Power Systems Analysis

Chapter 9 Unbalanced network: BIM

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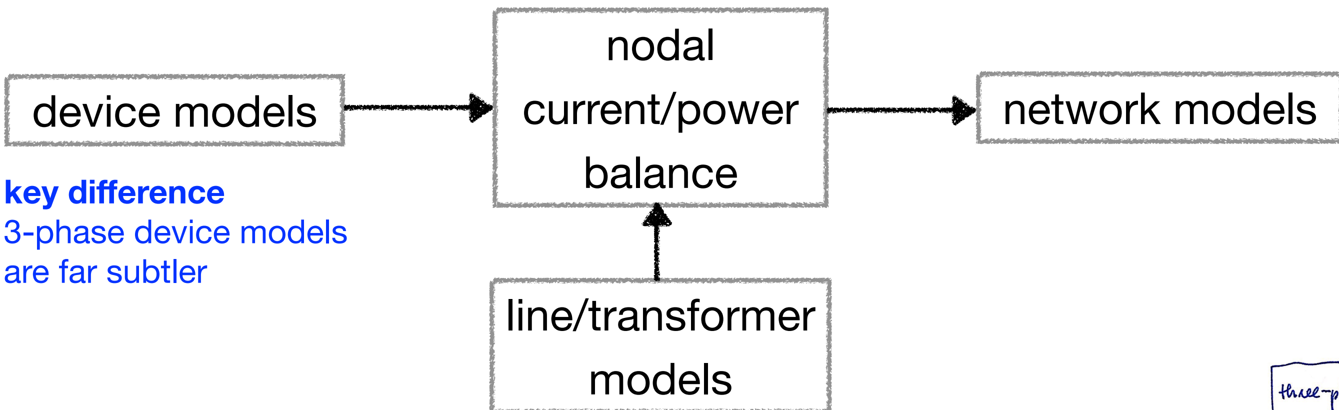
Outline

1. Network models
2. Three-phase analysis
3. Balanced network
4. Symmetric network

Outline

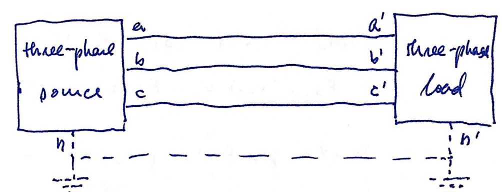
1. Network models: BIM
    - IV relation $(I = YV)$
    - sV relation (power flow equations)
    - Overall model (device + nodal balance)
2. Three-phase analysis
3. Balanced network
4. Symmetric network

Overview



key difference
3-phase device models
are far subtler

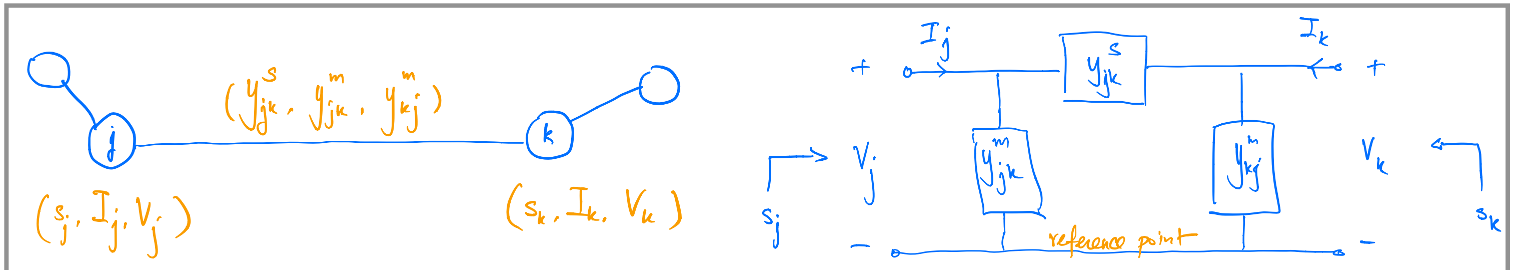
single-phase or 3-phase



Review: single-phase BIM

Network model

1. Network $G := (\overline{N}, E)$
   - $\overline{N} := \{0\} \cup N := \{0\} \cup \{1,\ldots,N\}$: buses/nodes
   - $E \subseteq \overline{N} \times \overline{N}$: lines/links/edges
2. Each line (j,k) is parameterized by $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right) \in \mathbb{C}^3$
   - y^s_{jk} : series admittance
   - y^m_{jk}, y^m_{kj} : shunt admittances, generally different



(a) Graph representation (b) Π equivalent circuit

Review: single-phase BIM

Admittance matrix $Y \in \mathbb{C}^{(N+1)\times(N+1)}$

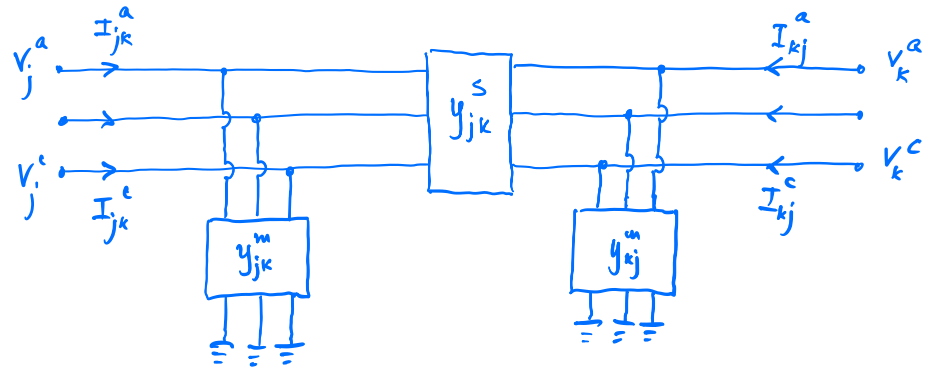
$$I_j = \sum_{k:j\sim k} I_{jk} = \left(\sum_{k:j\sim k} y_{jk}^s + y_{jj}^m \right) V_j - \sum_{k:j\sim k} y_{jk}^s V_k$$

In vector form:

$$I = YV \text{ where } Y_{jk} = \begin{cases} -y_{jk}^s, & j \sim k \quad (j \neq k) \\ \sum_{l:j\sim l} y_{jl}^s + y_{jj}^m, & j = k \\ 0 & \text{otherwise} \end{cases}$$

Assumption: 3-phase BIM

1. All lines are characterized by a 3-wire model
   - Only to simplify exposition
   - Valid if neutral lines are absent (e.g. connecting Δ devices) or grounded with $z_j^n = 0$ (Kron reduction)
   - Otherwise, 4-wire (including neutral line) or 5-wire (including earth return) models should be used.
   - They are conceptually similar to 3-wire model; see examples later



2. All transformers are modeled as 3-phase lines, characterized by a 3-wire model
   - We will henceforth talk about just lines in network models (even though they may be models for transformers)

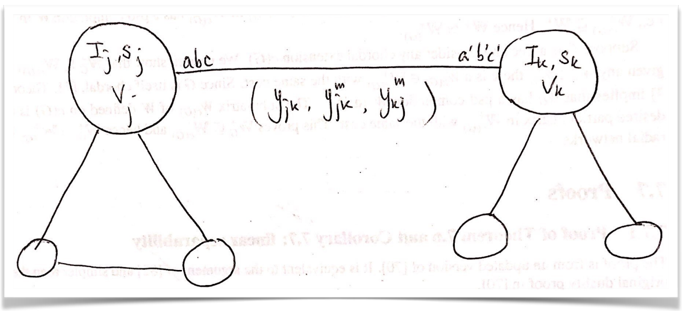
Bus injection model

Network model

1. A network of $N+1$ 3-phase devices connected by 3-phase lines is also modeled by a graph G

2. Each line in G is characterized by $\left(y_{jk}^s, y_{jk}^m, y_{kj}^m\right)$ where

   - $y_{jk}^s \in \mathbb{C}^{3\times3}$: 3×3 series phase admittance matrix
   - $y_{jk}^m, y_{kj}^m \in \mathbb{C}^3$: 3×3 shunt phase admittance matrices



Assumption: 3-phase Π circuit representation

3. Each bus (node) has 3 variables $\left(I_j, s_j, V_j\right) \in \mathbb{C}^9$

   - Only bus injections $\left(I_j, s_j\right)$ are involved
   - Branch flow models also involve branch variables $\left(I_{jk}, I_{kj}, S_{jk}, S_{kj}\right)$

Current balance

Series and shunt admittances

- 1-phase : scalars
- 3-phase : 3×3 (3-wire) or 4×4 (4-wire) matrices

1. 3-phase sending-end currents:

$$I_{jk} = y_{jk}^s \left(V_j - V_k\right) + y_{jk}^m V_j, \qquad I_{kj} = y_{jk}^s \left(V_k - V_j\right) + y_{kj}^m V_k$$

Current balance

Series and shunt admittances

- 1-phase : scalars
- 3-phase : 3×3 (3-wire) or 4×4 (4-wire) matrices

1. 3-phase sending-end currents:

$$I_{jk} = y_{jk}^s \left(V_j - V_k\right) + y_{jk}^m V_j, \qquad I_{kj} = y_{jk}^s \left(V_k - V_j\right) + y_{kj}^m V_k$$

2. Nodal current balance:

$$I_j = \sum_{k:j\sim k} I_{jk} = \sum_{k:j\sim k} y_{jk}^s (V_j - V_k) + \left(\sum_{k:j\sim k} y_{jk}^m\right) V_j$$

$$= \left(\left(\sum_{k:j\sim k} y_{jk}^s\right) + y_{jj}^m\right) V_j - \sum_{k:j\sim k} y_{jk}^s V_k$$

$$y_{jj}^m := \sum_{k:j\sim k} y_{jk}^m$$

Bus injection model

Bus admittance matrix Y

3. In terms of $3(N+1) \times 3(N+1)$ admittance matrix Y

$$I \;=\; Y\,V$$

$3(N+1)$ vector

where

$$Y_{jj} := \sum_{k:j\sim k} y_{jk}^s \;+\; y_{jj}^m$$

3×3 matrices

$$Y_{jk} := -\,y_{jk}^s$$

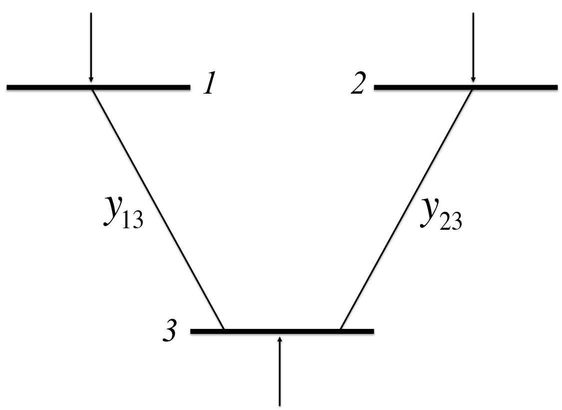
3×3 matrices

$$y_{jj}^m \;:=\; \sum_{k:j\sim k} y_{jk}^m$$

Y is admittance matrix of single-phase equivalent

Bus injection model

Bus admittance matrix Y



(a) 3-bus example.

$$
Y = \begin{bmatrix}
\begin{bmatrix} y_{13}^{aa} & y_{13}^{ab} & y_{13}^{ac} \\ y_{13}^{ba} & y_{13}^{bb} & y_{13}^{bc} \\ y_{13}^{ca} & y_{13}^{cb} & y_{13}^{cc} \end{bmatrix} & 0 & -\begin{bmatrix} y_{13}^{aa} & y_{13}^{ab} & y_{13}^{ac} \\ y_{13}^{ba} & y_{13}^{bb} & y_{13}^{bc} \\ y_{13}^{ca} & y_{13}^{cb} & y_{13}^{cc} \end{bmatrix} \\
0 & \begin{bmatrix} y_{23}^{aa} & y_{23}^{ab} & y_{23}^{ac} \\ y_{23}^{ba} & y_{23}^{bb} & y_{23}^{bc} \\ y_{23}^{ca} & y_{23}^{cb} & y_{23}^{cc} \end{bmatrix} & -\begin{bmatrix} y_{23}^{aa} & y_{23}^{ab} & y_{23}^{ac} \\ y_{23}^{ba} & y_{23}^{bb} & y_{23}^{bc} \\ y_{23}^{ca} & y_{23}^{cb} & y_{23}^{cc} \end{bmatrix} \\
-\begin{bmatrix} y_{13}^{aa} & y_{13}^{ab} & y_{13}^{ac} \\ y_{13}^{ba} & y_{13}^{bb} & y_{13}^{bc} \\ y_{13}^{ca} & y_{13}^{cb} & y_{13}^{cc} \end{bmatrix} & -\begin{bmatrix} y_{23}^{aa} & y_{23}^{ab} & y_{23}^{ac} \\ y_{23}^{ba} & y_{23}^{bb} & y_{23}^{bc} \\ y_{23}^{ca} & y_{23}^{cb} & y_{23}^{cc} \end{bmatrix} & \begin{bmatrix} y_{13}^{aa} + y_{23}^{aa} & y_{13}^{ab} + y_{23}^{ab} & y_{13}^{ac} + y_{23}^{ac} \\ y_{13}^{ba} + y_{23}^{ba} & y_{13}^{bb} + y_{23}^{bb} & y_{13}^{bc} + y_{23}^{bc} \\ y_{13}^{ca} + y_{23}^{ca} & y_{13}^{cb} + y_{23}^{cb} & y_{13}^{cc} + y_{23}^{cc} \end{bmatrix}
\end{bmatrix}
$$

(b) Admittance matrix Y.

Bus injection model

Bus admittance matrix Y

The $3(N+1) \times 3(N+1)$ admittance matrix Y leads to an equivalent circuit

which we call the single-phase equivalent of a 3-phase network

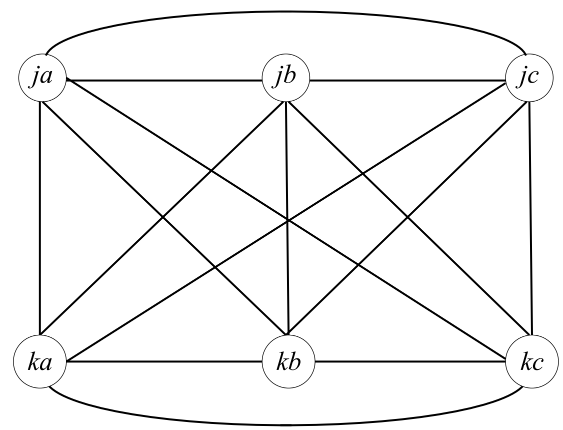
Bus injection model
Single-phase equivalent

Given: 3-phase network G with $3(N+1)$ buses, described by $3(N+1) \times 3(N+1)$ admittance matrix Y

Single-phase equivalent circuit $G^{3\phi}$ with $3(N+1)$ nodes

- Each node in $G^{3\phi}$ is identified by bus-phase pair (j, ϕ)
- Nodes (j, ϕ) and (k, ϕ') in $G^{3\phi}$ are connected if $Y_{j\phi,k\phi'} \neq 0$
- Each line (j, k) in G forms a 6-clique in the 1-phase equivalent $G^{3\phi}$

Single-phase analysis methods can be applied to single-phase equivalent $G^{3\phi}$ using Y



A clique in $G^{3\phi}$ corresponding to line (j, k) in G

Outline

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   - IV relation ($I = YV$)
   - sV relation (power flow equations)
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4. Symmetric network

Review: single-phase BIM

Complex line power

Using $S_{jk} := V_j I_{jk}^H$:

$$S_{jk} = \left(y_{jk}^s\right)^H \left(|V_j|^2 - V_j V_k^H \right) + \left(y_{jk}^m\right)^H |V_j|^2$$

$$S_{kj} = \left(y_{jk}^s\right)^H \left(|V_k|^2 - V_k V_j^H \right) + \left(y_{kj}^m\right)^H |V_k|^2$$

Line loss

$$S_{jk} + S_{kj} = \left(y_{jk}^s\right)^H \left| V_j - V_k \right|^2 + \left(y_{jk}^m\right)^H |V_j|^2 + \left(y_{kj}^m\right)^H |V_k|^2$$

series impedance

shunt impedances

Review: single-phase BIM

Power flow equation

Nodal power balance $s_j = \sum_{k:j\sim k} S_{jk}$:

$$s_j = \sum_{k:j\sim k} \left(|V_j|^2 - V_j V_k^H \right) \left(y_{jk}^s \right)^H + |V_j|^2 \left(y_{jj}^m \right)^H$$

In terms of admittance matrix Y

$$s_j = \sum_{k=1}^{N+1} Y_{jk}^H V_j V_k^H$$

$N+1$ complex equations in $2(N+1)$ complex variables $\left(s_j, V_j, j \in \overline{N} \right)$

Bus injection model

Single-phase equivalent

Bus injection model for 3-phase network:

$$s_j^\phi \quad = \quad \sum_{\substack{k \in \overline{N} \\ \phi' \in \{a,b,c\}}} Y_{j\phi,k\phi'}^H \, V_j^\phi \left(V_k^{\phi'} \right)^H$$

where $Y_{j\phi,k\phi'}$ are $(j\phi, k\phi')$th entry of the $3(N+1) \times 3(N+1)$ admittance matrix Y

This generalizes single-phase BIM:

$$s_j = \sum_{k=1}^{N+1} Y_{jk}^H \, V_j \, V_k^H$$

Bus injection model

Single-phase equivalent

Nodal power balance for 3-phase network

$$s_j \;=\; \sum_{k:j\sim k} \text{diag}\left(V_j(V_j - V_k)^H \left(y_{jk}^s\right)^H \;+\; V_j V_j^H \left(y_{jk}^m\right)^H \right)$$

$s_j = \text{diag}\left(V_j I_j^{\text{H}}\right)$

generalizes single-phase:

$$s_j = \sum_{k:j\sim k} \left(|V_j|^2 - V_j V_k^H \right) \left(y_{jk}^s\right)^H \;+\; |V_j|^2 \left(y_{jj}^m\right)^H$$

Overall model

Device + network

1. Device model for each 3-phase device
   - Internal model on $\left(V_j^{Y/\Delta}, I_j^{Y/\Delta}, s_j^{Y/\Delta}\right)$ + conversion rules
   - External model on $\left(V_j, I_j, s_j\right)$
   - Either can be used
   - Power source models are nonlinear; other devices are linear

2. Network model relates terminal vars (V, I, s)
   - Nodal current balance (linear): $I = YV$
   - Nodal power balance (nonlinear): $s_j = \sum_{k:j\sim k} \text{diag}\left(V_j(V_j - V_k)^{\mathsf{H}} y_{jk}^{s\mathsf{H}} + V_j V_j^{\mathsf{H}} y_{jk}^{m\mathsf{H}}\right)$
   - Either can be used

Overall model
Device + network

Overall model is linear if and only if voltage/current sources and impedances are present

- Power sources lead to nonlinear analysis
- … even though network equation $I = YV$ is linear, device models for power sources are nonlinear

Outline

1. Network models: BIM
2. Three-phase analysis
   - Device specification
   - Examples
   - General solution approach
3. Balanced network
4. Symmetric network

Three-phase analysis & optimization

At each bus j, there are 20 complex quantities for each 3-phase device

- External vars : $(V_j, I_j, s_j), \gamma_j$
- Internal vars : $\left(V_j^{Y\Delta}, I_j^{Y/\Delta}, s_j^{Y/\Delta}\right), \beta_j$

Analysis

Given: 3-phase devices & their specifications

- Voltage/current/power sources, impedances
- … in Y/Δ configuration

Calculate remaining variables

Solution:

- Write down device+network model
- Solve numerically

Optimization

Given: 3-phase devices & their specifications

- Voltage/current/power sources, impedances
- … in Y/Δ configuration

Minimize cost(controllable vars & state)

Solution:

- Write down device+network model
- Write down additional constraints
- Solve numerically

Device specification

Ideal devices

1. Voltage source $\left(E^Y, \gamma_j\right)$ or $\left(E^\Delta, \gamma_j, \beta_j\right)$
   - Y configuration: $\gamma_j := V_j^n$ neutral voltage
   - Δ configuration: $\gamma_j := \frac{1}{3} 1^\mathsf{T} V_j$ zero-seq terminal voltage, $\beta_j := \frac{1}{3} 1^\mathsf{T} I^\Delta$ zero-seq internal current
2. Current source $\left(J^Y, \gamma_j\right)$ or J^Δ
3. Power source $\left(\sigma^Y, \gamma_j\right)$ or $\left(\sigma^\Delta, \gamma_j \text{ or } \beta_j\right)$
   - Δ configuration: spec generally depends on details of the problem
4. Impedance $\left(z^Y, \gamma_j\right)$ or z^Δ

Device specification

Summary

	Buses j	Specification
{ buses with voltage sources in Y, Δ configurations }	N_v^Y	$V_j^Y := E_j^Y, \gamma_j$
	N_v^Δ	$V_j^\Delta := E_j^\Delta, \gamma_j, \beta_j,$
{ buses with current sources in Y, Δ configurations }	N_c^Y	$I_j^Y := J_j^Y, \gamma_j$
	N_c^Δ	$I_j^\Delta := J_j^\Delta$
{ buses with impedances in Y, Δ configurations }	N_i^Y	z_j^Y, γ_j
	N_i^Δ	z_j^Δ
{ buses with power sources in Y, Δ configurations }	N_p^Y	σ_j^Y, γ_j
	N_p^Δ	$\sigma_j^\Delta, \gamma_j$

Device specification

Neutral voltage γ_j for Y-configured devices

1. Neutral voltage $\gamma_j := V_j^n$ for every Y-configured device
    - γ_j may be specified directly (e.g. $\gamma_j := 1$ pu)
    - γ_j may be determined from other information (more likely)
2. Indirect specification of γ_j
    - Assumption C8.1 (neutral is grounded and voltage ref is ground): $\gamma_j = V_j^n = -z_j^n \left(1^\top I_j\right)$
    - Assumption C8.1 with $z_j^n = 0$: $\gamma_j = V_j^n = 0$
    - Neutral not grounded but $1^\top V_j^Y = 0$: $\gamma_j = \frac{1}{3} 1^\top V_j$
    - Such indirect specification provides additional equations to solve for γ_j
3. Neutral voltage γ_j and zero-seq voltage
    - For Y-configured device: $V_j \;=\; V_j^Y + V_j^n 1$
    - $\gamma_j := V_j^n \;=\; \frac{1}{3} 1^\top V_j$ if and only if $1^\top V_j^Y = 0$

Device specification

Zero-sequence voltage γ_j for Δ-configured devices

1. For Δ-configured voltage sources, zero-seq voltages $\gamma_j := \frac{1}{3} 1^\mathsf{T} V_j$ need to be specified
   - γ_j may be specified by one of its terminal voltages, say, V_j^a
2. For Δ-configured current sources and impedances, γ_j need not be specified
   - $\gamma_j := \frac{1}{3} 1^\mathsf{T} V_j$ can be determined once its terminal voltages V_j is determined from network equations

Three-phase analysis problem

Given:

- device spec in blue
- line model

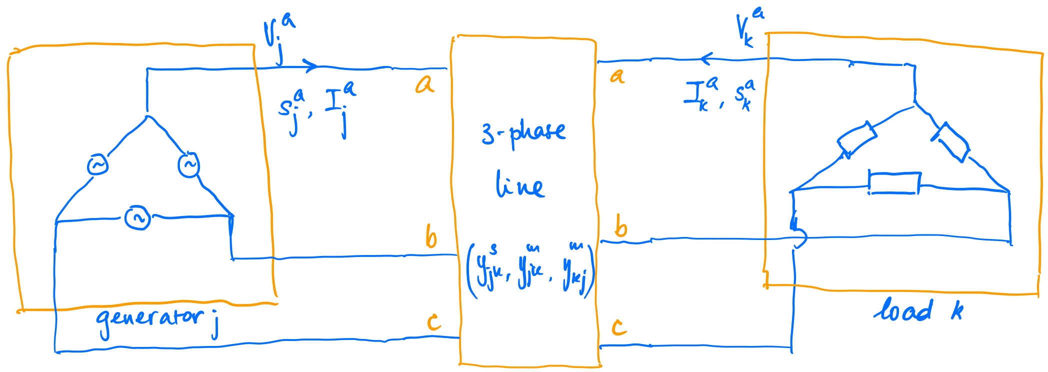
Determine:

- Some or all of internal variables
- Some or all of terminal variables

Buses j	Specification	Unknowns
N_v^Y	$V_j^Y := E_j^Y,\ \gamma_j$	$\left(I_j^Y, s_j^Y\right),\ (V_j, I_j, s_j)$
N_v^Δ	$V_j^\Delta := E_j^\Delta,\ \gamma_j, \beta_j,$	$\left(I_j^\Delta, s_j^\Delta\right),\ (V_j, I_j, s_j)$
N_c^Y	$I_j^Y := J_j^Y, \gamma_j$	$\left(V_j^Y, s_j^\Delta\right),\ (V_j, I_j, s_j)$
N_c^Δ	$I_j^\Delta := J_j^\Delta$	$\left(V_j^\Delta, s_j^\Delta, \beta_j\right),\ (V_j, I_j, s_j, \gamma_j)$
N_i^Y	$z_j^Y,\ \gamma_j$	$\left(V_j^Y, I_j^Y, s_j^Y\right),\ (V_j, I_j, s_j)$
N_i^Δ	z_j^Δ	$\left(V_j^\Delta, I_j^\Delta, s_j^\Delta, \beta_j\right),\ (V_j, I_j, s_j, \gamma_j)$
N_p^Y	$\sigma_j^Y,\ \gamma_j$	$\left(V_j^Y, I_j^Y\right),\ (V_j, I_j, s_j)$
N_p^Δ	$\sigma_j^\Delta,\ \gamma_j$	$\left(V_j^\Delta, I_j^\Delta, \beta_j\right),\ (V_j, I_j, s_j)$

Example

Δ-configuration



Given:

- Voltage source $\left(E_j^\Delta, \gamma_j, \beta_j\right)$
- Impedance z_k^Δ
- Line parameters $\left(y_{jk}^s, y_{jk}^m = y_{kj}^m = 0\right)$

Show:

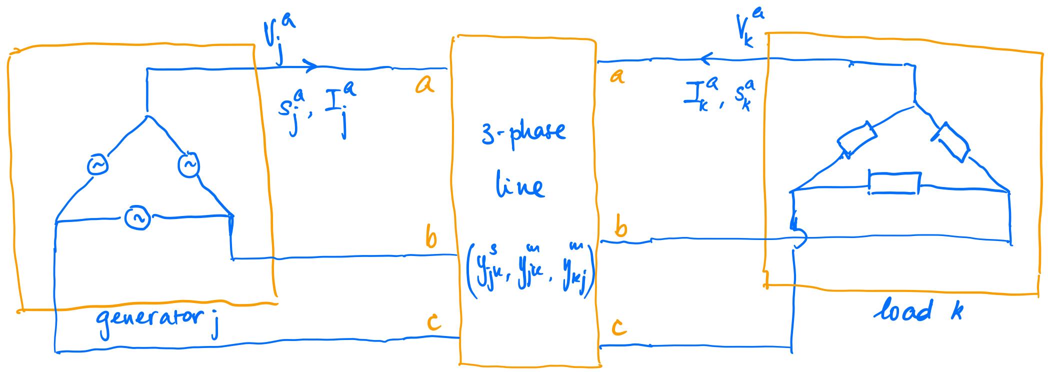
$$I_k^\Delta = Z_{\text{Th}}^{-1} E_j^\Delta, \qquad V_k^\Delta = z_k^\Delta Z_{\text{Th}}^{-1} E_j^\Delta$$

voltage divider rule

where $Z_{\text{Th}} := \Gamma z_{jk}^s \Gamma^{\mathsf{T}} + z_k^\Delta$ is the Thevenin equivalent of line in series with load

Example

Δ-configuration



Given:

- Voltage source $\left(E_j^\Delta, \gamma_j, \beta_j\right)$
- Impedance z_k^Δ
- Line parameters $\left(y_{jk}^s, y_{jk}^m = y_{kj}^m = 0\right)$

Solution:

- Current balance: $V_k = V_j - z_{jk}^s I_j$
- Conversion rule : $V_k^\Delta = \Gamma V_k, \quad E_j^\Delta = \Gamma V_j, \quad I_j = -I_k = \Gamma^\mathsf{T} I_k^\Delta$

$$\implies \quad V_k^\Delta = E_j^\Delta - \Gamma z_{jk}^s \Gamma^\mathsf{T} I_k^\Delta$$

- Internal model: $V_k^\Delta = z_k^\Delta I_k^\Delta$

$$\implies \left(\Gamma z_{jk}^s \Gamma^\mathsf{T} + z_k^\Delta\right) I_k^\Delta = E_j^\Delta$$

General solution approach

Buses j	Specification
N_v^Y	$V_j^Y := E_j^Y,\ \gamma_j$
N_v^Δ	$V_j^\Delta := E_j^\Delta,\ \gamma_j, \beta_j,$
N_c^Y	$I_j^Y := J_j^Y, \gamma_j$
N_c^Δ	$I_j^\Delta := J_j^\Delta$
N_i^Y	$z_j^Y,\ \gamma_j$
N_i^Δ	z_j^Δ

Solution procedure

1. Write down current balance equation that relates terminal vars (V, I)
2. Write down internal models and conversion rules (or external device models)
3. Solve numerically for desired vars

General solution approach

Buses j	Specification
N_v^Y	$V_j^Y := E_j^Y, \gamma_j$
N_v^Δ	$V_j^\Delta := E_j^\Delta, \gamma_j, \beta_j,$
N_c^Y	$I_j^Y := J_j^Y, \gamma_j$
N_c^Δ	$I_j^\Delta := J_j^\Delta$
N_i^Y	z_j^Y, γ_j
N_i^Δ	z_j^Δ

Details: overall system of equations

1. Network equation: $$\begin{bmatrix} I_v \\ I_c \\ I_i \end{bmatrix} = \underbrace{\begin{bmatrix} Y_{vv} Y_{vc} Y_{vi} \\ Y_{cv} Y_{cc} Y_{ci} \\ Y_{iv} Y_{ic} Y_{ii} \end{bmatrix}}_{Y} \begin{bmatrix} V_v \\ V_c \\ V_i \end{bmatrix}$$

2. Voltage sources: $V_v \; := \; \Gamma_v^\dagger E_v \; + \; \gamma_v \otimes 1$

3. Current sources: $I_c \; := \; -\Gamma_c^\mathsf{T} J_c$

4. Impedances: $V_i^{\text{int}} \; = \; Z_i I_i^{\text{int}}$

$$I_i = -\Gamma_i^\mathsf{T} I_i^{\text{int}}$$

$$\Gamma_i V_i = V_i^{\text{int}} + \gamma_i \otimes 1$$

General solution approach

Buses j	Specification
N_v^Y	$V_j^Y := E_j^Y,\ \gamma_j$
N_v^Δ	$V_j^\Delta := E_j^\Delta,\ \gamma_j, \beta_j,$
N_c^Y	$I_j^Y := J_j^Y, \gamma_j$
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N_i^Δ	z_j^Δ

Solution procedure

1. Write down current balance equation that relates terminal vars (V, I)
2. Write down internal models and conversion rules (or external device models)
3. Solve numerically for desired vars

Remark (nonlinearity)

1. Can always use $I = YV$ in Step 1 (instead of nonlinear power flow equations)
2. If there is no power source, device models are linear ⇒ overall system is linear
3. Otherwise, power sources models are nonlinear ⇒ overall system is nonlinear

General solution approach

Buses j	Specification
N_v^Y	$V_j^Y := E_j^Y,\ \gamma_j$
N_v^Δ	$V_j^\Delta := E_j^\Delta,\ \gamma_j, \beta_j,$
N_c^Y	$I_j^Y := J_j^Y, \gamma_j$
N_c^Δ	$I_j^\Delta := J_j^\Delta$
N_i^Y	$z_j^Y,\ \gamma_j$
N_i^Δ	z_j^Δ

Solution procedure

1. Write down current balance equation that relates terminal vars (V, I)
2. Write down internal models and conversion rules (or external device models)
3. Solve numerically for desired vars

Reduced system

1. Solve for $\left(V_c, I_i^{\text{int}}\right)$

$$\begin{bmatrix} \mathbb{I}_c \otimes \mathbb{I} & Z_{ci}\Gamma_i^{\mathsf{T}} \\ 0 & \Gamma_i Z_{ii}\Gamma_i^{\mathsf{T}} + Z_i \end{bmatrix} \begin{bmatrix} V_c \\ I_i^{\text{int}} \end{bmatrix} = \begin{bmatrix} Z_{cc} \\ \Gamma_i Z_{ic} \end{bmatrix} I_c - \begin{bmatrix} A_{cv} \\ A_{iv} \end{bmatrix} V_v - \begin{bmatrix} 0 \\ \gamma_i \otimes 1 \end{bmatrix}$$

2. Derive all other variables analytically in terms of $\left(V_c, I_i^{\text{int}}\right)$

General solution approach

Buses j	Specification
N_v^Y	$V_j^Y := E_j^Y,\ \gamma_j$
N_v^Δ	$V_j^\Delta := E_j^\Delta,\ \gamma_j, \beta_j,$
N_c^Y	$I_j^Y := J_j^Y,\ \gamma_j$
N_c^Δ	$I_j^\Delta := J_j^\Delta$
N_i^Y	$z_j^Y,\ \gamma_j$
N_i^Δ	z_j^Δ
N_p^Y	$\sigma_j^Y,\ \gamma_j$
N_p^Δ	$\sigma_j^\Delta,\ \gamma_j$

With power sources

1. Write down current balance equation that relates terminal vars (V, I)
2. Write down internal models and conversion rules (or external device models)
3. Solve numerically for desired vars

General solution approach

Buses j	Specification
N_v^Y	$V_j^Y := E_j^Y,\ \gamma_j$
N_v^Δ	$V_j^\Delta := E_j^\Delta,\ \gamma_j, \beta_j,$
N_c^Y	$I_j^Y := J_j^Y,\ \gamma_j$
N_c^Δ	$I_j^\Delta := J_j^\Delta$
N_i^Y	$z_j^Y,\ \gamma_j$
N_i^Δ	z_j^Δ
N_p^Y	$\sigma_j^Y,\ \gamma_j$
N_p^Δ	$\sigma_j^\Delta,\ \gamma_j$

Details: overall system of equations

1. Network equation: $$\begin{bmatrix} I_v \\ I_c \\ I_i \end{bmatrix} = \underbrace{\begin{bmatrix} Y_{vv} Y_{vc} Y_{vi} \\ Y_{cv} Y_{cc} Y_{ci} \\ Y_{iv} Y_{ic} Y_{ii} \end{bmatrix}}_{Y} \begin{bmatrix} V_v \\ V_c \\ V_i \end{bmatrix}$$
2. Voltage sources: $V_v \ := \ \Gamma_v^\dagger E_v \ + \ \gamma_v \otimes 1$
3. Current sources: $I_c \ := \ -\Gamma_c^\mathsf{T} J_c$
4. Impedances: $V_i^{\text{int}} \ = \ Z_i I_i^{\text{int}}$

$$I_i = -\Gamma_i^\mathsf{T} I_i^{\text{int}}$$

$$\Gamma_i V_i = \ V_i^{\text{int}} + \gamma_i \otimes 1$$

same equations as before

General solution approach

Buses j	Specification
N_v^Y	$V_j^Y := E_j^Y,\ \gamma_j$
N_v^Δ	$V_j^\Delta := E_j^\Delta,\ \gamma_j, \beta_j,$
N_c^Y	$I_j^Y := J_j^Y,\ \gamma_j$
N_c^Δ	$I_j^\Delta := J_j^\Delta$
N_i^Y	$z_j^Y,\ \gamma_j$
N_i^Δ	z_j^Δ
N_p^Y	$\sigma_j^Y,\ \gamma_j$
N_p^Δ	$\sigma_j^\Delta,\ \gamma_j$

Details: overall system of equations

5. Power sources: internal model

$$\sigma_p = \text{diag}\left(V_p^{\text{int}} I_p^{\text{int}\,\text{H}} \right)$$

Conversion rules:

$$I_p = -\Gamma_p I_p^{\text{int}}$$

$$\Gamma_p V_p = V_p^{\text{int}} + \gamma_p \otimes 1$$

additional (nonlinear) equations for power sources

General solution approach

Buses j	Specification
N_v^Y	$V_j^Y := E_j^Y,\ \gamma_j$
N_v^Δ	$V_j^\Delta := E_j^\Delta,\ \gamma_j, \beta_j,$
N_c^Y	$I_j^Y := J_j^Y, \gamma_j$
N_c^Δ	$I_j^\Delta := J_j^\Delta$
N_i^Y	$z_j^Y,\ \gamma_j$
N_i^Δ	z_j^Δ

Solution procedure

1. Write down current balance equation that relates terminal vars (V, I)
2. Write down internal models and conversion rules (or external device models)
3. Solve numerically for desired vars

Remark (nonlinearity)

1. Can always use $I = YV$ in Step 1 (instead of nonlinear power flow equations)
2. If there is no power source, device models are linear ⇒ overall system is linear
3. Otherwise, power sources models are nonlinear ⇒ overall system is nonlinear

General solution approach

Buses j	Specification
N_v^Y	$V_j^Y := E_j^Y,\ \gamma_j$
N_v^Δ	$V_j^\Delta := E_j^\Delta,\ \gamma_j, \beta_j,$
N_c^Y	$I_j^Y := J_j^Y, \gamma_j$
N_c^Δ	$I_j^\Delta := J_j^\Delta$
N_i^Y	$z_j^Y,\ \gamma_j$
N_i^Δ	z_j^Δ

Solution procedure

1. Write down current balance equation that relates terminal vars (V, I)
2. Write down internal models and conversion rules (or external device models)
3. Solve numerically for desired vars

Reduced system

1. Solve for $\left(V_c, I_i^{\text{int}}\right)$

$$\begin{bmatrix} \mathbb{I}_c \otimes \mathbb{I} & Z_{ci}\Gamma_i^{\mathsf{T}} \\ 0 & \Gamma_i Z_{ii}\Gamma_i^{\mathsf{T}} + Z_i \end{bmatrix} \begin{bmatrix} V_c \\ I_i^{\text{int}} \end{bmatrix} = \begin{bmatrix} Z_{cc} \\ \Gamma_i Z_{ic} \end{bmatrix} I_c - \begin{bmatrix} A_{cv} \\ A_{iv} \end{bmatrix} V_v - \begin{bmatrix} 0 \\ \gamma_i \otimes 1 \end{bmatrix}$$

2. Derive all other variables analytically in terms of $\left(V_c, I_i^{\text{int}}\right)$

Outline

1. Network models: BIM
2. Three-phase analysis
3. Balanced network
   - Three-phase analysis
   - Per-phase network
   - Per-phase analysis
4. Symmetric network

Three-phase analysis

At each bus j, there are 20 complex quantities for each 3-phase device

- External vars : $(V_j, I_j, s_j), \gamma_j$
- Internal vars : $\left(V_j^{Y\Delta}, I_j^{Y/\Delta}, s_j^{Y/\Delta}\right), \beta_j$

Analysis

Given: 3-phase devices & their specifications

- Voltage/current/power sources, impedances
- … in Y/Δ configuration

Calculate remaining variables

Solution:

- Write down device+network model
- Solve numerically

Special case:

- Devices are balanced
- Lines are balanced & decoupled

Balanced devices

Buses j	Specification
N_v^Y	$V_j^Y := E_j^Y,\ \gamma_j$
N_v^Δ	$V_j^\Delta := E_j^\Delta,\ \gamma_j, \beta_j,$
N_c^Y	$I_j^Y := J_j^Y,\ \gamma_j$
N_c^Δ	$I_j^\Delta := J_j^\Delta$
N_i^Y	$z_j^Y,\ \gamma_j$
N_i^Δ	z_j^Δ

Device specification

1. Devices are balanced positive-seq sets:

$$\begin{aligned} V_j^{Y/\Delta} &:= \lambda_j \alpha_+, & j \in N_v \\ I_j^{Y/\Delta} &:= \mu_j \alpha_+, & j \in N_c \\ z_j^{Y/\Delta} &:= \zeta_j \mathbb{I}, & j \in N_i \end{aligned}$$

where $\lambda_j, \mu_j, \zeta_j \in \mathbb{C}$

2. External model of voltage sources:

$$V_v = \hat{\lambda}_v \otimes \alpha_+ + \gamma_v \otimes 1$$

3. External model of current sources:

$$I_c = -\hat{\mu}_c \otimes \alpha_+$$

Balanced devices

Buses j	Specification
N_v^Y	$V_j^Y := E_j^Y,\ \gamma_j$
N_v^Δ	$V_j^\Delta := E_j^\Delta,\ \gamma_j, \beta_j,$
N_c^Y	$I_j^Y := J_j^Y, \gamma_j$
N_c^Δ	$I_j^\Delta := J_j^\Delta$
N_i^Y	$z_j^Y,\ \gamma_j$
N_i^Δ	z_j^Δ

Device specification

4. Internal impedance model and conversion rules

$$V_i^{\text{int}} = \left(\zeta_i \otimes \mathbb{I}\right) I_i^{\text{int}}$$

$$I_i = -\Gamma_i^{\mathsf{T}} I_i^{\text{int}}$$

$$\Gamma_i V_i = V_i^{\text{int}} + \gamma_i^0 \otimes 1$$

Balanced lines

1. All lines are balanced, i.e.

$$y_{jk}^s \;=\; \eta_{jk}^s \mathbb{I}, \qquad y_{jk}^m \;=\; \eta_{jk}^m \mathbb{I}, \qquad y_{kj}^m \;=\; \eta_{kj}^m \mathbb{I}, \qquad \eta_{jk}^s, \eta_{jk}^m, \eta_{kj}^m \in \mathbb{C}$$

2. Define per-phase admittance matrix $Y^{1\phi} \in \mathbb{C}^{(N+1)\times(N+1)}$

$$Y_{jk}^{1\phi} := \begin{cases} -\eta_{jk}^s, & (j,k) \in E, \quad (j \neq k) \\ \sum_{k:j\sim k} \left(\eta_{jk}^s + \eta_{jk}^m\right), & j = k \\ 0 & \text{otherwise} \end{cases}$$

3. $3(N+1) \times 3(N+1)$ admittance matrix Y becomes $Y = Y^{1\phi} \otimes \mathbb{I}$

4. Current balance equation $I = YV$ becomes

$$I \;=\; \left(Y^{1\phi} \otimes \mathbb{I}\right) Y$$

Three-phase analysis problem

Buses j	Specification
N_v^Y	$V_j^Y := E_j^Y,\ \gamma_j$
N_v^Δ	$V_j^\Delta := E_j^\Delta,\ \gamma_j, \beta_j,$
N_c^Y	$I_j^Y := J_j^Y,\ \gamma_j$
N_c^Δ	$I_j^\Delta := J_j^\Delta$
N_i^Y	$z_j^Y,\ \gamma_j$
N_i^Δ	z_j^Δ

Given

- device spec in blue
- line model

Determine

- some or all remaining vars

Balanced voltages & currents

Theorem

1. Any solution x consists of generalized balanced vectors in positive sequence, i.e., $x_j = a_j\alpha_+ + b_j 1$ for some $a_j, b_j \in \mathbb{C}$

2. All $x_j = a_j\alpha_+$ are balanced if
   - For all voltage sources: $\gamma_v = 0$
   - For all Y configured impedances: $\gamma_i^Y := \left(V_j^n, j \in N_i^Y\right) = 0$

Balanced voltages & currents

Proof sketch

1. Use reduced system to show $\left(V_c, I_i^{\text{int}}\right)$ consists of generalized balanced vectors

2. Derive all other vars in terms of $\left(V_c, I_i^{\text{int}}\right)$ and show that they consist of generalized balanced vectors

Balanced voltages & currents

Proof sketch: step 1

Lemma

Reduced system becomes

$$\underbrace{\begin{bmatrix} \mathbb{I}_c \otimes \mathbb{I} & \left(Z^{1\phi}_{ci} \otimes \mathbb{I}\right) \Gamma_i^{\mathsf{T}} \\ 0 & \Gamma_i \left(Z^{1\phi}_{ii} \otimes \mathbb{I}\right) \Gamma_i^{\mathsf{T}} + \left(\zeta_i \otimes \mathbb{I}\right) \end{bmatrix}}_{M} \begin{bmatrix} V_c \\ I_i^{\text{int}} \end{bmatrix} \;=\; a' \otimes \alpha_+ \;+\; b' \otimes 1$$

Balanced voltages & currents

Proof sketch: step 1

Lemma

Reduced system becomes

$$\underbrace{\begin{bmatrix} \mathbb{I}_c \otimes \mathbb{I} & \left(Z_{ci}^{1\phi} \otimes \mathbb{I}\right) \Gamma_i^{\mathsf{T}} \\ 0 & \Gamma_i \left(Z_{ii}^{1\phi} \otimes \mathbb{I}\right) \Gamma_i^{\mathsf{T}} + \left(\zeta_i \otimes \mathbb{I}\right) \end{bmatrix}}_{M} \begin{bmatrix} V_c \\ I_i^{\text{int}} \end{bmatrix} \quad = \quad a' \otimes \alpha_+ \ + \ b' \otimes 1$$

Lemma

Each 3×3 block of M^{-1} is of the form $\left[M^{-1}\right]_{jk} \quad := \quad v_{jk}\,\mathbb{I} \ + \ w_{jk}\, W_{jk}$

where $v_{jk}, w_{jk} \in \mathbb{C}$ and $W_{jk} \in \mathbb{C}^{3\times 3}$ is one of $\mathbb{I}, \Gamma, \Gamma^{\mathsf{T}}, \Gamma\Gamma^{\mathsf{T}}, \Gamma^{\mathsf{T}}\Gamma$

Balanced voltages & currents

Proof sketch: step 1

j-th 3×3 block of $\left(V_c, I_k^\Delta\right)$ is of the form

$$\begin{aligned}\sum_k \left[M^{-1}\right]_{jk} \left(a_k' \alpha_+ + b_k' \mathbf{1}\right) \ &= \ \sum_k a_k' \left(v_{jk}\mathbb{I} + w_{jk} W_{jk}\right) \alpha_+ \ + \ \sum_k b_k' \left(v_{jk}\mathbb{I} + w_{jk} W_{jk}\right) \mathbf{1} \\ &= \ a_j \alpha_+ \ + \ b_j \mathbf{1}\end{aligned}$$

because

$$W_{jk}\alpha_+ \ = \ \begin{cases} \alpha_+ & \text{if} \quad W_{jk} = \mathbb{I} \\ (1-\alpha)\alpha_+ & \text{if} \quad W_{jk} = \Gamma \\ (1-\alpha^2)\alpha_+ & \text{if} \quad W_{jk} = \Gamma^{\mathsf{T}} \\ 3\alpha_+ & \text{if} \quad W_{jk} = \Gamma\Gamma^{\mathsf{T}} \text{ or } \Gamma^{\mathsf{T}}\Gamma \end{cases}$$

$$W_{jk}\mathbf{1} \ = \ \begin{cases} \mathbf{1} & \text{if} \quad W_{jk} = \mathbb{I} \\ 0 & \text{else} \end{cases}$$

Decoupling & per-phase analysis

Positive-seq per-phase network

Reduced system implies (α_+ coordinate):

$$\begin{bmatrix} \hat{i}_v \\ -\hat{\mu}_c \\ \hat{i}_i \end{bmatrix} = \begin{bmatrix} Y^{1\phi}_{vv} Y^{1\phi}_{vc} Y^{1\phi}_{vi} \\ Y^{1\phi}_{cv} Y^{1\phi}_{cc} Y^{1\phi}_{ci} \\ Y^{1\phi}_{iv} Y^{1\phi}_{ic} Y^{1\phi}_{ii} \end{bmatrix} \begin{bmatrix} \hat{\lambda}_v \\ \hat{v}_c \\ \hat{v}_i \end{bmatrix}$$

Defines per-phase network

- Admittance matrix: $Y^{1\phi}$
- Voltage sources: $\hat{\lambda}_v$
- Current sources: $-\hat{\mu}_c$
- Impedances $\hat{\eta}_i$: $\hat{i}_i = -\hat{\eta}_i \hat{v}_i$

4 sets of equations in 4 sets of vars $\left(\hat{v}_c, \hat{v}_i, \hat{i}_v, \hat{i}_i\right)$

Decoupling & per-phase analysis

Zero-seq per-phase network

Reduced system implies (1 coordinate):

$$\begin{bmatrix} \hat\beta_v \\ 0 \\ \hat\beta_i \end{bmatrix} = \begin{bmatrix} Y_{vv}^{1\phi} Y_{vc}^{1\phi} Y_{vi}^{1\phi} \\ Y_{cv}^{1\phi} Y_{cc}^{1\phi} Y_{ci}^{1\phi} \\ Y_{iv}^{1\phi} Y_{ic}^{1\phi} Y_{ii}^{1\phi} \end{bmatrix} \begin{bmatrix} \gamma_v \\ \hat\gamma_c \\ \hat\gamma_i \end{bmatrix}$$

Defines per-phase network

- Admittance matrix: $Y^{1\phi}$
- Voltage sources: γ_v
- Current sources: 0 (no device at buses j where current sources are connected)
- Impedances $\hat\eta_i$: $\hat\beta_i = -\hat\eta_i \left(\hat\gamma_i - \gamma_i\right)$

4 sets of equations in 4 sets of vars $\left(\hat\gamma_c, \hat\gamma_i, \hat\beta_v, \hat\beta_i\right)$

Decoupling & per-phase analysis

Zero-seq per-phase network

Set $\hat{\gamma}_{-v} := 0$ and $\hat{\beta}_{-c} := 0$ if

- For all voltage sources: $\gamma_v = 0$
- For all Y configured impedances: $\gamma_i^Y := \left(V_j^n, j \in N_i^Y \right) = 0$

Decoupling & per-phase analysis

Per-phase analysis

1. Solve positive-seq per-phase network for $\left(\hat{v}_c, \hat{v}_i, \hat{i}_v, \hat{i}_i\right)$
2. Solve zero-seq per-phase network for $\left(\hat{\gamma}_c, \hat{\gamma}_i, \hat{\beta}_v, \hat{\beta}_i\right)$
3. Derive terminal voltages V_{-v}

$$V_j =: \; v_j^{\text{int}}\alpha_+ + \left(\gamma_j^{\text{int}} + \gamma_j\right)1, \qquad j \in N_c^Y \cup N_i^Y$$

$$V_j =: \; v_j\alpha_+ + \gamma_j 1, \qquad j \in N_c^\Delta \cup N_i^\Delta$$

4. Determine terminal currents I_{-c}

$$I_j =: \; i_j^{\text{int}}\alpha_+ + \beta_j^{\text{int}}1, \qquad j \in N_v^Y \cup N_i^Y$$

$$I_j =: \; i_j\alpha_+, \qquad j \in N_v^\Delta \cup N_i^\Delta$$

Outline

1. Network models: BIM
2. Three-phase analysis
3. Balanced network
4. Symmetric network
   - Sequence impedances and sources
   - Sequence line
   - Three-phase analysis

Symmetric components

1. In an unbalanced network, phases are coupled and per-phase analysis is generally not applicable
2. If the network has certain symmetry, similarity transformation may lead to sequence networks that are decoupled
   - e.g., impedances are balanced, lines are transposed (even if lines are coupled and generations and loads are unbalanced)
3. Single-phase analysis can then be applied to each of the decoupled sequence networks. This is most useful for fault analysis
4. Without any symmetry, symmetric components may offer no advantage.

Recall: similarity transformation

Sequence variables

1. Complex symmetric Fortescue matrix F and its inverse $F^{-1} = \overline{F}$:

$$F = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \alpha_+ & \alpha_- \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1^\mathsf{T} \\ \alpha_+^\mathsf{T} \\ \alpha_-^\mathsf{T} \end{bmatrix} := \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix}$$

$$\overline{F} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \alpha_- & \alpha_+ \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1^\mathsf{T} \\ \alpha_-^\mathsf{T} \\ \alpha_+^\mathsf{T} \end{bmatrix} := \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2 \end{bmatrix}$$

Recall: similarity transformation

Sequence variables

2. F defines a similarity transformation:

$$x \;=\; F\tilde{x}, \qquad \tilde{x} \;:=\; F^{-1}x \;=\; \overline{F}x$$

3. $\tilde{x}$ is called the sequence variable of x. Its components are

$$\tilde{x}_0 \;:=\; \frac{1}{\sqrt{3}}1^{\mathsf{H}}x, \qquad \tilde{x}_+ \;:=\; \frac{1}{\sqrt{3}}\alpha_+^{\mathsf{H}}x, \qquad \tilde{x}_- \;:=\; \frac{1}{\sqrt{3}}\alpha_-^{\mathsf{H}}x$$

zero-sequence — positive-sequence — negative-sequence

They are also called symmetric components.

4. Sequence voltage and current:

$$\tilde{V} \;=\; \overline{F}V, \qquad \tilde{I} \;=\; \overline{F}I$$

Sequence networks

General method to derive sequence networks: for each (linear) device or line/transformer

1. Write its external model $I = AV$ that relates terminal voltage and current (V, I)
    - e.g., impedances are balanced, lines are transposed (even if lines are coupled and generations and loads are unbalanced)
2. Substitute $V = F\tilde{V}$ and $I = F\tilde{I}$ to obtain the external model $\tilde{I} = \left(\overline{F}AF\right)\tilde{V}$ relating the sequence vars $\left(\tilde{V}, \tilde{I}\right)$
3. With symmetry, $\tilde{F}AF$ turns out to be diagonal and hence can be interpreted as 3 separate devices on 3 decoupled networks called sequence networks
4. Each sequence network can be analyzed separately like a single-phase network

Sequence impedance

Y configuration $\left(z^Y, z^n\right)$

1. External model (from Ch 8) is, under assumption C8.1:

$$V \;=\; -Z^Y I \quad \text{with} \quad Z^Y := z^Y \;+\; z^n\, 1 1^{\mathsf{T}} = \begin{bmatrix} z^{an}+z^n & z^n & z^n \\ z^n & z^{an}+z^n & z^n \\ z^n & z^n & z^{cn}+z^n \end{bmatrix}$$

2. Substituting $V = F\tilde{V},\ I = F\tilde{I}$ to get sequence impedance matrix $\tilde{Z}^Y$:

$$\tilde{V} \;=\; -\underbrace{\overline{F} Z^Y F}_{\tilde{Z}^Y}\, \tilde{I} \;=\; -\tilde{Z}^Y \tilde{I}$$

Sequence impedance

Y configuration $\left(z^Y, z^n\right)$

1. If impedance is balanced, i.e., $z^{an} = z^{bn} = z^{cn}$, then

$$\tilde{Z}^Y \quad = \quad \begin{bmatrix} z^{an} + 3z^n & 0 & 0 \\ 0 & z^{an} & 0 \\ 0 & 0 & z^{an} \end{bmatrix}$$

2. External model $\tilde{V} = -\tilde{Z}^Y \tilde{I}$ in sequence coordinate becomes decoupled

$$\begin{bmatrix} \tilde{V}_0 \\ \tilde{V}_+ \\ \tilde{V}_- \end{bmatrix} \quad = \quad - \begin{bmatrix} z^{an} + 3z^n & 0 & 0 \\ 0 & z^{an} & 0 \\ 0 & 0 & z^{an} \end{bmatrix} \begin{bmatrix} \tilde{I}_0 \\ \tilde{I}_+ \\ \tilde{I}_- \end{bmatrix}$$

diagonal = decoupled !

Sequence impedance

Y configuration $\left(z^Y, z^n\right)$

Interpretation

1. The external model $\tilde V = -\tilde Z^Y \tilde I$ defines sequence impedances on 3 separate (decoupled) sequence networks:

zero-seq impedance: $\tilde V_0 = -\left(z^{an} + 3z^n\right)\tilde I_0$

positive-seq impedance: $\tilde V_+ = -z^{an}\tilde I_+$

negative-seq impedance: $\tilde V_- = -z^{an}\tilde I_-$

2. Each of these decoupled sequence networks can be analyzed like a single-phase network

Sequence impedance

Δ configuration $\left(z^\Delta, z^n\right)$

1. External model (from Ch 8) is:

$$V \;=\; -Z^\Delta I \;+\; \gamma 1, \qquad 1^\mathsf{T} I \;=\; 0 \qquad \text{with}$$

$$Z^\Delta \;:=\; \frac{1}{9}\Gamma^\mathsf{T} \underbrace{z^\Delta \left(\mathbb{I} - \frac{1}{\zeta} 1\, \tilde{z}^{\Delta\mathsf{T}}\right)}_{\hat{z}^\Delta} \Gamma$$

2. Substituting $V = F\tilde{V},\ I = F\tilde{I}$ to convert to sequence coordinate:

$$\tilde{V} \;=\; -\underbrace{\left(\overline{F} Z^\Delta F\right)}_{\tilde{Z}^\Delta} \tilde{I} \;+\; \gamma\, \overline{F} 1, \qquad 1^\mathsf{T} F \tilde{I} \;=\; 0$$

Sequence impedance

Δ configuration (z^Δ, z^n)

1. If impedance is balanced, i.e., $z^{ab} = z^{bc} = z^{ca}$, then

$$Z^\Delta = \frac{z^{ab}}{3}\left(\mathbb{I} - \frac{1}{3}11^\mathsf{T}\right), \qquad \tilde{Z}^\Delta = \frac{z^{ab}}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. External model in sequence coordinate becomes decoupled

$$\begin{bmatrix} 0 \\ \tilde{V}_+ \\ \tilde{V}_- \end{bmatrix} = -\frac{z^{ab}}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \tilde{I}_0 \\ \tilde{I}_+ \\ \tilde{I}_- \end{bmatrix}, \qquad \tilde{I}_0 = \frac{1}{\sqrt{3}}\left(I_a + I_b + I_c\right) = 0$$

$\tilde{I}_0 = 0$ is KCL because there is no neutral wire

Sequence impedance

Δ configuration $\left(z^{\Delta}, z^{n}\right)$

Interpretation

1. The relation $\tilde{V} = -\tilde{Z}^{\Delta}\tilde{I}$ defines sequence impedances on 2 decoupled sequence networks:

zero-seq impedance: null $(\tilde{I}_0 = 0, \tilde{Z}_0 = \infty,$ open circuit)

positive-seq impedance: $$\tilde{V}_+ = -\frac{z^{ab}}{3}\tilde{I}_+$$

negative-seq impedance: $$\tilde{V}_- = -\frac{z^{ab}}{3}\tilde{I}_-$$

2. $\tilde{I}_0 = 0$ means zero-seq impedance is open-circuited (no device) in the zero-seq network
3. Positive and negative-seq impedances are $z^{ab}/3$, as in a balanced network

Sequence voltage source

Y **configuration** $\left(E^Y, z^Y, z^n\right)$

1. External model (from Ch 8) is, under assumption C8.1:

$$V \;=\; E^Y - Z^Y I \quad \text{with} \quad Z^Y := z^Y \;+\; z^n\, 11^\mathsf{T} = \begin{bmatrix} z^{an} + z^n & z^n & z^n \\ z^n & z^{an} + z^n & z^n \\ z^n & z^n & z^{cn} + z^n \end{bmatrix}$$

2. Substituting $V = F\tilde{V},\ I = F\tilde{I}$ to convert to sequence coordinate

$$\tilde{V} \;=\; \underbrace{\overline{F}E^Y}_{\tilde{E}^Y} \;-\; \underbrace{\overline{F}Z^Y F}_{\tilde{Z}^Y}\,\tilde{I} \;=:\; \tilde{E}^Y \;-\; \tilde{Z}^Y \tilde{I}$$

Sequence voltage source

Y **configuration** $\left(E^Y, z^Y, z^n\right)$

1. If impedance is balanced, i.e., $z^{an} = z^{bn} = z^{cn}$ (internal voltage E^Y may be unbalanced), then external model in sequence coordinate becomes decoupled:

$$\begin{bmatrix} \tilde{V}_0 \\ \tilde{V}_+ \\ \tilde{V}_- \end{bmatrix} = \begin{bmatrix} \tilde{E}_0^Y \\ \tilde{E}_+^Y \\ \tilde{E}_-^Y \end{bmatrix} - \begin{bmatrix} z^{an} + 3z^n & 0 & 0 \\ 0 & z^{an} & 0 \\ 0 & 0 & z^{an} \end{bmatrix} \begin{bmatrix} \tilde{I}_0 \\ \tilde{I}_+ \\ \tilde{I}_- \end{bmatrix}$$

2. **Interpretation:** voltage sources on 3 decoupled sequence networks:

zero-seq voltage source: $\tilde{V}_0 = \tilde{E}_0^Y - \left(z^{an} + 3z^n\right) \tilde{I}_0$

positive-seq voltage source: $\tilde{V}_+ = \tilde{E}_+^Y - z^{an}\tilde{I}_+$

negative-seq voltage source: $\tilde{V}_- = \tilde{E}_-^Y - z^{an}\tilde{I}_-$

Sequence voltage source

Y **configuration** $\left(E^Y, z^Y, z^n\right)$

1. If $z^{an} = z^{bn} = z^{cn}$ and $E^Y = E^{an}\alpha_+$ is balanced:

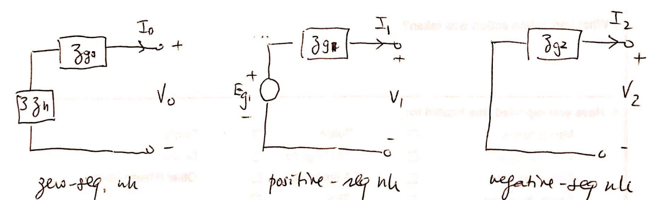
$$\begin{bmatrix} \tilde{V}_0 \\ \tilde{V}_+ \\ \tilde{V}_- \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{3}\, E^{an} \\ 0 \end{bmatrix} - \begin{bmatrix} z^{an} + 3z^n & 0 & 0 \\ 0 & z^{an} & 0 \\ 0 & 0 & z^{an} \end{bmatrix} \begin{bmatrix} \tilde{I}_0 \\ \tilde{I}_+ \\ \tilde{I}_- \end{bmatrix}$$

2. **Interpretation:** voltage source and impedances on decoupled sequence networks:

zero-seq impedance: $\tilde{V}_0 = -\left(z^{an} + 3z^n\right)\tilde{I}_0$

positive-seq voltage source: $\tilde{V}_+ = \sqrt{3}\, E^{an} - z^{an}\tilde{I}_+$

negative-seq mpedance: $\tilde{V}_- = -z^{an}\tilde{I}_-$



Sequence voltage source

Δ configuration (E^Δ, z^Δ)

1. External model (from Ch 8) is:

$$V \;=\; \hat{\Gamma} E^\Delta - Z^\Delta I + \gamma \mathbb{1}, \qquad \mathbb{1}^\mathsf{T} I \;=\; 0 \qquad \text{with}$$

$$\hat{\Gamma} \;:=\; \frac{1}{3}\Gamma^\mathsf{T}\left(\mathbb{I} - \frac{1}{\zeta}\,\tilde{z}^\Delta\,\mathbb{1}^\mathsf{T}\right), \qquad Z^\Delta \;:=\; \frac{1}{9}\Gamma^\mathsf{T} z^\Delta \left(\mathbb{I} - \frac{1}{\zeta}\,\mathbb{1}\,\tilde{z}^{\Delta\mathsf{T}}\right)\Gamma$$

2. Substituting $V = F\tilde{V}$, $I = F\tilde{I}$ to convert to sequence domain

$$\tilde{V} \;=\; \underbrace{\overline{F}\hat{\Gamma}E^\Delta}_{\tilde{E}^\Delta} - \underbrace{\overline{F}Z^\Delta F}_{\tilde{Z}^\Delta}\,\tilde{I} \;+\; \gamma\overline{F}\mathbb{1} \;\;=:\;\; \tilde{E}^\Delta \;-\; \tilde{Z}^\Delta\tilde{I} \;+\; \tilde{V}_0 e_1, \qquad \sqrt{3}\tilde{I}_0 \;=\; 0$$

$\tilde{I}_0 = 0$ is KCL because there is no neutral wire

Sequence voltage source

Δ configuration $\left(E^{\Delta}, z^{\Delta}\right)$

1. If impedance is balanced, i.e., $z^{ab} = z^{bc} = z^{ca}$ (internal voltage E^Y may be unbalanced), then external model in sequence domain becomes decoupled:

$$\begin{bmatrix} 0 \\ \tilde{V}_+ \\ \tilde{V}_- \end{bmatrix} = \begin{bmatrix} 0 \\ (1-\alpha)^{-1}\tilde{E}_+^{\Delta} \\ (1-\alpha^2)^{-1}\tilde{E}_-^{\Delta} \end{bmatrix} - \frac{z^{ab}}{3}\begin{bmatrix} 0 \\ \tilde{I}_+ \\ \tilde{I}_- \end{bmatrix}, \qquad \tilde{I}_0 = 0$$

2. **Interpretation:** voltage sources on positive and negative-sequence networks:

zero-seq voltage source: null $(\tilde{I}_0 = 0, \tilde{Z}_0 = \infty,$ open circuit)

positive-seq voltage source: $\tilde{V}_+ = \dfrac{E_+^{\Delta}}{1-\alpha} - \dfrac{z^{ab}}{3}\tilde{I}_+$ voltage source

negative-seq voltage source: $\tilde{V}_- = \dfrac{E_-^{\Delta}}{1-\alpha^2} - \dfrac{z^{ab}}{3}\tilde{I}_-$ voltage source

Sequence voltage source

Δ configuration $\left(E^{\Delta}, z^{\Delta}\right)$

1. If $z^{ab} = z^{bc} = z^{ca}$ and $E^{\Delta} = E^{ab}\alpha_+$ is balanced:

$$\begin{bmatrix} 0 \\ \tilde{V}_+ \\ \tilde{V}_- \end{bmatrix} = \begin{bmatrix} 0 \\ e^{-i\pi/6} E^{ab} \\ 0 \end{bmatrix} - \frac{z^{ab}}{3} \begin{bmatrix} 0 \\ \tilde{I}_+ \\ \tilde{I}_- \end{bmatrix}$$

2. **Interpretation:** voltage source in positive-seq network and impedance on negative-seq network:

zero-seq voltage source: null $\left(\tilde{I}_0 = 0, \tilde{Z}_0 = \infty,\right.$ open circuit)

positive-seq voltage source: $\tilde{V}_+ = e^{-i\pi/6} E^{ab} - \frac{z^{ab}}{3}\tilde{I}_+$ voltage source

negative-seq impedance: $\tilde{V}_- = -\frac{z^{ab}}{3}\tilde{I}_-$ impedance

Sequence current source

Y configuration $\left(J^Y, y^Y, z^n\right)$

1. External model (from Ch 8) is

$$I \;=\; -J^Y \;-\; y^Y\left(V - V^n 1\right)$$

2. Substituting $V = F\tilde{V}$, $I = F\tilde{I}$ to convert to sequence coordinate

$$\tilde{I} \;=\; -\underbrace{\overline{F}J^Y}_{\tilde{J}^Y} - \underbrace{\overline{F}y^Y F}_{\tilde{Y}^Y}\tilde{V} \;+\; V^n\overline{F}y^Y 1$$

Sequence current source

Y configuration $\left(J^Y, y^Y, z^n\right)$

1. If admittance $y^Y := y^{an}\mathbb{I}$ is balanced, then under assumption C8.1, external model in sequence coordinate becomes decoupled (though unbalanced):

$$\begin{bmatrix} \left(1 + 3\, y^{an}\, z^n\right) \tilde{I}_0 \\ \tilde{I}_+ \\ \tilde{I}_- \end{bmatrix} = - \begin{bmatrix} \tilde{J}_0^Y \\ \tilde{J}_+^Y \\ \tilde{J}_-^Y \end{bmatrix} - y^{an} \begin{bmatrix} \tilde{V}_0 \\ \tilde{V}_+ \\ \tilde{V}_- \end{bmatrix}$$

2. **Interpretation:** current sources on 3 decoupled sequence networks:

zero-seq current source: $$\tilde{I}_0 = - \frac{\tilde{J}_0^Y}{1 + 3\, y^{an}\, z^n} - \frac{y^{an}}{1 + 3\, y^{an}\, z^n} \tilde{V}_0$$

positive-seq current source: $$\tilde{I}_+ = - \tilde{J}_+^Y - y^{an} \tilde{V}_+$$

negative-seq current source: $$\tilde{I}_- = - \tilde{J}_-^Y - y^{an} \tilde{V}_-$$

Sequence current source

Y configuration $\left(J^Y, y^Y, z^n\right)$

1. If $y^Y := y^{an}\mathbf{1}$ and $J^Y := J^{an}\alpha_+$ is balanced then the sequence networks become

zero-seq admittance: $\tilde{I}_0 \;=\; -\dfrac{y^{an}}{1 + 3\,y^{an}\,z^n}\,\tilde{V}_0$ — admittance

positive-seq current source: $\tilde{I}_+ = -\sqrt{3}J^{an} \;-\; y^{an}\tilde{V}_+$ — current source

negative-seq admittance: $\tilde{I}_- = -\,y^{an}\tilde{V}_-$ — admittance

Sequence current source

Δ configuration $\left(J^{\Delta}, y^{\Delta}\right)$

1. External model (from Ch 8) is

$$I \;=\; -\left(\Gamma^{\mathsf{T}} J^{\Delta} \;+\; Y^{\Delta} V\right)$$

2. Substituting $V = F\tilde{V}$, $I = F\tilde{I}$ to convert to sequence coordinate

$$\tilde{I} \;=\; -\left(\underbrace{\overline{F}\Gamma^{\mathsf{T}} J^{\Delta}}_{\tilde{J}^{\Delta}} + \underbrace{\overline{F} Y^{\Delta} F}_{\tilde{Y}^{\Delta}} \tilde{V}\right) \;=:\; -\left(\tilde{J}^{\Delta} \;+\; \tilde{Y}^{\Delta}\tilde{V}\right)$$

Sequence current source

Δ configuration $\left(J^{\Delta}, y^{\Delta}\right)$

1. If admittance $y^{\Delta} := y^{ab}\mathbb{1}$ is balanced, then external model in sequence coordinate becomes decoupled (though unbalanced):

$$\begin{bmatrix} \tilde{I}_0 \\ \tilde{I}_+ \\ \tilde{I}_- \end{bmatrix} = - \begin{bmatrix} \tilde{J}_0^{\Delta} \\ \tilde{J}_+^{\Delta} \\ \tilde{J}_-^{\Delta} \end{bmatrix} - 3y^{ab} \begin{bmatrix} 0 \\ \tilde{V}_+ \\ \tilde{V}_- \end{bmatrix}$$

2. **Interpretation:** current sources on 3 decoupled sequence networks:

zero-seq current source:	$\tilde{I}_0 = -\tilde{J}_0^{\Delta}$	ideal current source
positive-seq current source:	$\tilde{I}_+ = -\tilde{J}_+^{\Delta} - 3y^{ab}\tilde{V}_+$	non-ideal current source
negative-seq current source:	$\tilde{I}_- = -\tilde{J}_-^{\Delta} - 3y^{ab}\tilde{V}_-$	non-ideal current source

Sequence current source

Δ configuration $\left(J^{\Delta}, y^{\Delta}\right)$

1. If $y^{\Delta} := y^{ab}\mathbb{I}$ and $J^{\Delta} := J^{ab}\alpha_+$ is balanced then the sequence networks become

zero-seq current source:	null $\quad (\tilde{I}_0 = 0)$	open circuit (no device)
positive-seq current source:	$\tilde{I}_+ = -3e^{-i\pi/6}J^{ab} - 3y^{ab}\tilde{V}_+$	current source
negative-seq admittance:	$\tilde{I}_- = -3y^{ab}\tilde{V}_-$	admittance $3y^{ab}$

Sequence line

1. Line model with zero shunt admittances

$$V_j \;-\; V_k \;=\; z^s_{jk}\, I_{jk}$$

2. Substituting $V = F\tilde{V},\ I = F\tilde{I}$ to convert to sequence coordinate

$$\tilde{V}_j \;-\; \tilde{V}_k \;=\; \underbrace{\left(\overline{F} z^s_{jk} F\right)}_{\tilde{z}^s_{jk}} \tilde{I}_{jk} \;=:\; \tilde{z}^s_{jk}\, \tilde{I}_{jk}$$

Sequence line

1. If phase impedance matrix z_{jk}^s is symmetric:

$$z_{jk}^s = \begin{bmatrix} z^1 & z^2 & z^2 \\ z^2 & z^1 & z^2 \\ z^2 & z^2 & z^1 \end{bmatrix}$$

then the sequence impedance matrix $\tilde{z}_{jk}^s$ is diagonal (decoupled):

$$\tilde{z}_{jk}^s = \begin{bmatrix} z^1 + 2z^2 & 0 & 0 \\ 0 & z^1 - z^2 & 0 \\ 0 & 0 & z^1 - z^2 \end{bmatrix}$$

Sequence line

2. **Interpretation:** the 3-phase line becomes 3 separate (decoupled) sequence networks

zero-seq impedance: $\tilde{V}_{j,0} - \tilde{V}_{k,0} = \left(z^1 + 2z^2\right) \tilde{I}_{jk,0}$

positive-seq impedance: $\tilde{V}_{j,+} - \tilde{V}_{k,+} = \left(z^1 - z^2\right) \tilde{I}_{jk,+}$

negative-seq impedance: $\tilde{V}_{j,-} - \tilde{V}_{k,-} = \left(z^1 - z^2\right) \tilde{I}_{jk,-}$

Outline

1. Network models: BIM
2. Three-phase analysis
3. Balanced network
4. Symmetric network
    - Sequence impedances and sources
    - Sequence line
    - Three-phase analysis

Symmetric network

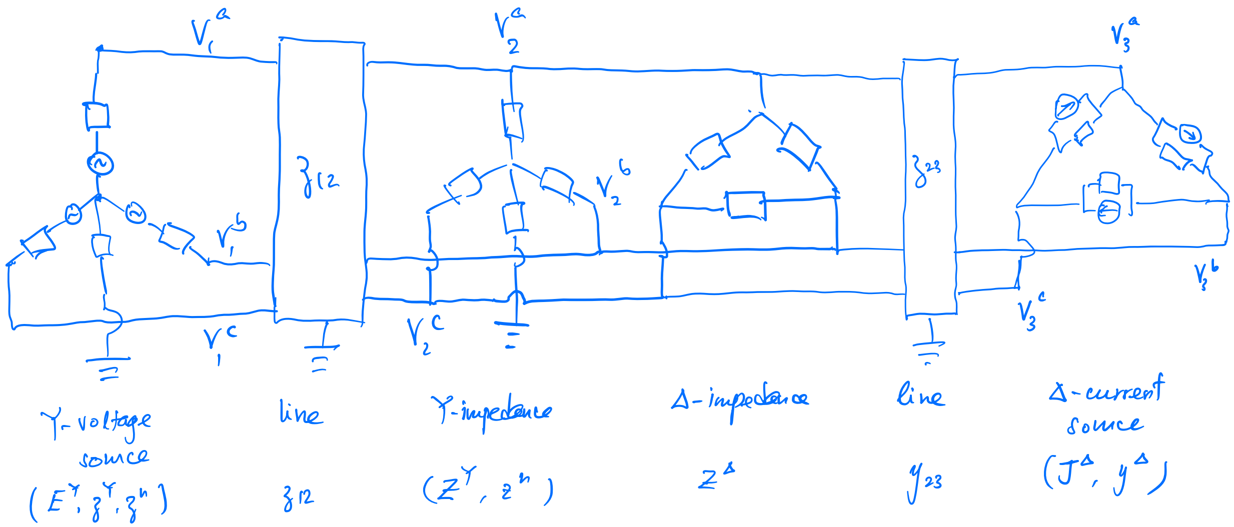
A 3-phase network is symmetric if

1. All impedances are symmetric, $z_j^{Y/\Delta} = z_j^{an/ab}\mathbb{1}$
2. All voltage sources have symmetric series impedances $z_j^{Y/\Delta} = z_j^{an/ab}\mathbb{1}$
3. All current sources have symmetric shunt admittances $y_j^{Y/\Delta} = y_j^{an/ab}\mathbb{1}$
4. All lines (j, k) have symmetric series impedances $z_{jk}^s = \begin{bmatrix} z_{jk}^1 & z_{jk}^2 & z_{jk}^2 \\ z_{jk}^2 & z_{jk}^1 & z_{jk}^2 \\ z_{jk}^2 & z_{jk}^2 & z_{jk}^1 \end{bmatrix}$ and zero shunt admittances

It can be shown that its sequence networks are decoupled (see textbook)

Example

Symmetric network

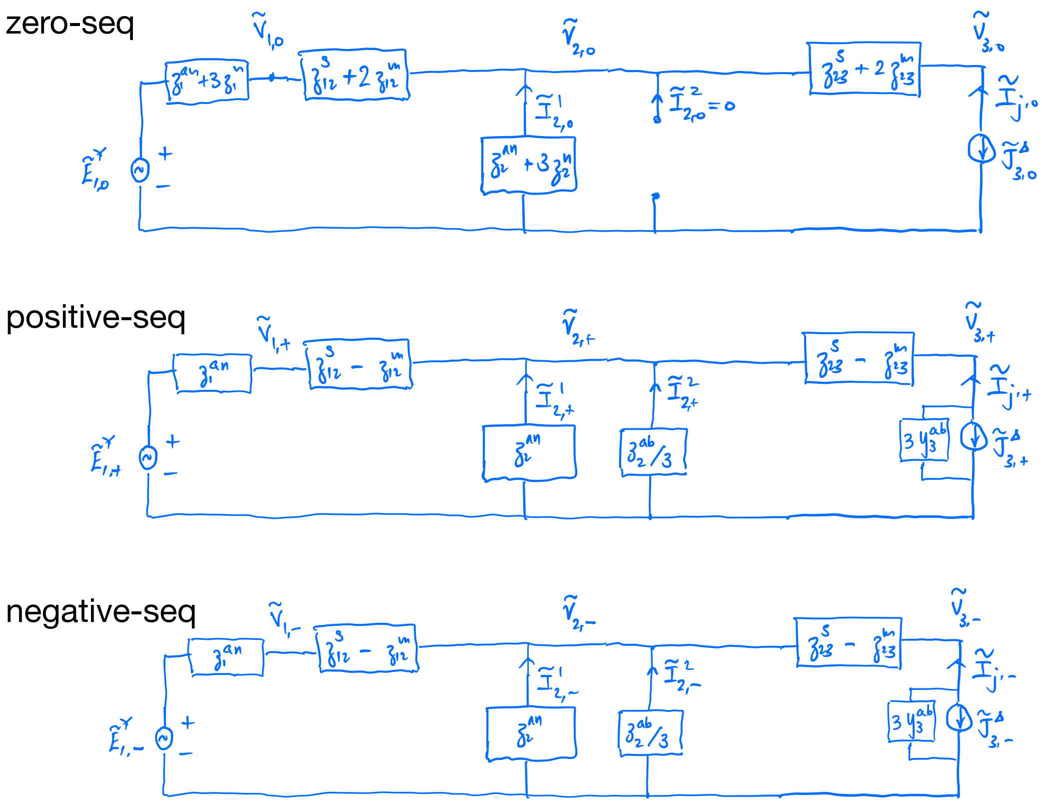


Calculate

1. Terminal load voltage $V_2 := \left(V_2^a, V_2^b, V_2^c\right)$
2. Internal current $I_2^Y := \left(I_2^{an}, I_2^{bn}, I_2^{cn}\right)$ and total complex power $1^\mathsf{T} s_2^Y$ delivered to Y-configured load
3. Internal current $I_2^\Delta := \left(I_2^{ab}, I_2^{bc}, I_2^{ca}\right)$ and total complex power $1^\mathsf{T} s_2^\Delta$ delivered to Δ-configured load

Example

Sequence networks



Solution strategy

1. Construct sequence networks (decoupled)
2. Determine terminal sequence voltage $\tilde{V}_2$ and terminal sequence currents $\tilde{I}_2^1, \tilde{I}_2^2$
3. Terminal phase variables are then

$$V_2 = F\tilde{V}_2, \quad I_2^1 = F\tilde{I}_2^1, \quad I_2^2 = F\tilde{I}_2^2$$

4. Determine internal currents $\left(I_2^Y, I_2^\Delta\right)$ and power $\left(s_2^Y, s_2^\Delta\right)$ using conversion rules

Example

Solution sketch

1. Determine terminal sequence voltage $\tilde{V}_2$ by analyzing each sequence network separately
2. Terminal sequence load currents are then, in terms of $\tilde{V}_2$

$$\tilde{I}^1_{2,0} = -\frac{\tilde{V}_{2,0}}{z_2^{an} + 3z_2^n}, \quad \tilde{I}^1_{2,+} = -\frac{\tilde{V}_{2,+}}{z_2^{an}}, \quad \tilde{I}^1_{2,-} = -\frac{\tilde{V}_{2,-}}{z_2^{an}} \quad \tilde{I}^2_{2,0} = 0, \quad \tilde{I}^2_{2,+} = -\frac{3\tilde{V}_{2,+}}{z_2^{ab}}, \quad \tilde{I}^2_{2,-} = -\frac{3\tilde{V}_{2,-}}{z_2^{ab}}$$

3. Terminal phase variables are then

$$V_2 = F\tilde{V}_2, \;\; I_2^1 = F\tilde{I}_2^1, \;\; I_2^2 = F\tilde{I}_2^2$$

4. Internal voltages are (under assumption C8.1) and currents are

$$V_2^Y = V_2 - V_2^n \mathsf{1} = V_2 + z_2^n \left(\mathsf{1}\mathsf{1}^{\mathsf{T}}\right) I_2^1, \qquad V_2^\Delta = \Gamma V_2$$

$$I_2^Y = -I_2^1, \qquad I_2^\Delta = -\frac{1}{3}\Gamma I_2^2 + \beta_2 \mathsf{1}$$

5. Hence load powers are (total power $\mathsf{1}^{\mathsf{T}} s_2^\Delta$ is independent of β_2

$$s_2^Y := \text{diag}\left(V_2^Y I_2^{Y\mathsf{H}}\right) = -\text{diag}\left(V_2 I_2^{1\mathsf{H}} + z_2^n \left(\mathsf{1}\mathsf{1}^{\mathsf{T}}\right) I_2^1 I_2^{1\mathsf{H}}\right)$$

$$s_2^\Delta := \text{diag}\left(V_2^\Delta I_2^{\Delta\mathsf{H}}\right) = -\text{diag}\left(\Gamma V_2 I_2^{2\mathsf{H}} \Gamma^\dagger\right) + \bar{\beta}_2 \Gamma V_2$$