_{2\text{pu}}\, e^{-j\angle K(n)}$$

$$\tilde{I}_{1\text{pu}} = \frac{\tilde{I}_1}{\tilde{I}_{1B}} = \frac{K^*(n) I_2}{|K(n)|\, I_{2B}} = I_{2\text{pu}}\, e^{-j\angle K(n)}$$

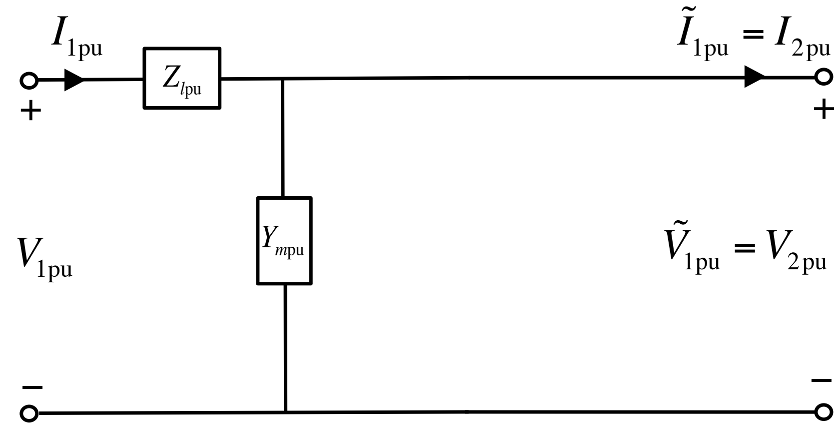
If $\angle K(n) = 0$ then

$$\tilde{V}_{1\text{pu}} = V_{2\text{pu}}, \quad \tilde{I}_{1\text{pu}} = I_{2\text{pu}}$$

Across ideal transformer



area 1 area 2



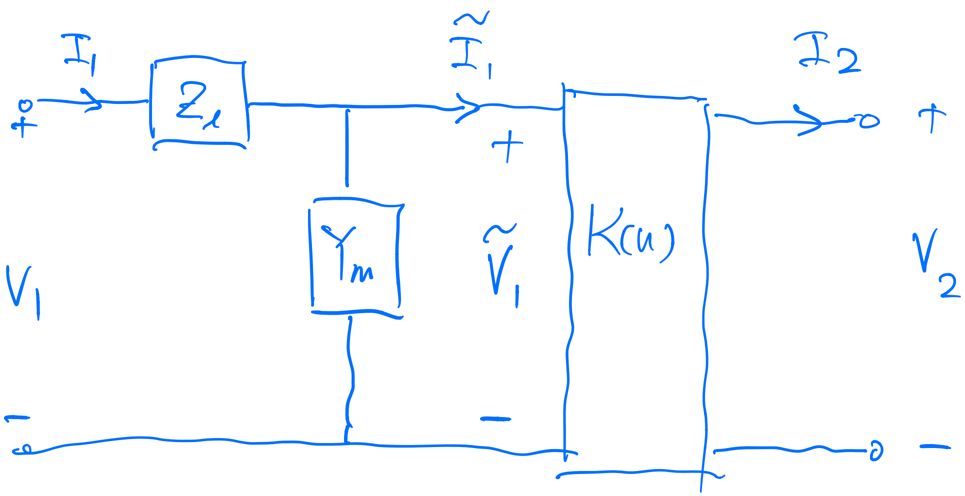
per-unit circuit

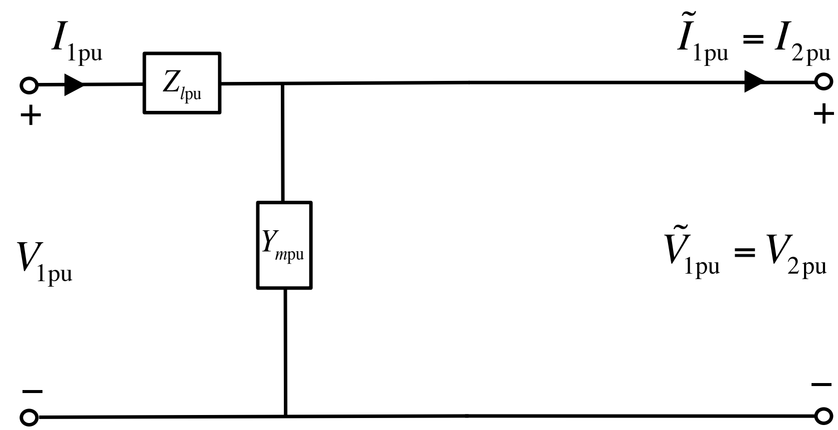
If $\angle K(n) = 0$ then

$$\tilde{V}_{1\text{pu}} = V_{2\text{pu}}, \quad \tilde{I}_{1\text{pu}} = I_{2\text{pu}}$$

Ideal transformer has disappeared !

Across ideal transformer



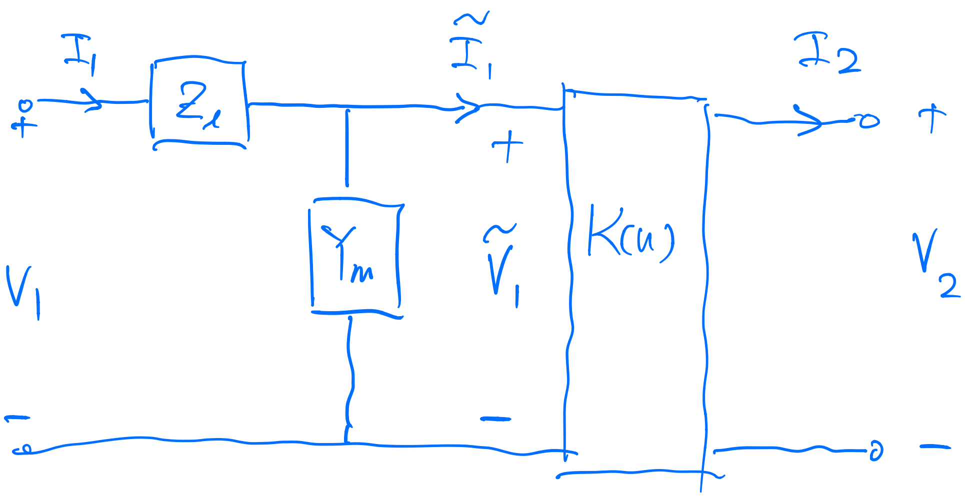


per-unit circuit

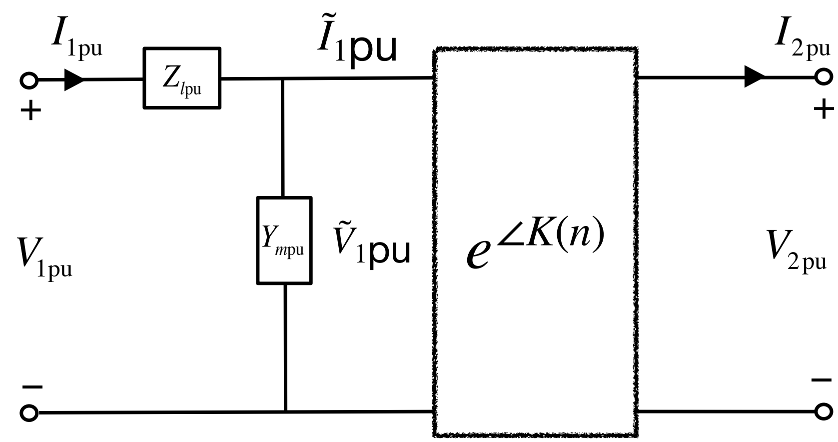
$\angle K(n) = 0$ if

- 1ϕ or balanced 3ϕ in YY or $\Delta\Delta$
- Normal systems where connection-induced phase shifts can be ignored

Across ideal transformer



area 1 area 2



per-unit circuit

Otherwise

- pu circuit contains an off-nominal phase-shifting transformer

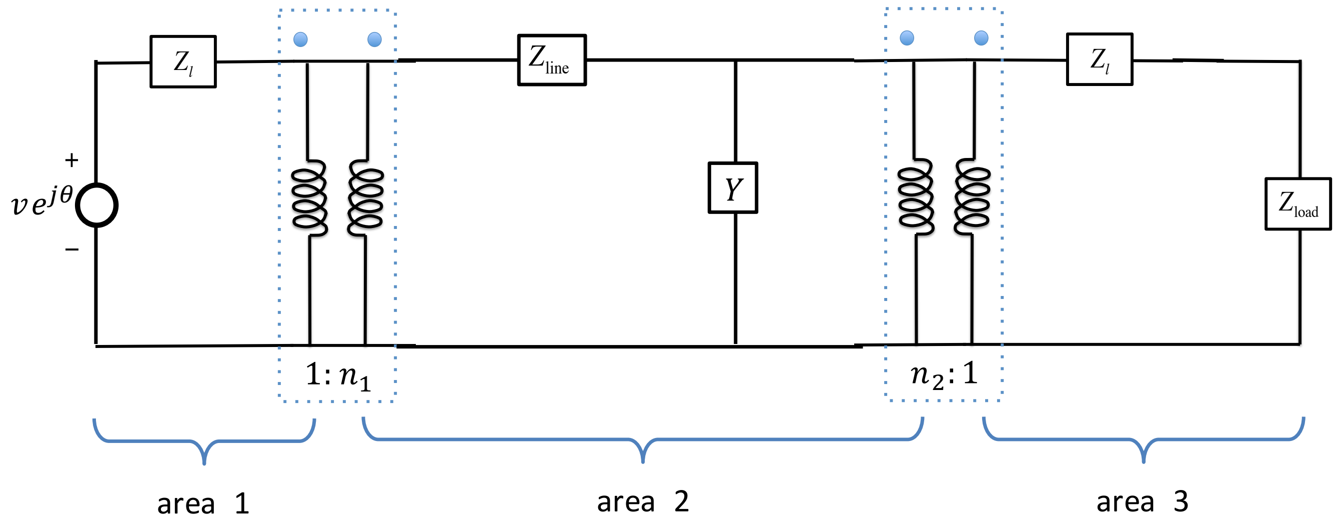
Across ideal transformer

Example

Given nameplate rating of generator

- Voltage v V
- Apparent power s VA

Calculate base values



Voltage base $V_{1B} := v$, power base $S_B := s$

- Area 1: $I_{1B} := s/v, \;\; Z_{1B} := v^2/s$
- Area 2: $V_{2B} := n_1 v, \; I_{2B} := s/(n_1 v), \; Z_{2B} := (n_1 v)^2/s, \; Y_{2B} := s/(v_1 v)^2$
- Area 3: $V_{3B} := n_1 v/n_2, \; I_{3B} := n_2 s/(n_1 v), \; Z_{3B} := (n_1 v)^2/(n_2^2 s), \; Y_{3B} := (n_2^2 s)/(v_1 v)^2$

3ϕ quantities

Given 1ϕ devices (generators, lines, loads) with

- with 1ϕ quantities $\left(S^{1\phi}, V^{1\phi}, I^{1\phi}, Z^{1\phi}\right)$
- and their base values

Construct balanced 3ϕ devices from these 1ϕ devices

- What are 3ϕ quantities of interest?
- What are base values so that 3ϕ quantities equal to 1ϕ quantities in p.u.?

Base values should satisfy the same 3ϕ relationships as actual quantities
Values depend on the configuration, Y or Δ

3ϕ quantities
Y configuration

In terms of

$\left(S^{1\phi}, V^{1\phi}, I^{1\phi}, Z^{1\phi}\right)$

and their base values

- 3ϕ power (total power to/from 3 1ϕ devices):

$$S^{3\phi} = 3S^{1\phi},$$

- Line-to-line voltage

$$V^{\mathsf{ll}} = \sqrt{3}e^{i\pi/6}\, V^{\mathsf{ln}},$$

- Line current

$$I^{3\phi} = I_{an} = I^{1\phi},$$

- Line-to-neutral voltage

$$V^{\mathsf{ln}} = V^{1\phi},$$

- Impedance

$$Z^{3\phi} = Z^{1\phi},$$

3ϕ quantities
Y configuration

In terms of

$\left(S^{1\phi}, V^{1\phi}, I^{1\phi}, Z^{1\phi}\right)$

and their base values

- 3ϕ power (total power to/from 3 1ϕ devices):

$$S^{3\phi} = 3S^{1\phi}, \qquad S_B^{3\phi} = 3S_B^{1\phi}$$

- Line-to-line voltage

$$V^{\text{ll}} = \sqrt{3}e^{i\pi/6}\,V^{\text{ln}}, \qquad V_B^{\text{ll}} = \sqrt{3}\,V_B^{\text{ln}}$$

- Line current

$$I^{3\phi} = I_{an} = I^{1\phi}, \qquad I_B^{3\phi} = I_B^{1\phi}$$

- Line-to-neutral voltage

$$V^{\text{ln}} = V^{1\phi}, \qquad V_B^{\text{ln}} = V_B^{1\phi}$$

- Impedance

$$Z^{3\phi} = Z^{1\phi}, \qquad Z_B^{3\phi} = Z_B^{1\phi}$$

Calculation

Base values satisfy the same relationship

3ϕ quantities

Δ configuration

In terms of

$\left(S^{1\phi}, V^{1\phi}, I^{1\phi}, Z^{1\phi}\right)$

and their base values

- 3ϕ power (total power to/from 3 1ϕ devices):

$$S^{3\phi} = 3S^{1\phi},$$

- Line-to-line voltage

$$V^{\text{ll}} = \sqrt{3}e^{i\pi/6}\, V^{\text{ln}},$$

- Line current

$$I^{3\phi} = I_{ab} - I_{ca} = \sqrt{3}\; e^{-i\pi/6}\, I^{1\phi},$$

- Line-to-neutral voltage

$$V^{\text{ln}} = \left(\sqrt{3}\; e^{i\pi/6}\right)^{-1} V^{1\phi},$$

- Impedance

$$Z^{3\phi} = Z^{1\phi}/3,$$

Note:

V^{ln}, $Z^{3\phi}$ are voltage and & impedance in Y equivalent circuit

3ϕ quantities

Δ configuration

In terms of

$\left(S^{1\phi}, V^{1\phi}, I^{1\phi}, Z^{1\phi}\right)$

and their base values

- 3ϕ power (total power to/from 3 1ϕ devices):

$$S^{3\phi} = 3S^{1\phi}, \qquad S_B^{3\phi} = 3S_B^{1\phi}$$

- Line-to-line voltage

$$V^{\text{ll}} = \sqrt{3}e^{i\pi/6}\,V^{\text{ln}}, \qquad V_B^{\text{ll}} = \sqrt{3}\,V_B^{\text{ln}}$$

- Line current

$$I^{3\phi} = I_{ab} - I_{ca} = \sqrt{3}\,e^{-i\pi/6}\,I^{1\phi}, \qquad I_B^{3\phi} = \sqrt{3}\,I_B^{1\phi}$$

- Line-to-neutral voltage

$$V^{\text{ln}} = \left(\sqrt{3}\,e^{i\pi/6}\right)^{-1} V^{1\phi}, \qquad V_B^{\text{ln}} = (\sqrt{3})^{-1}V_B^{1\phi}$$

- Impedance

$$Z^{3\phi} = Z^{1\phi}/3, \qquad Z_B^{3\phi} = Z_B^{1\phi}/3$$

Note:

V^{ln}, $Z^{3\phi}$ are voltage and & impedance in Y equivalent circuit

Per-unit quantities

Per-unit quantities satisfy

$$S_{\text{pu}}^{3\phi} = S_{\text{pu}}^{1\phi}, \qquad V_{\text{pu}}^{\text{ll}} = V_{\text{pu}}^{\text{ln}}, \qquad Z_{\text{pu}}^{3\phi} = Z_{\text{pu}}^{1\phi}$$

$$\left| V_{\text{pu}}^{\text{ln}} \right| = \left| V_{\text{pu}}^{1\phi} \right|, \qquad \left| I_{\text{pu}}^{3\phi} \right| = \left| I_{\text{pu}}^{1\phi} \right|$$

- 3ϕ quantities equal 1ϕ quantities in p.u.
- modulo phase shifts in Δ configuration:

$$V_{\text{pu}}^{\text{ln}} := \frac{V^{\text{ln}}}{V_B^{\text{ln}}} = \frac{\left(\sqrt{3}e^{i\pi/6}\right)^{-1} V^{1\phi}}{\left(\sqrt{3}\right)^{-1} V_B^{1\phi}} = e^{-i\pi/6}\, V_{\text{pu}}^{1\phi}$$

Per-unit per-phase analysis

1. For single-phase system, pick power base $S_B^{1\phi}$ for entire system and voltage base $V_{1B}^{1\phi}$ in area 1, e.g., induced by nameplate ratings of transformer
2. For balanced 3ϕ system, pick 3ϕ power base $S_B^{3\phi}$ and line-to-line voltage base V_B^{ll} induced by nameplate ratings of 3ϕ transformer. Then choose power & voltage bases for per-phase equivalent circuit:

$$S_B^{1\phi} := S_B^{3\phi} \Big/ 3, \qquad V_{1B}^{1\phi} := V_{1B}^{ll} \Big/ \sqrt{3}$$

$S_{1B}^{1\phi}$ will be power base for entire per-phase circuit.

3. Calculate current and impedance bases in that area:

$$I_{1B} := \frac{S_B^{1\phi}}{V_{1B}^{1\phi}}, \qquad Z_{1B} := \frac{\left(V_{1B}^{1\phi}\right)^2}{S_B^{1\phi}}$$

Per-unit per-phase analysis

4. Calculate base values for voltages, currents, and impedances in areas i connected to area 1 using the magnitude n_i of transformer gains (assume area 1 is primary):

$$V_{iB}^{1\phi} := n_i V_{1B}^{1\phi}, \qquad V_{iB}^{||} := n_i V_{1B}^{||}, \qquad I_{iB} := \frac{1}{n_i} I_{1B}, \qquad Z_{iB} := n_i^2 Z_{1B}$$

Continue this process to calculate the voltage, current, and impedance base values for all areas

Per-unit per-phase analysis

5. For real, reactive, apparent power in entire system, use $S_B^{1\phi}$ as base value.

   For resistances and reactances, use Z_{iB} as base value in area i.

   For admittances, conductances, and susceptancesq, use $Y_{iB} := 1/Z_{iB}$ as base value in area i

6. Draw impedance diagram of entire system, and solve for desired per-unit quantities

7. Convert back to actual quantities if desired

Summary

1. Single-phase transformer
   - Ideal transformer gain n, equivalent circuit
2. Three-phase transformer
   - $YY, \Delta\Delta, \Delta Y, Y\Delta$: external behavior, YY equivalent
3. Equivalent impedance
   - Short cut for analyzing circuits containing transformers
   - Transmission matrix, driving-point impedance
4. Per-phase analysis
5. Per-unit normalization
   - Physical laws, across transformer, 3ϕ quantities, per-unit per-phase analysis