Course info: EE/CS/EST 135

Text:

Power System Analysis

Analytical tools and structural properties

STEVEN H. LOW

California Institute of Technology
slow@caltech.edu

Draft at: https://netlab.caltech.edu/book_reg/

Units: (lecture-lab-prep) = (3-3-3), letter or P/F
Lectures: Tue/Thur 10:30-11:55am, Rm 314 ANB
Instructor: Steven Low, slow@caltech.edu, Rm 219 ANB
Office hour: Just email me any time to arrange

Grading: attendance/participation, homework, project

Course info

Course AI Policy: based on honor code

Use AI tools if and only if it is net help to your learning of course material

Implications:

- You should learn not just power system knowledge, but also how to think about power system problems
  $\Longrightarrow$ If it replaces your thinking through the materials, it impedes your learning
- One size cannot fit all: It is impossible to formulate precise rules that fit all situations, so use your discretion to interpret “net”, “help”, “learning”, “material” in each situation

Pace: You drive the pace

- Default plan: Part I of PSA (mostly)
- I will adapt to your speed
- Let me know how the course can be made useful to your goals!

Why take this course

1. The topic is important?
2. The topic is interesting?
3. I want to pursue PhD in energy transition?
4. I want to pursue entrepreneurship in energy transition?
5. I want to work in energy industry (e.g. utilities)?
6. Others: ____________________________

Power System Analysis

Chapter 1 Basic concepts

Steven Low Caltech (Oct 3, 2025)

Outline

1. Single-phase systems
2. Balanced three-phase systems
3. Complex power

Outline

1. Single-phase systems
    - Phasor representation
    - Single-phase devices
    - Linear circuit analysis
    - One-line diagram and equivalent circuit
2. Balanced three-phase systems
3. Complex power

Voltage and current phasors

Steady state behavior

Quantities of interest in power systems

- Voltage $v(t)$ at a point: energy required to move a unit of charge from an (arbitrary but fixed) reference point to that point (Volt, V)
- Current $i(t)$ at a point: flow rate of electric charge through that point (Ampere, A)
- Instantaneous power $p(t) := v(t)i(t)$: rate of energy transfer when a unit of charge is moved through a voltage (potential difference) between two points (Watt, W)

In an AC (alternating current) system, they are sinusoidal functions of time with frequency ω (Hz)

- Voltage $v(t) = V_{\text{max}} \cos(\omega t + \theta_V)$
- Current $i(t) = I_{\text{max}} \cos(\omega t + \theta_I)$
- Power $p(t) := v(t)i(t)$

Steady state = frequency ω is fixed (constant over time) and the same everywhere in the system

- The voltage $v(t)$ and current $i(t)$ are completely specified by their amplitude and (phase) angle
- Nominal frequency: 60 Hz in US, 50 Hz in Europe, China

Voltage phasor

1. Voltage: $v(t) = V_{\text{max}} \cos(\omega t + \theta_V) = \text{Re}\left\{V_{\text{max}} e^{i\theta_V} \cdot e^{i\omega t}\right\}$
    - ω : nominal system frequency
    - V_{max} : amplitude
    - θ_V : phase angle
2. Phasor: $V := \dfrac{V_{\text{max}}}{\sqrt{2}} e^{i\theta_V}$ volt (V)
3. Relationship: $v(t) = \text{Re}\{\sqrt{2} V \cdot e^{i\omega t}\}$
4. Voltage magnitude $|V|$: the root-mean-square (RMS) value

$$|V| = \sqrt{\frac{1}{T}\int_0^{\text{T}} v^2(t)dt} = \sqrt{\frac{1}{T}\int_0^{\text{T}} V_{\text{max}}^2 \cos^2(\omega t + \theta_V)dt}$$

Current phasor

1. Voltage: $i(t) = I_{\text{max}} \cos(\omega t + \theta_I) = \text{Re}\left\{I_{\text{max}} e^{i\theta_I} \cdot e^{i\omega t}\right\}$
   - ω : nominal system frequency
   - I_{max} : amplitude
   - θ_I : phase angle
2. Phasor: $I := \dfrac{I_{\text{max}}}{\sqrt{2}} e^{i\theta_I}$ ampere (A)
3. Relationship: $i(t) = \text{Re}\{\sqrt{2} I \cdot e^{i\omega t}\}$
4. Current magnitude $|I|$: the root-mean-square (RMS) value

$$|I| = \sqrt{\frac{1}{T}\int_0^{\text{T}} i^2(t)dt} = \sqrt{\frac{1}{T}\int_0^{\text{T}} I_{\text{max}}^2 \cos^2(\omega t + \theta_I)dt}$$

Single-phase devices

Impedance z

Resistor: $v(t) = ri(t)$ (Ohm's law)

Inductor: $v(t) = l\frac{di(t)}{dt}$

Capacitor: $i(t) = c\frac{dv(t)}{dt}$

these are basic circuit elements to model the grid

Single-phase devices

Impedance z

Resistor: $v(t) = ri(t)$ (Ohm's law)

- Hence $\text{Re}\left(\sqrt{2}\,|V| \cdot e^{i(\omega t+\theta_V)}\right) = \text{Re}\left\{rI \cdot \sqrt{2}e^{i\omega t}\right\}$
- In phasor domain: $V = rI$

Inductor: $v(t) = l\dfrac{di(t)}{dt}$

- Hence $\text{Re}\left(\sqrt{2}\,|V| \cdot e^{i(\omega t+\theta_V)}\right) = \text{Re}\left\{i\omega lI \cdot \sqrt{2}e^{i\omega t}\right\}$
- In phasor domain: $V = (i\omega l)I$

Capacitor: $i(t) = c\dfrac{dv(t)}{dt}$

- Hence $\text{Re}\left\{I \cdot \sqrt{2}e^{i\omega t}\right\} = \text{Re}\left(i\omega c\,|V|\sqrt{2} \cdot e^{i(\omega t+\theta_V)}\right)$
- In phasor domain: $V = (i\omega c)^{-1}I$

these are basic circuit elements to model the grid

Single-phase devices

Impedance z

Impedance or admittance: $V = zI,\ I = yV$

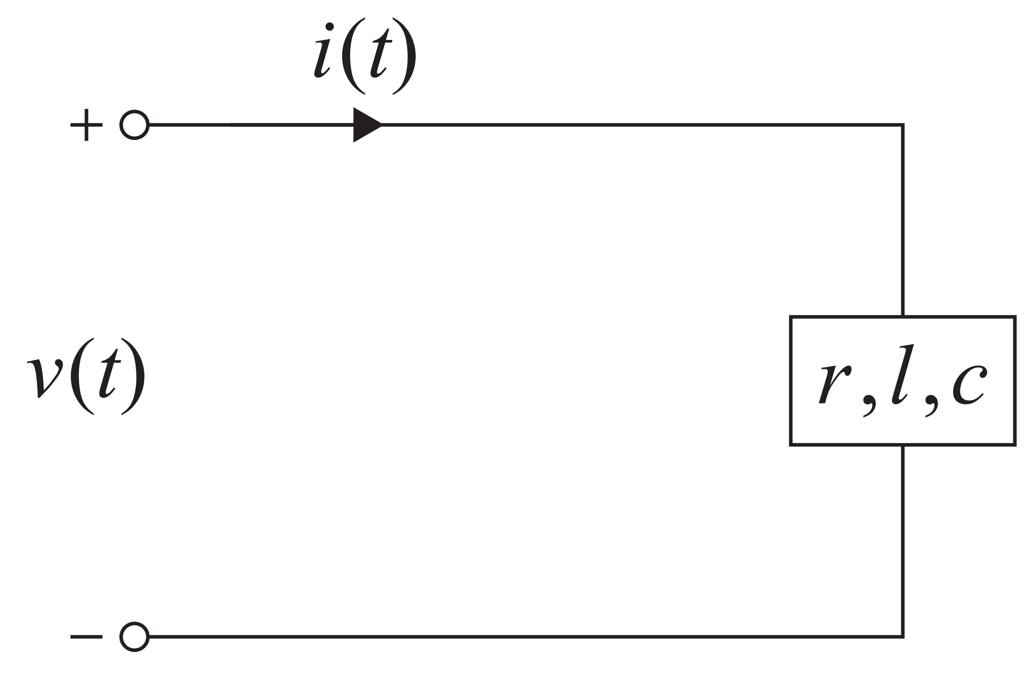
- resistor: $z_r := r$, inductor: $z_l := i\omega l$, capacitor: $z_c := \dfrac{1}{i\omega c}$

In general, impedance $z = r + ix$

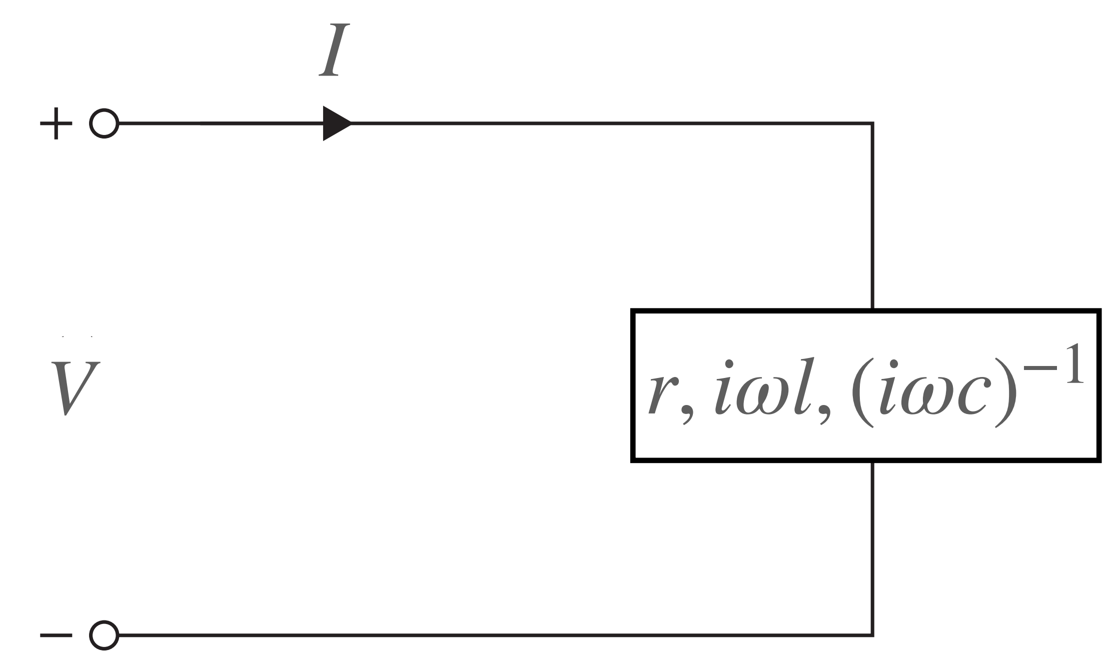
- r : resistance Ω
- x : reactance Ω

Admittance $y = z^{-1} = g + ib$

- $g := r/(r^2 + x^2)$: conductance Ω^{-1}
- $b := -x/(r^2 + x^2)$: susceptance Ω^{-1}



(a) Time domain



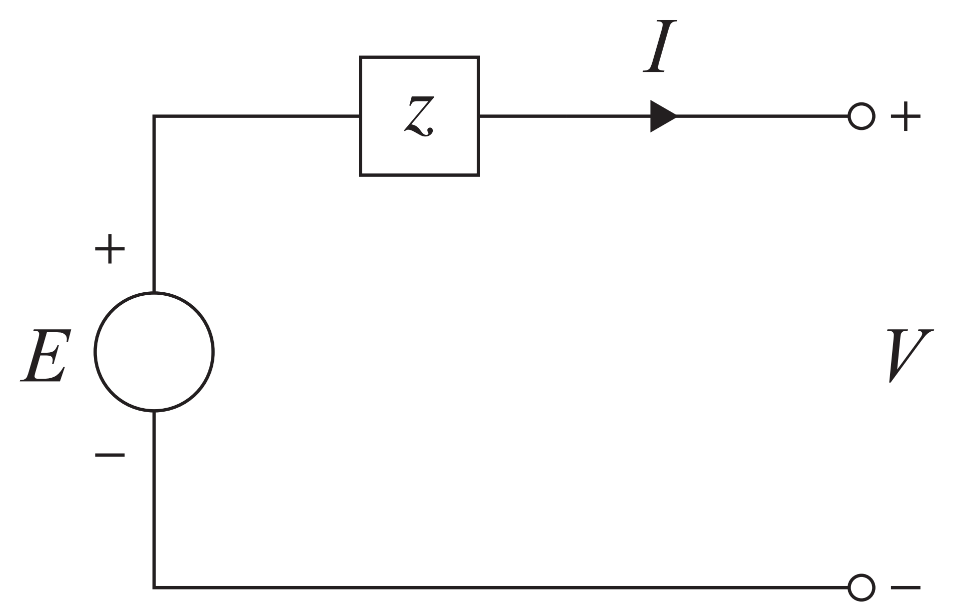
(b) Phasor domain

Single-phase devices

Sources

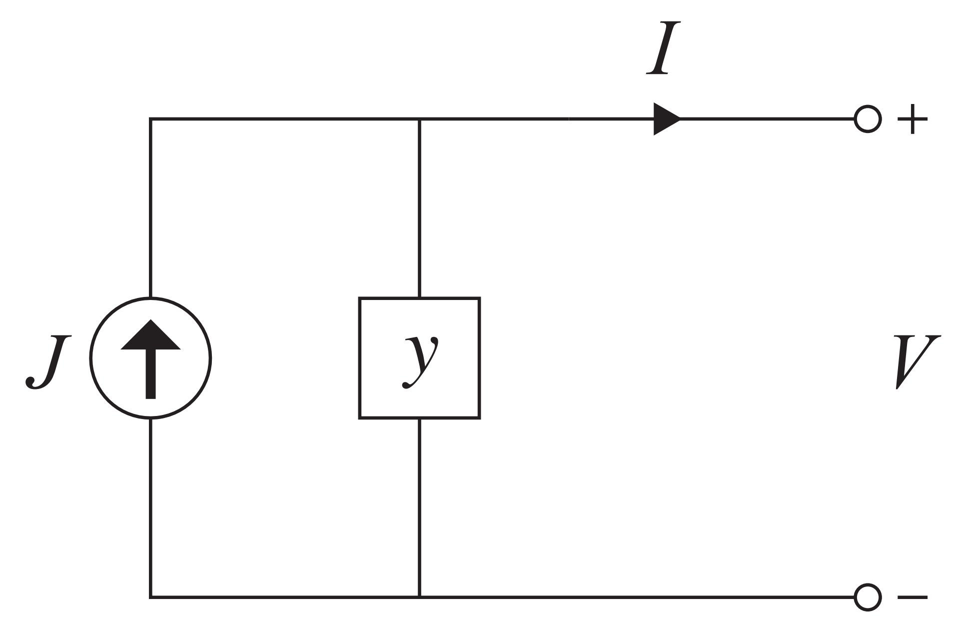
Voltage source (E, z)

- Internal variables: internal voltage E; series impedance z
- Terminal variables: terminal voltage V; terminal current I
- External model: $V = E - zI$



Current source (J, y)

- Internal variables: internal current J; shunt admittance y
- Terminal variables: terminal voltage V; terminal current I
- External model: $I = J - yV$



Voltage source (E, z) and current source (J, y) are equivalent (same external model) if

$$J = \frac{E}{z}, \qquad y = \frac{1}{z}$$

Ideal sources: $z = 0, \ y = 0$

Devices are models

Circuit elements commonly used for modeling generators, loads, lines and transformers

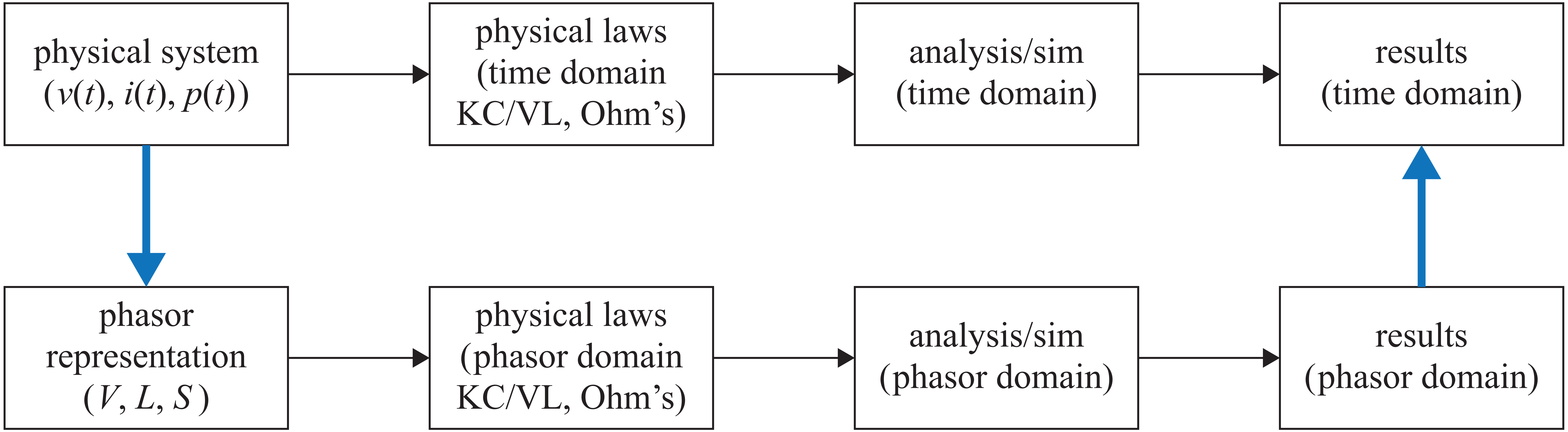
Device	Circuit model
Generator	Voltage source, current source, power source
Load	Impedance, current source, voltage source, power source
Line	Impedance (Chapter 2)
Transformer	Impedance, voltage/current gain (Chapter 3)

Outline

1. Single-phase systems
   - Phasor representation
   - Single-phase devices
   - Linear circuit analysis
   - One-line diagram and equivalent circuit
2. Balanced three-phase systems
3. Complex power

Circuit analysis

Phasor domain



Circuit analysis: review

A brief review of circuit analysis for EE students

Mathematical background required

- Basic algebraic graph theory (see Appendix: Linear algebra preliminaries)

Circuit analysis

Circuit model

A circuit is represented by a directed graph $\hat{G} := (\hat{N}, \hat{E})$ with arbitrary orientation

- $\hat{N} := \{\text{nodes/buses}\}$
- $\hat{E} := \{\text{lines/links/branches/edges}\} \subseteq \hat{N} \times \hat{N}$

links denoted by (j, k) or $j \to k$

- There can be multiple links between two nodes, e.g., $l_1 = j \to k$, $l_2 = k \to j$

Associated with each link $l = j \to k$ are two vars

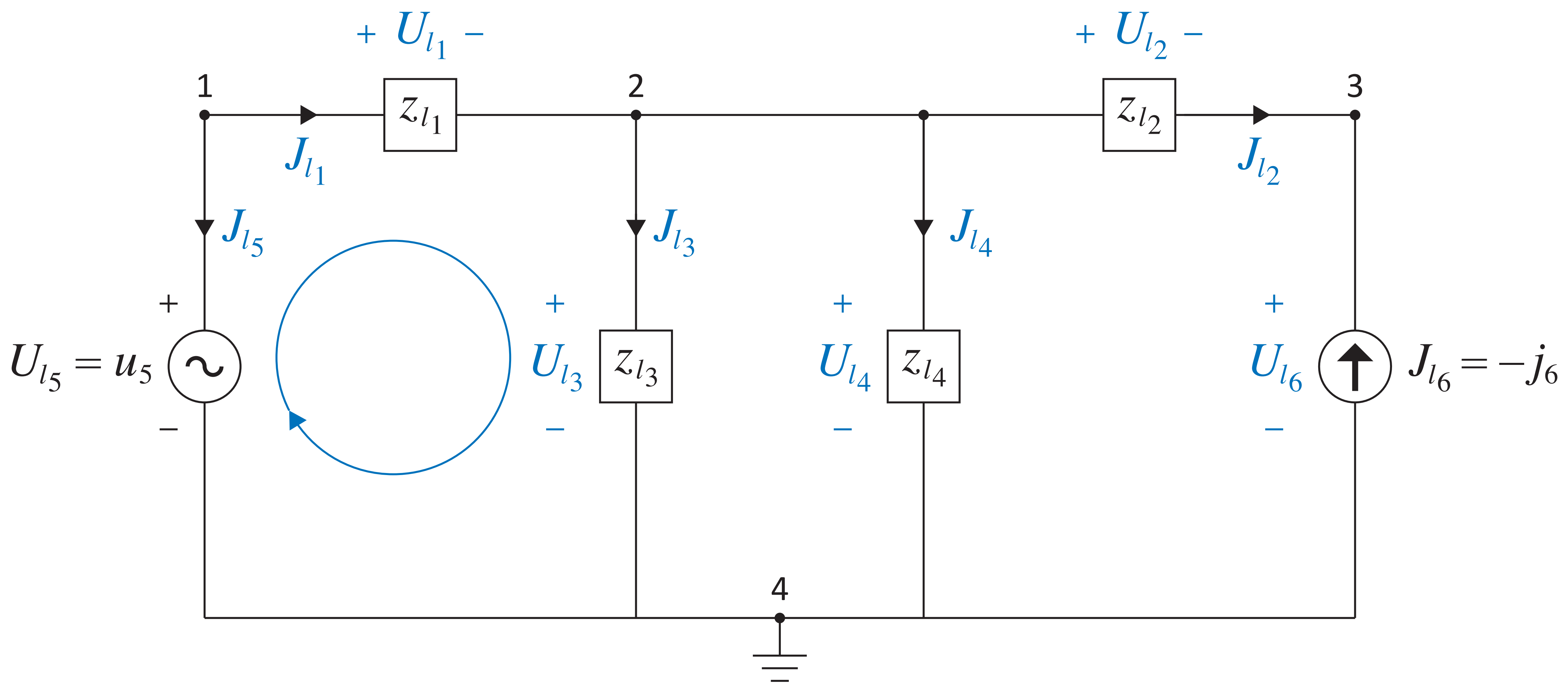
- U_l : branch voltage across link l in direction of l
- J_l : branch current from j to k

Each link l represents one device

- Impedance z_l : $U_l = z_l J_l$ (Ohm's law)
- (Ideal) voltage source u_l : $U_l = u_l$ given
- (Ideal) current source j_l : $J_l = j_l$ given

Circuit analysis

Example



(a) Circuit

$$
\hat{C} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{cccc|c|c}
l_1 & l_2 & l_3 & l_4 & l_5 & l_6 \\
\hline
1 & & & & 1 & \\
-1 & 1 & 1 & 1 & & \\
 & -1 & & & & 1 \\
 & & -1 & -1 & -1 & -1
\end{array}
$$

$\underbrace{\qquad\qquad}_{\hat{C}_1}$ $\underbrace{\quad}_{\hat{C}_2}$ $\underbrace{\quad}_{\hat{C}_3}$

(b) Incidence matrix

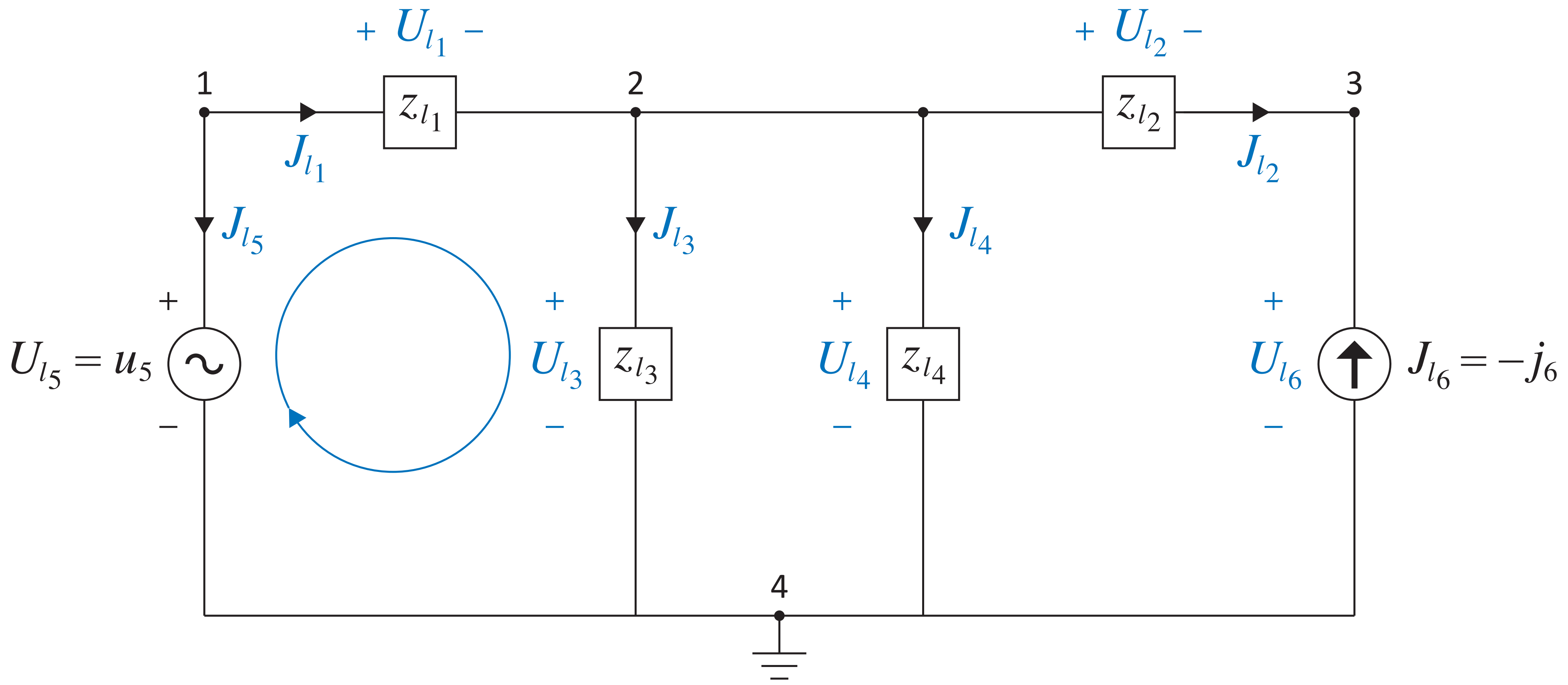
Figure 1.4 A circuit represented as a directed graph $\hat{G}$ where each link l is either an impedance z_l, a voltage source U_l, or a current source J_l. The voltage source $U_{l_5} = u_5$ and current source $J_{l_6} = -j_6$ are given. Its incidence matrix $\hat{C}$ is partitioned into $\hat{C}_1, \hat{C}_2, \hat{C}_3$ corresponding to the impedances, the voltage source, and the current source respectively.

Circuit analysis

KCL, KVL

Kirchhoff's current law (KCL): incident currents at any node j sum to zero

- At all nodes j: $$-\sum_{i:i\to j\in\hat{E}} J_{ij} + \sum_{k:j\to k\in\hat{E}} J_{jk} = 0$$
- Example: at node 2, $-J_{l_1} + J_{l_2} + J_{l_3} + J_{l_4} = 0$



Circuit analysis

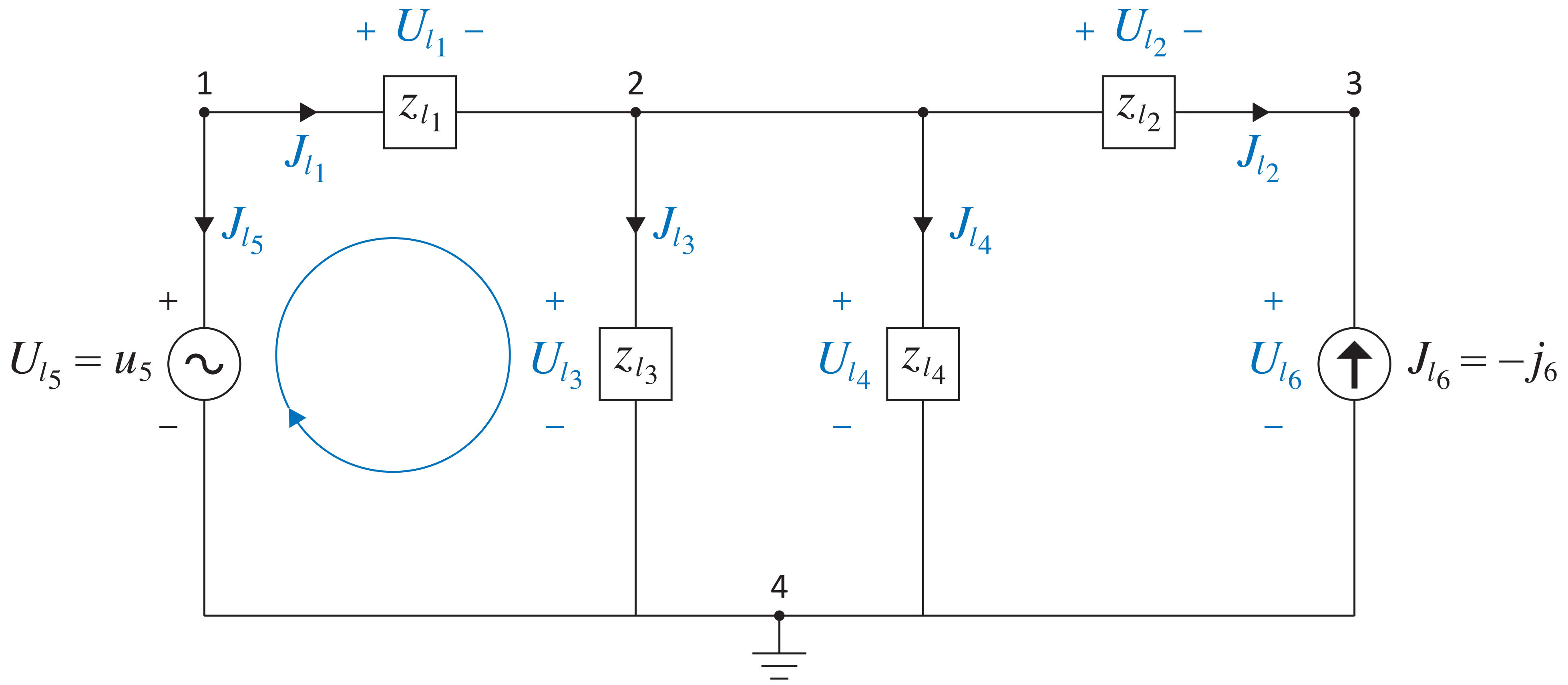
KCL, KVL

Kirchhoff's current law (KCL): incident currents at any node j sum to zero

- At all nodes j: $$-\sum_{i:i\to j\in\hat{E}} J_{ij} + \sum_{k:j\to k\in\hat{E}} J_{jk} \;=\; 0$$
- Example: at node 2, $-J_{l_1} + J_{l_2} + J_{l_3} + J_{l_4} = 0$

Kirchhoff's voltage law (KCL): voltage drops around any cycle c sum to zero

- Around all cycles c: $$\sum_{l\in c} U_l - \sum_{-l\in c} U_l \;=\; 0$$
- Example: $U_{l_1} + U_{l_3} - U_{l_5} = 0$



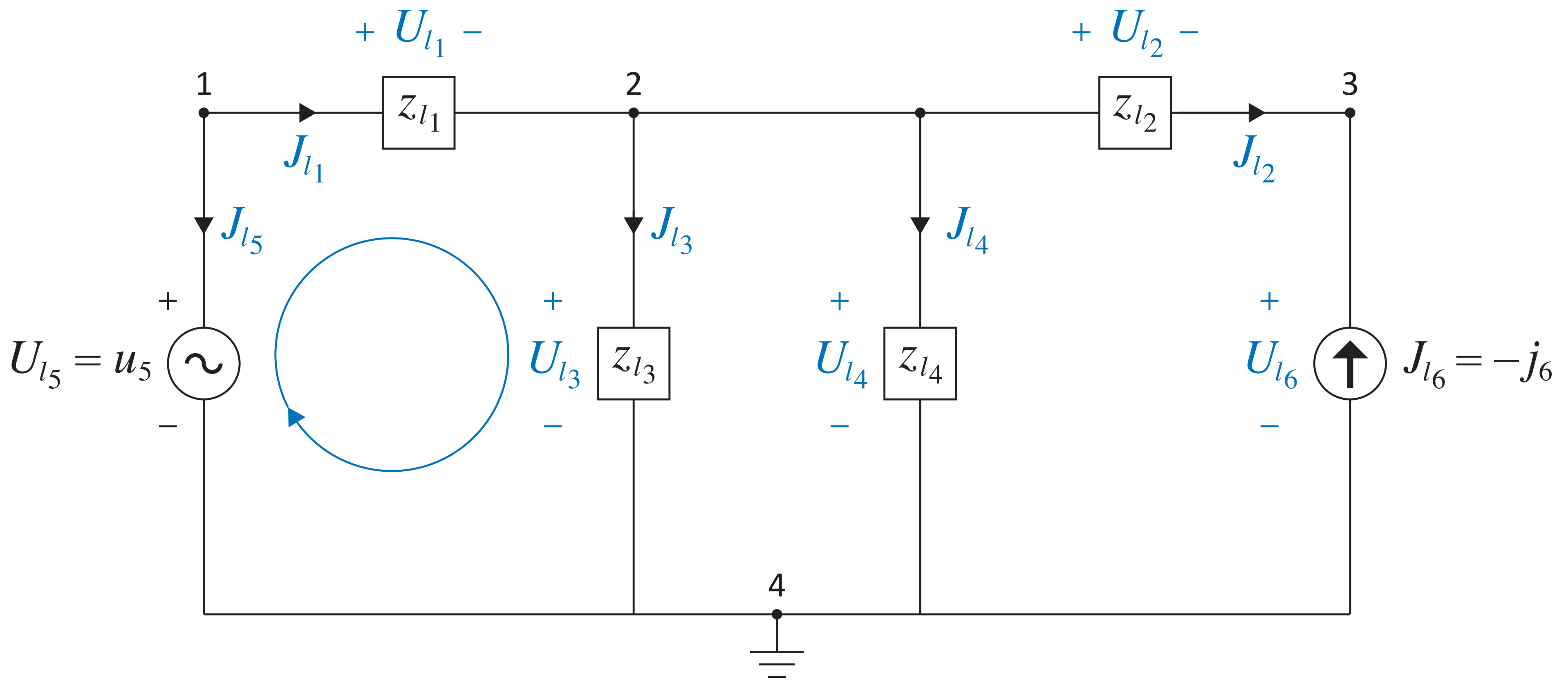
Circuit analysis

KCL, KVL

Can represent KCL, KVL compactly in vector form

Let $\hat{C} \in \{-1,0,1\}^{|\hat{N}|\times|\hat{E}|}$ be the node-by-link incidence matrix

$$\hat{C}_{jl} := \begin{cases} 1 & \text{if } l = j \to k \text{ for some bus } k \\ -1 & \text{if } l = i \to j \text{ for some bus } i \\ 0 & \text{otherwise} \end{cases}, \qquad j \in \hat{N},\ l \in \hat{E}$$



(a) Circuit

$$\hat{C} = \begin{array}{c|cccc:c:c|} & l_1 & l_2 & l_3 & l_4 & l_5 & l_6 \\ 1 & 1 & & & & 1 & \\ 2 & -1 & 1 & 1 & 1 & & \\ 3 & & -1 & & & & 1 \\ 4 & & & -1 & -1 & -1 & -1 \end{array}$$

$\hat{C}_1$ (l_1–l_4), $\hat{C}_2$ (l_5), $\hat{C}_3$ (l_6)

(b) Incidence matrix

Circuit analysis

KCL, KVL

Can represent KCL, KVL compactly in vector form

Let $\hat{C} \in \{-1,0,1\}^{|\hat{N}| \times |\hat{E}|}$ be the node-by-link incidence matrix

$$\hat{C}_{jl} := \begin{cases} 1 & \text{if } l = j \to k \text{ for some bus } k \\ -1 & \text{if } l = i \to j \text{ for some bus } i \\ 0 & \text{otherwise} \end{cases}, \qquad j \in \hat{N},\ l \in \hat{E}$$

Then

KCL: $\hat{C}J = 0$

KVL: $U = \hat{C}^{\mathsf{T}} V$ for some nodal voltage $V \in \mathbb{C}^{|\hat{N}|}$ (wrt reference node)

Arbitrary reference: WLOG let $\hat{N}$ be reference node, i.e., $V_{\hat{N}} := 0$

Circuit analysis

Problem formulation

Given: Circuit represented by $|\hat{N}| \times |\hat{E}|$ incidence matrix $\hat{C}$

- For every link $l \in \hat{E}$
  - Impedance z_l : $U_l = z_l J_l$ (Ohm's law)
  - (Ideal) voltage source u_l : $U_l = u_l$ given
  - (Ideal) current source j_l : $J_l = j_l$ given
- KCL: $\hat{C}J \ = \ 0$
- KVL: $U \ = \ \hat{C}^{\mathsf{T}} V$
- Reference voltage: $V_{\hat{N}} := 0$

Solve for: (V, J, U)

- $|\hat{N}| + 2|\hat{E}| + 1$ (complex) equations in $|\hat{N}| + 2|\hat{E}|$ unknowns, at most $|\hat{N}| + 2|\hat{E}|$ are linearly independent

Circuit analysis

Problem formulation

Partition lines into $\hat{E} =: \hat{E}_1 \cup \hat{E}_2 \cup \hat{E}_3$ with

- $\hat{E}_1$: impedances
- $\hat{E}_2$: voltage sources
- $\hat{E}_3$: current sources

Circuit analysis

Problem formulation

Solve for: (V, J, U)

$$\begin{bmatrix} 0 & \hat{C} & 0 \\ 0 & -Z & \mathbb{I}_{|\hat{E}_1|} \\ 0 & 0 & \mathbb{I}_{|\hat{E}_2|} \\ 0 & \mathbb{I}_{|\hat{E}_3|} & 0 \\ \hat{C}^\top & 0 & -\mathbb{I}_{|\hat{E}|} \\ e_{|\hat{N}|}^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} V \\ J \\ U \end{bmatrix} = \begin{bmatrix} 0_{|\hat{N}|} \\ 0_{|\hat{E}_1|} \\ u \\ j \\ 0_{|\hat{E}|} \\ 0_1 \end{bmatrix}$$

where $Z := \text{Diag}\left(z_l, l \in \hat{E}\right)$

Circuit analysis

Solution

Partition lines into $\hat{E} =: \hat{E}_1 \cup \hat{E}_2 \cup \hat{E}_3$ with

- $\hat{E}_1$: impedances
- $\hat{E}_2$: voltage sources
- $\hat{E}_3$: current sources

Let $(|\hat{N}| - 1) \times |\hat{E}|$ reduced incidence matrix C without reference row $|\hat{N}|$ and

- $C \; =: \; \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix}$ according to lines in $\hat{E}_1, \hat{E}_2, \hat{E}_3$

Partition variables in the same order

- $U \; := \; \begin{bmatrix} U_1 \\ u \\ U_3 \end{bmatrix}, \qquad J \; := \; \begin{bmatrix} J_1 \\ J_2 \\ j \end{bmatrix}$

ideal current sources (given)

ideal current sources (given)

Circuit analysis

Solution

Then problem becomes

$$\underbrace{\begin{bmatrix} 0 & C_1 & C_2 & 0 \\ C_1^\top & -Z & 0 & 0 \\ C_2^\top & 0 & 0 & 0 \\ C_3^\top & 0 & 0 & -\mathbb{I}_{|\hat{E}_3|} \end{bmatrix}}_{M} \begin{bmatrix} V_{-\hat{N}} \\ J_1 \\ J_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} -C_3 j \\ 0 \\ u \\ 0 \end{bmatrix}$$

where $Z := \text{Diag}\left(z_l, l \in \hat{E}\right)$

Theorem

Suppose graph $\hat{G}$ is connected. M is invertible if

- $Y_1 := C_1 Z^{-1} C_1^\top$ is invertible; and — $|\hat{N}| - 1$ admittance matrix of subgraph induced by impedances
- $C_2^\top Y_1^{-1} C_2$ is invertible — $|\hat{E}_2| \times |\hat{E}_2|$ matrix corresponding to voltage sources

Circuit analysis

Solution

Theorem [existence & uniqueness of solution]

Suppose graph $\hat{G}$ is connected. M is invertible if

- $Y_1 := C_1 Z^{-1} C_1^{\mathsf{T}}$ is invertible; and
- $C_2^{\mathsf{T}} Y_1^{-1} C_2$ is invertible

$|\hat{N}| - 1$ admittance matrix of subgraph induced by impedances

$|\hat{E}_2| \times |\hat{E}_2|$ matrix corresponding to voltage sources

Conditions imply:

- Subgraph with all non-reference nodes induced by impedances is connected (Then, Y_1 is always invertible if Z is real, i.e., resistive subnetwork)
- Current sources in $\hat{E}_3$ do not contain a cut set; hence j cannot violate KCL (currents on any cut sum to 0)
- C_2 is of full column rank, i.e., no voltage sources in $\hat{E}_2$ form a cycle; hence u cannot violate KVL
- These conditions are "almost" sufficient

Tellegen’s theorem

Tellegen’s theorem is consequence of 3 facts

- $\hat{C}^{|\hat{E}|} = \text{null}(\hat{C}) \oplus \text{range}\left(\hat{C}^{\mathsf{T}}\right)$ is direct sum
- KCL: $\hat{C}J = 0$, i.e., $J \in \text{null}(\hat{C})$
- KVL: $U = \hat{C}^{\mathsf{T}}V$, i.e., $U \in \text{range}\left(\hat{C}^{\mathsf{T}}\right)$

Therefore branch currents J and branch voltages U are orthogonal:

- $J^{\mathsf{H}}U = 0$ (Tellegen’s theorem)

J and U can be from different networks as long as they have the same incidence matrix $\hat{C}$ (topology) !

Outline

1. Single-phase systems
   - Phasor representation
   - Single-phase devices
   - Linear circuit analysis
   - One-line diagram and equivalent circuit
2. Balanced three-phase systems
3. Complex power

One-line diagram

A power system is often specified by a one-line diagram, not as a circuit

- The behavior of a one-line diagram is defined by its equivalent circuit

We formally define a one-line diagram and its equivalent circuit

Definition [One-line diagram]

A one-line diagram is a pair $(G, \mathbb{Y})$ where

- $G := (\overline{N}, E)$ is a graph
- $\mathbb{Y} := \left(y_{jk}^s, y_{jk}^m, y_{kj}^m,\ l = (j,k) \in E\right)$ is a set of line admittances
- $y_{jk}^s \in \mathbb{C}$: series admittance; $(y_{jk}^m, y_{kj}^m) \in \mathbb{C}^2$: shunt admittances

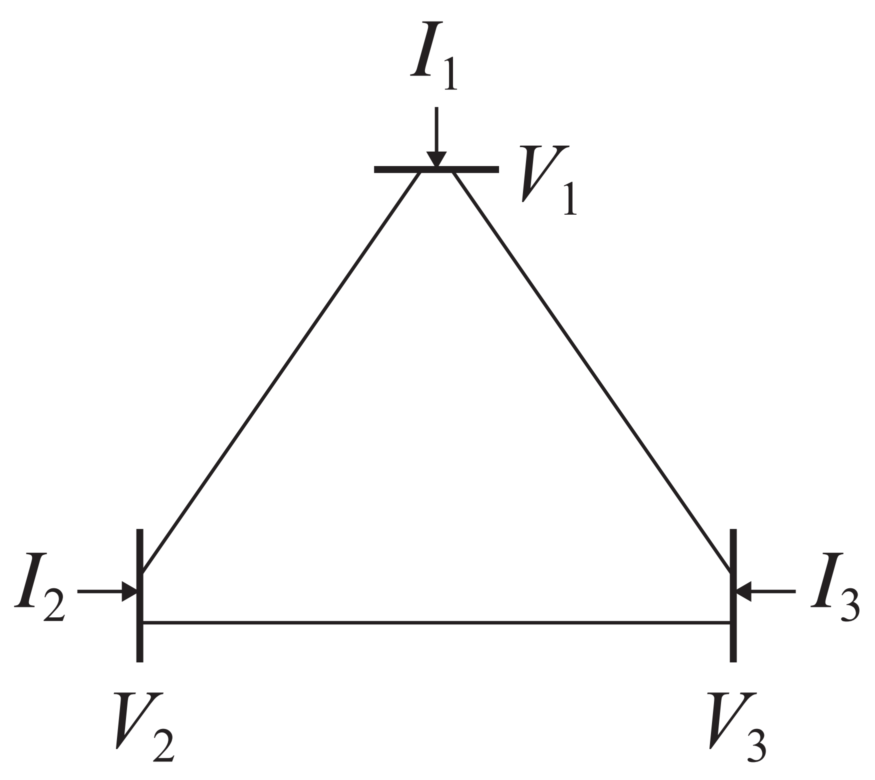
Can accommodate transformers with $\mathbb{Y} := \left((y_{jk}^s, y_{jk}^m), (y_{kj}^s, y_{kj}^m),\ l = (j,k) \in E\right)$

Equivalent circuit

Nodal devices

Associated with each node $j \in \overline{N}$ are two variables

- $V_j \in \mathbb{C}$: nodal voltage wrt common reference point (e.g., the ground)
- $I_j \in \mathbb{C}$: nodal (net) current injection (from node j to rest of network)



(a) Graph $G = (\overline{N}, E)$

$$\mathbb{Y} := \Big(\big(y_{12}^s, y_{12}^m, y_{21}^m\big), \big(y_{23}^s, y_{23}^m, y_{32}^m\big), \big(y_{31}^s, y_{31}^m, y_{13}^m\big) \Big)$$

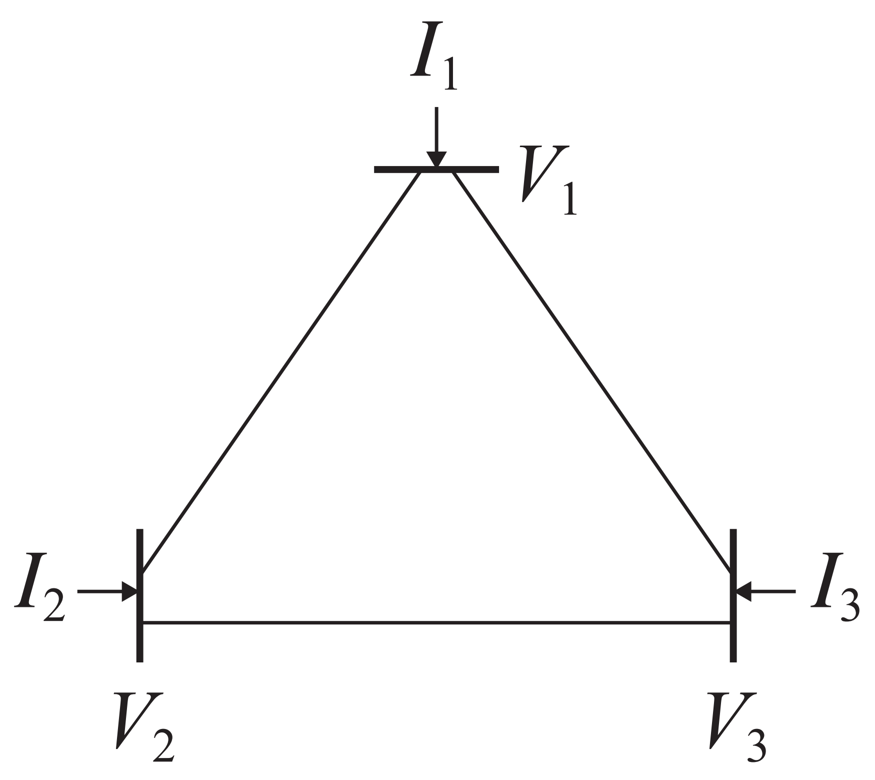
(b) Line parameters $\mathbb{Y}$

Equivalent circuit

Nodal devices

Associated with each node $j \in \overline{N}$ are two variables

- $V_j \in \mathbb{C}$: nodal voltage wrt common reference point (e.g., the ground)
- $I_j \in \mathbb{C}$: nodal (net) current injection (from node j to rest of network)
- Interpretation: a nodal device(s) is connected between node j and the common voltage reference point
  - Impedance z_j : $V_j = z_j I_j$
  - Ideal voltage source v_j : $V_j = v_j$
  - Ideal current source i_j : $I_j = i_j$



(a) Graph $G = (\overline{N}, E)$

$$\mathbb{Y} := \Big(\big(y_{12}^s, y_{12}^m, y_{21}^m\big), \big(y_{23}^s, y_{23}^m, y_{32}^m\big), \big(y_{31}^s, y_{31}^m, y_{13}^m\big) \Big)$$

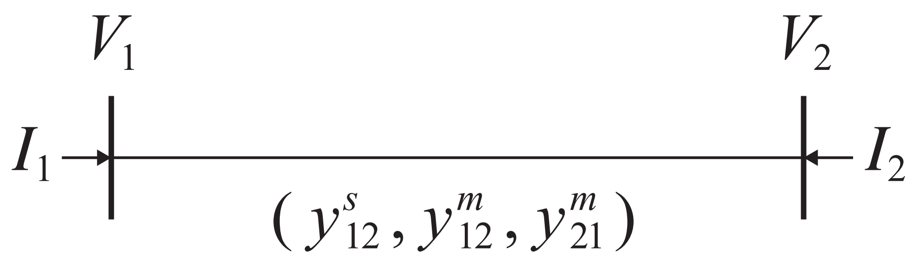
(b) Line parameters $\mathbb{Y}$

Equivalent circuit

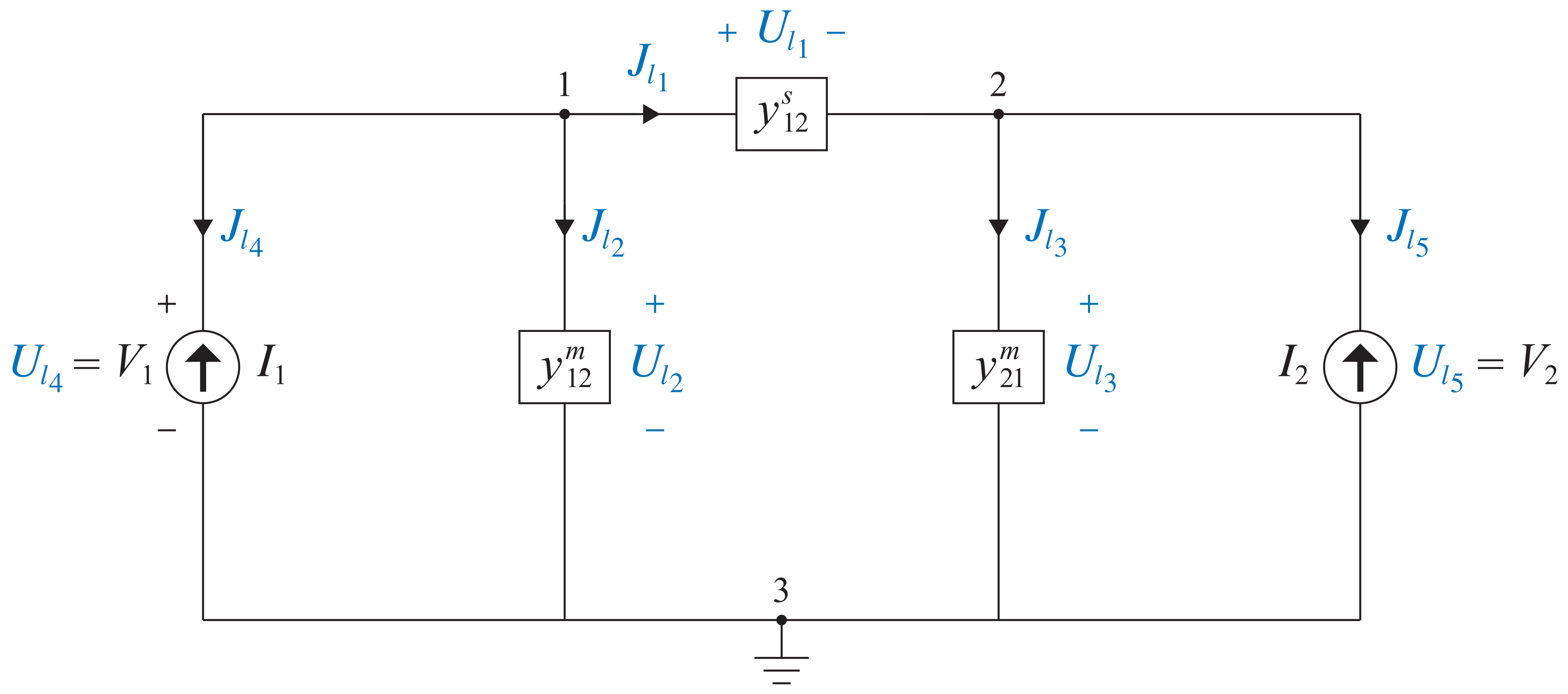
Single line $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$

Behavior of one-line diagram is defined by its equivalent circuit

- Equivalent circuit of entire diagram is determined by equivalent circuit of single line $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$



One-line diagram (single line)



Equivalent circuit of $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$

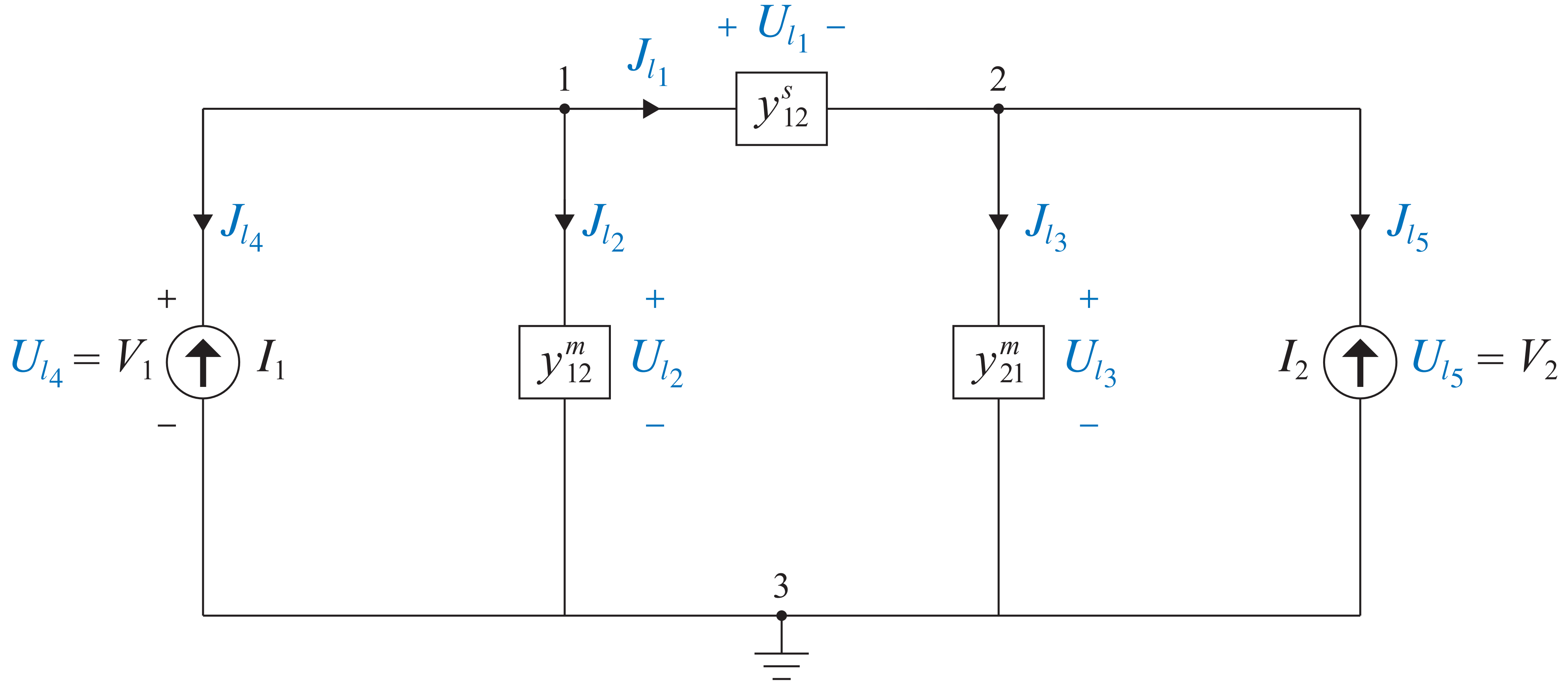
- Includes reference point (i.e., eq circuit has $|\hat{N}| := |\overline{N}| + 1$ nodes)
- Nodal injection I_j from reference point (node 3) to node j
- Suppose an ideal current source I_j is between node j and ref node 3

Equivalent circuit of single line

Circuit analysis

Let directed graph $\hat{G} := (\hat{N}, \hat{E})$ represent the equivalent circuit

- $\hat{N} := \{1,2,3\}, \quad \hat{E} := \{l_1 := 1 \to 2, l_2 := 1 \to 3, l_3 := 2 \to 3, l_4 := 1 \to 3, l_5 := 2 \to 3\}$ $\quad \hat{C} := \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 1 \\ 0 & -1 & -1 & -1 & -1 \end{bmatrix}$
- Links $l_1 : \; J_{l_1} = y^s_{12} U_{l_1}, \quad l_2 : \; J_{l_2} = y^m_{12} U_{l_2}, \quad l_3 : \; J_{l_3} = y^m_{21} U_{l_3}$
- Nodal devices : $l_4 : \; J_{l_4} = -I_1, \quad l_5 : \; J_{l_5} = -I_2$
- KCL: $\hat{C}J = 0$, KVL: $\exists \; V := (V_1, V_2, V_3)$ s.t. $U = \hat{C}^{\mathsf{T}} V$



Eliminate branch vars (U, J) relates nodal vars (I, V)

$$I = YV$$

where the admittance matrix is

$$Y := \begin{bmatrix} y^s_{12} + y^m_{12} & -y^s_{12} \\ -y^s_{12} & y^s_{12} + y^m_{21} \end{bmatrix}$$

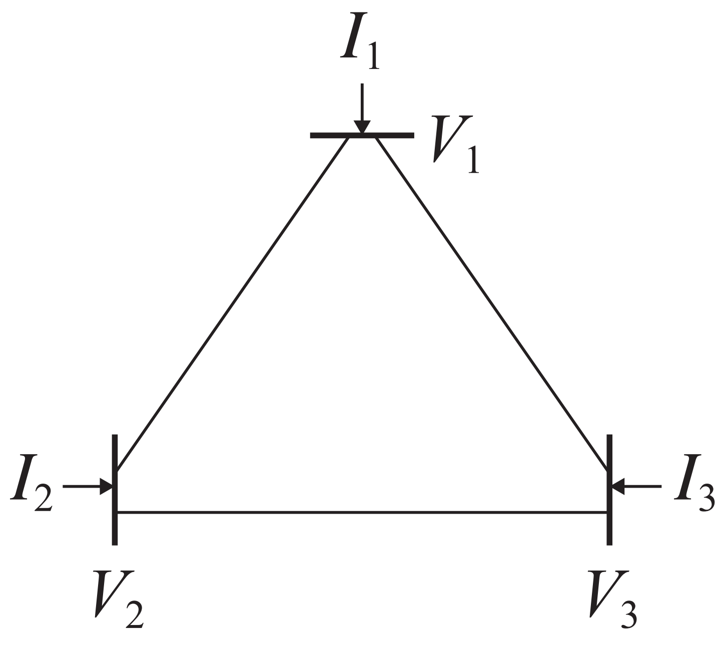
Equivalent circuit

One-line diagram $(G, \mathbb{Y})$

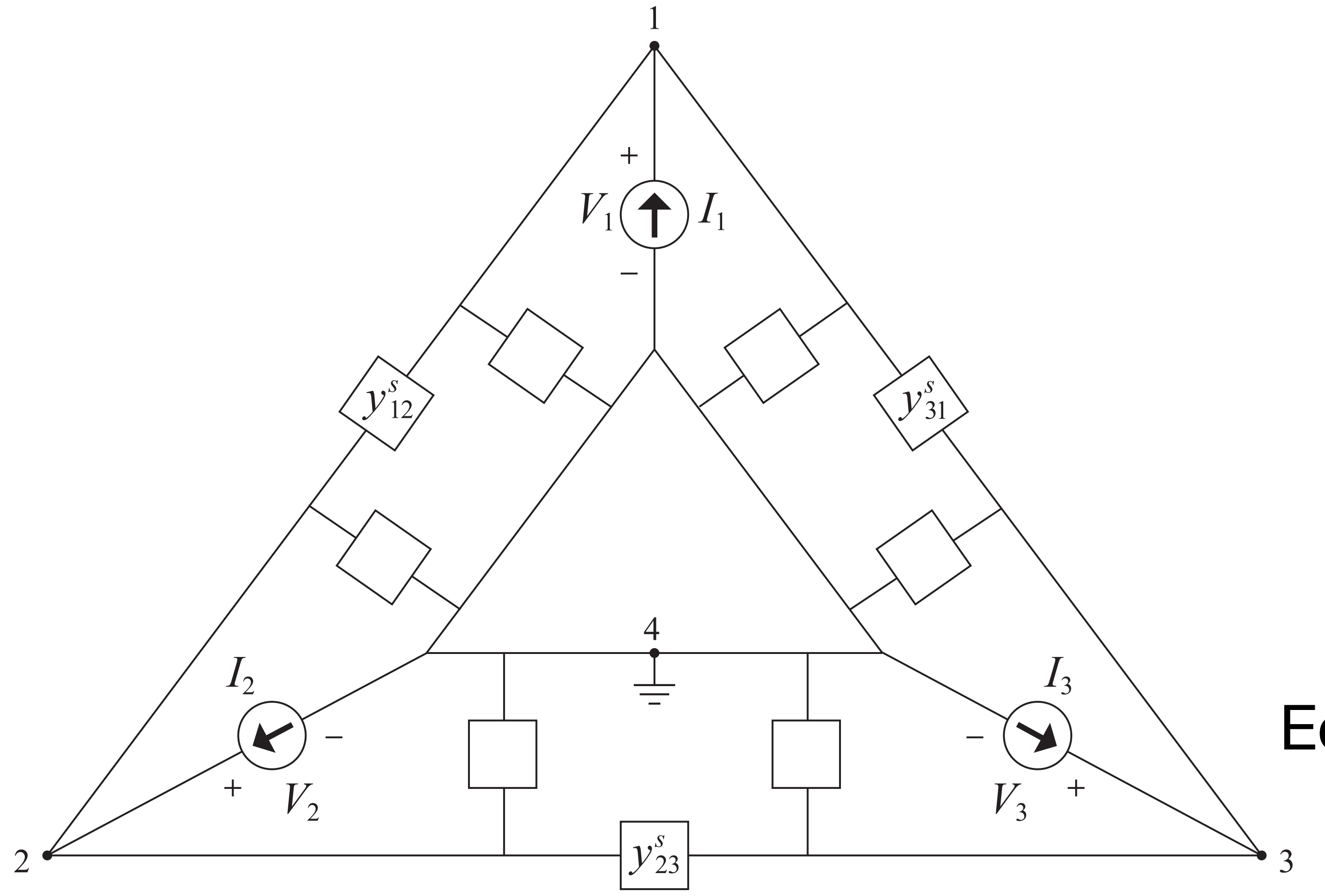
Equivalent circuit represented by directed graph $\hat{G} := (\hat{N}, \hat{E})$ constructed from $G := (\overline{N}, E)$

- $\hat{N} \ := \ \overline{N} \cup \{\, |\overline{N}| + 1\}$ with node $\hat{N} := |\overline{N}| + 1$ as reference point $(V_{\hat{N}} := 0)$
- Each $j \in \overline{N}$ in one-line diagram G gives rise to a link $l := j \to \hat{N}$ in equivalent circuit $\hat{G}$
- Each line $(j, k) \in E$ in one-line diagram G gives rise to 3 links $\left(l_{jk}, l_{j\hat{N}}, l_{k\hat{N}}\right)$ in equivalent circuit $\hat{G}$
- $\hat{E} \ := \ E \cup \hat{E}_{\hat{N}}$ where $\hat{E}_{\hat{N}} := \{\text{links } j \to \hat{N}\}$

Same analysis relates nodal vars (I, V) : $\ I = YV\ $ where Y is admittance matrix



(a) Graph $G = (\overline{N}, E)$



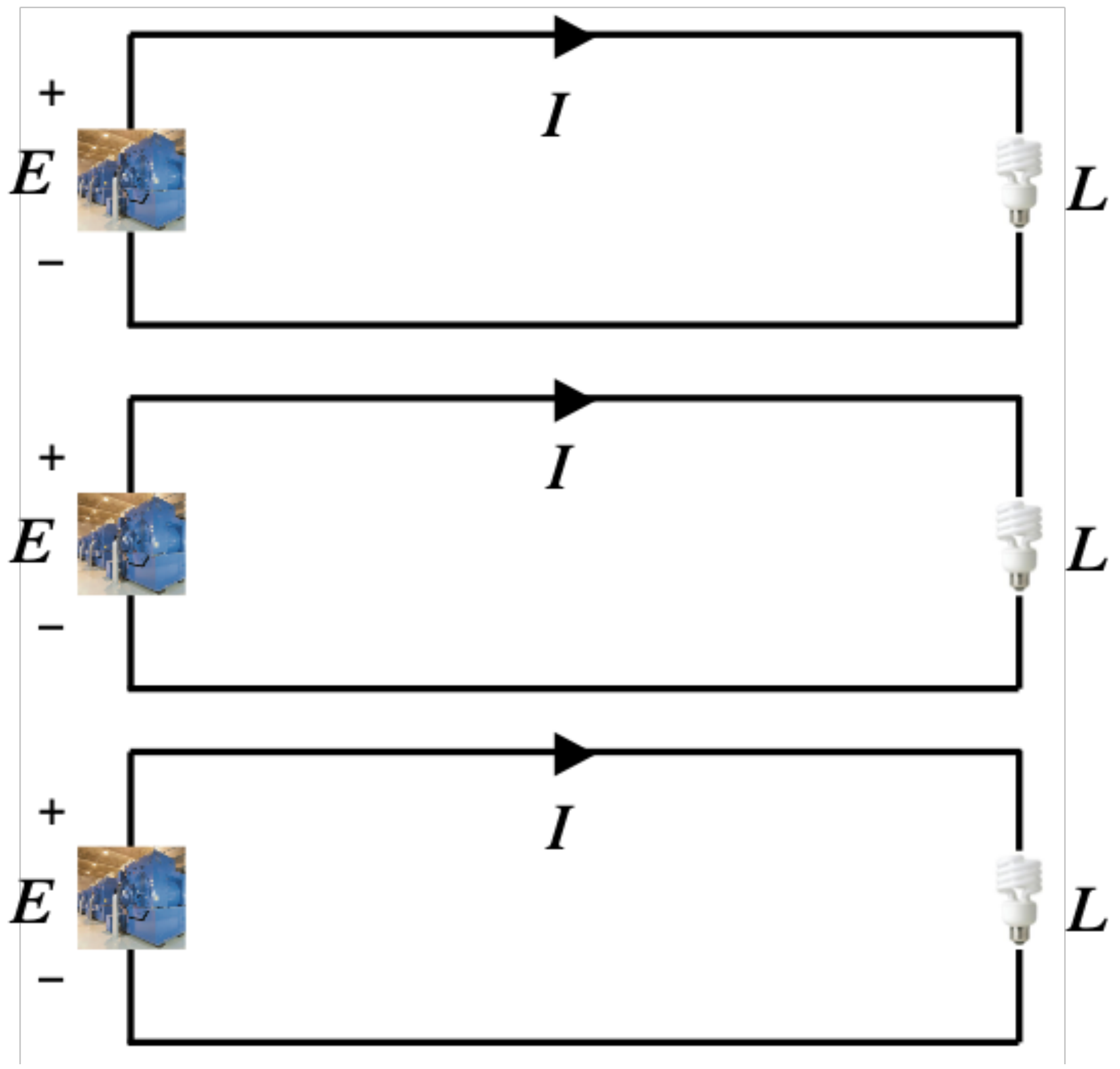
Equivalent circuit $\hat{G} := (\hat{N}, \hat{E})$

Outline

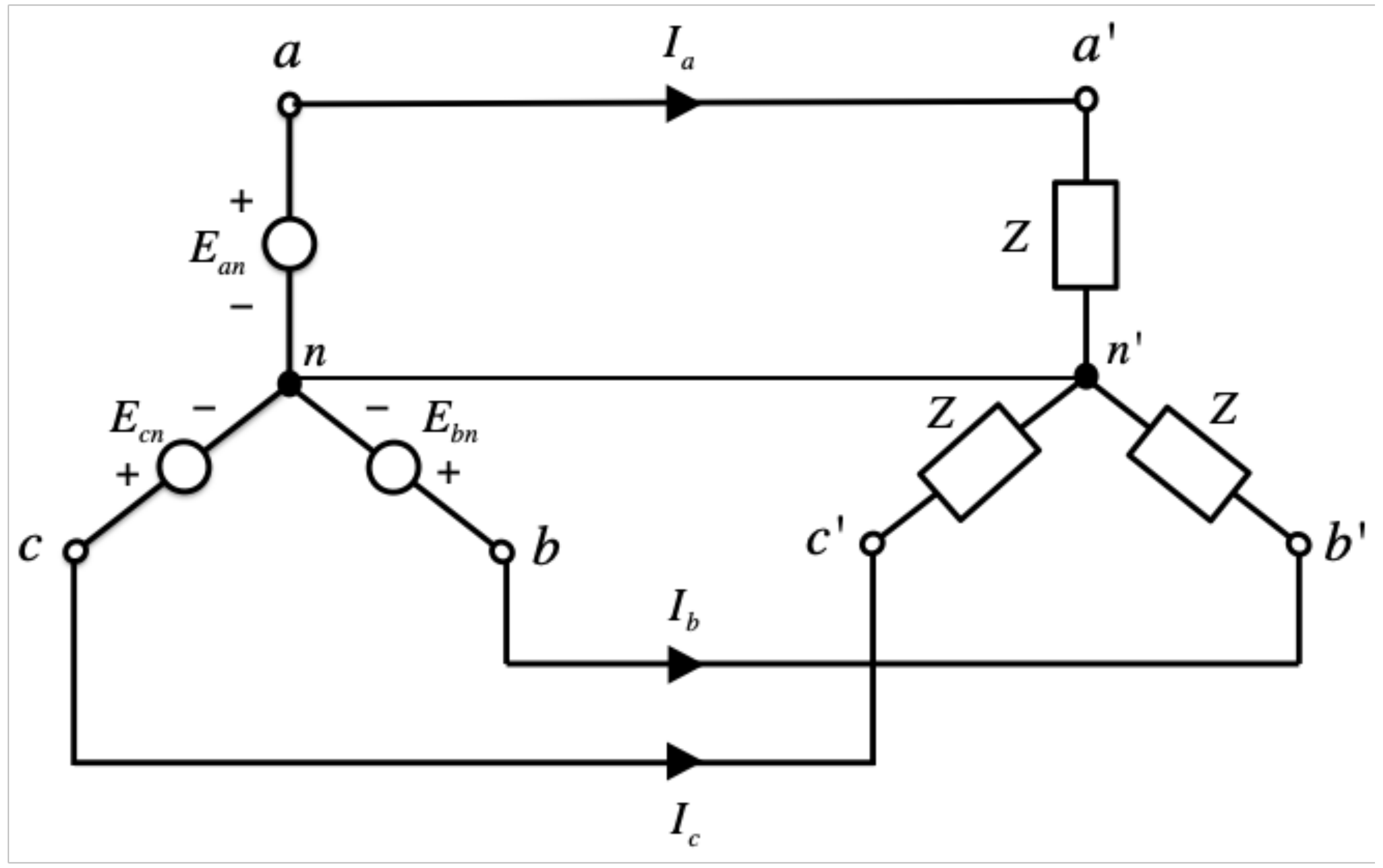
1. Single-phase systems
2. Balanced three-phase systems
   - Internal and terminal vars
   - Balanced vectors and conversion matrices $\Gamma, \Gamma^{\mathsf{T}}$
   - Balanced systems in Y configurations
   - Balanced systems in Δ configurations
   - Per-phase analysis
3. Complex power

Balanced 3-phase system

3 single-phase system:



single 3-phase system:

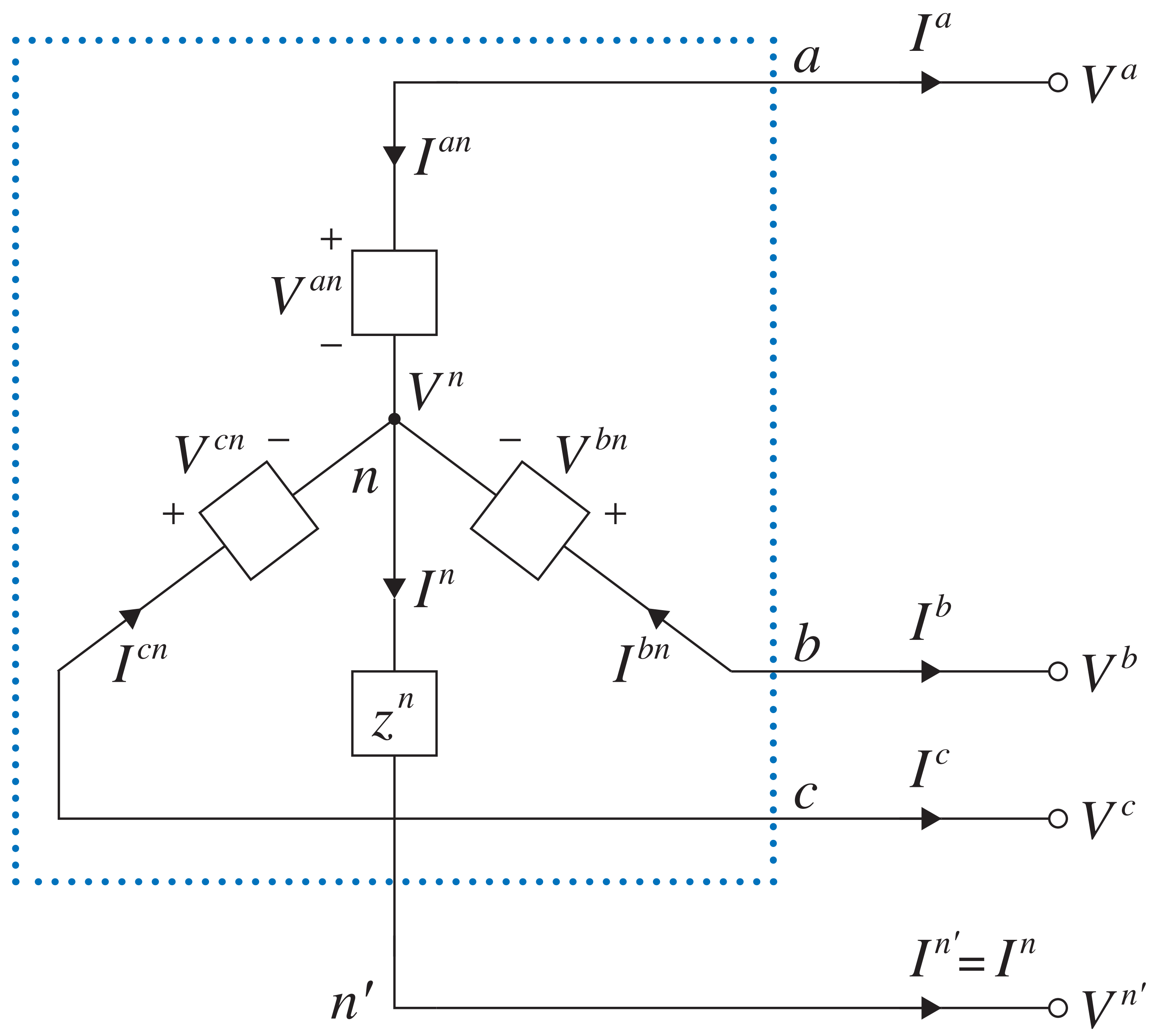


Internal variables

Y configuration

Each single-phase device can be

- Voltage source, current source, impedance, ideal or not



Internal (line-to-neutral or phase) voltage, current, power:

$$V^Y := \begin{bmatrix} V^{an} \\ V^{bn} \\ V^{cn} \end{bmatrix}, \ I^Y := \begin{bmatrix} I^{an} \\ I^{bn} \\ I^{cn} \end{bmatrix}, \ s^Y := \begin{bmatrix} s^{an} \\ s^{bn} \\ s^{cn} \end{bmatrix} := \begin{bmatrix} V^{an}\bar{I}^{an} \\ V^{bn}\bar{I}^{bn} \\ V^{cn}\bar{I}^{cn} \end{bmatrix}$$

neutral voltage (wrt common reference pt) $V^n \in \mathbb{C}$

neutral current (away from neutral) $I^n \in \mathbb{C}$

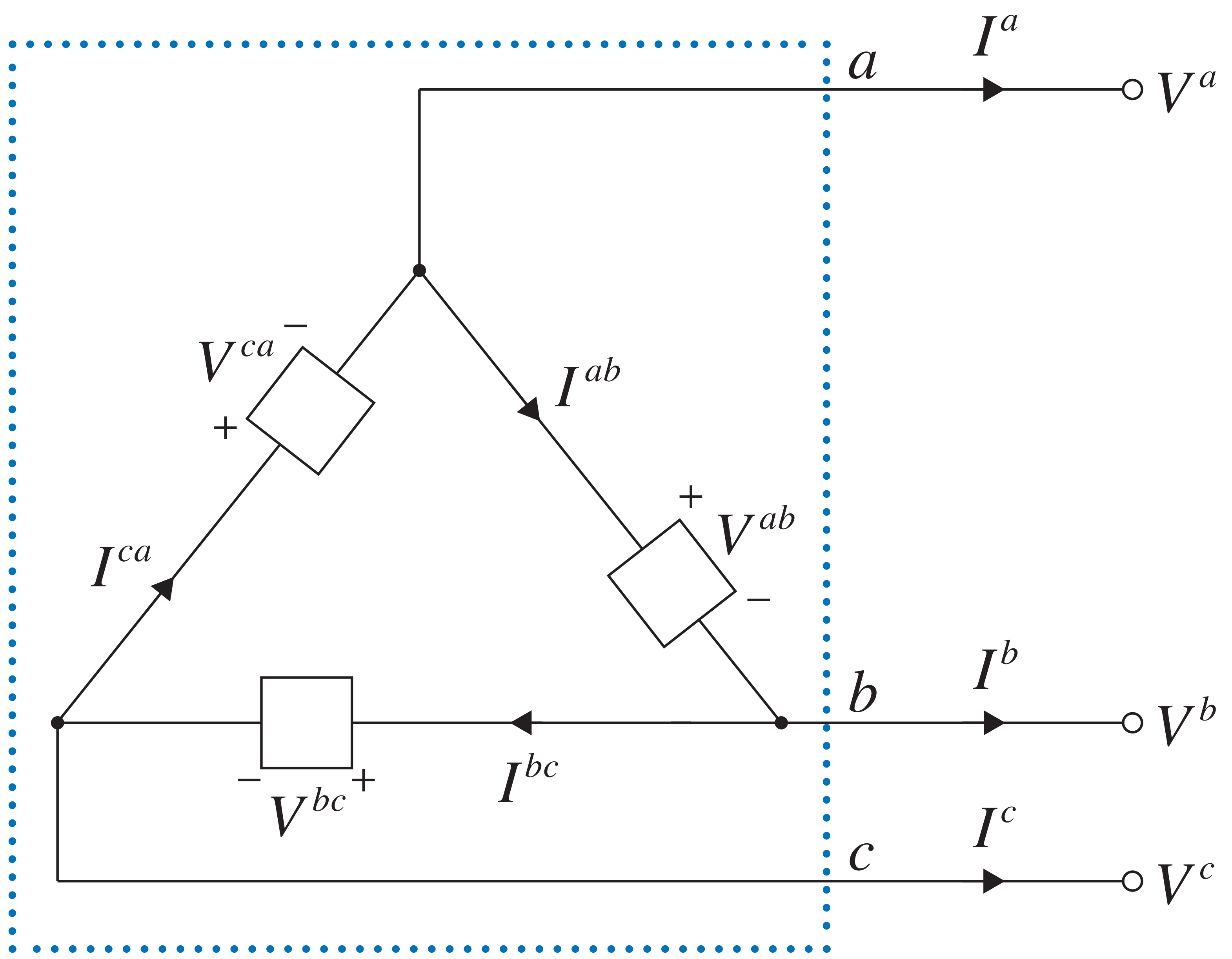
Device may or may not be grounded, and neutral impedance z^n may or may not be zero

Internal variables

Δ configuration

Internal (line-to-line or phase-to-phase) voltage, current, power:

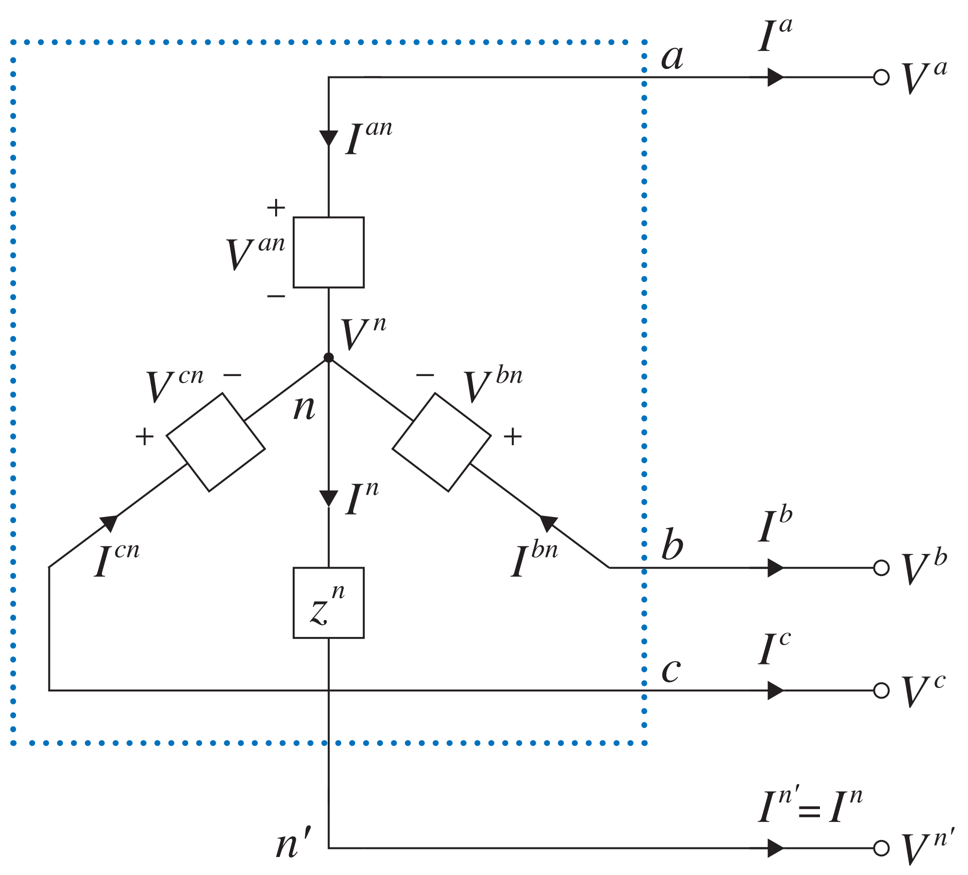
$$V^{\Delta} := \begin{bmatrix} V^{ab} \\ V^{bc} \\ V^{ca} \end{bmatrix}, \; I^{\Delta} := \begin{bmatrix} I^{ab} \\ I^{bc} \\ I^{ca} \end{bmatrix}, \; s^{\Delta} := \begin{bmatrix} s^{ab} \\ s^{bc} \\ s^{ca} \end{bmatrix} := \begin{bmatrix} V^{ab}\bar{I}^{ab} \\ V^{bc}\bar{I}^{bc} \\ V^{ca}\bar{I}^{ca} \end{bmatrix}$$



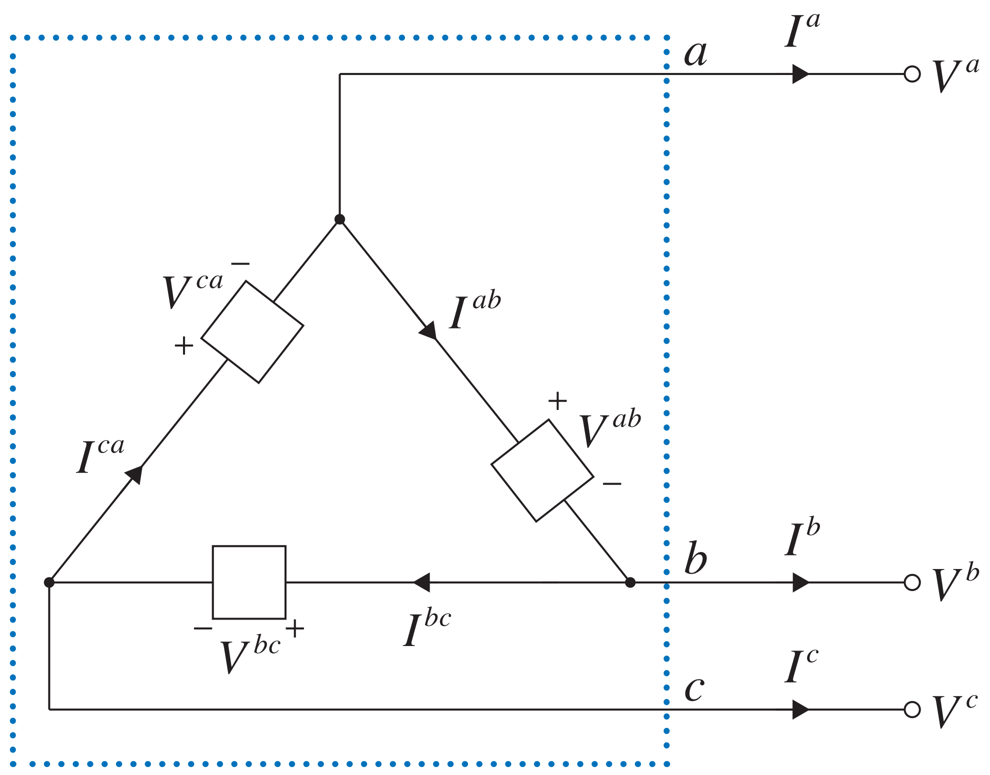
Terminal variables

Terminal voltage, current, power (for both Y and Δ):

$$V := \begin{bmatrix} V^a \\ V^b \\ V^c \end{bmatrix}, \quad I := \begin{bmatrix} I^a \\ I^b \\ I^c \end{bmatrix}, \quad s := \begin{bmatrix} s^a \\ s^b \\ s^c \end{bmatrix} := \begin{bmatrix} V^a \bar{I}^a \\ V^b \bar{I}^b \\ V^c \bar{I}^c \end{bmatrix}$$



- V is with respect to an arbitrary common reference point, e.g. the ground
- I and s are in the direction out of the device

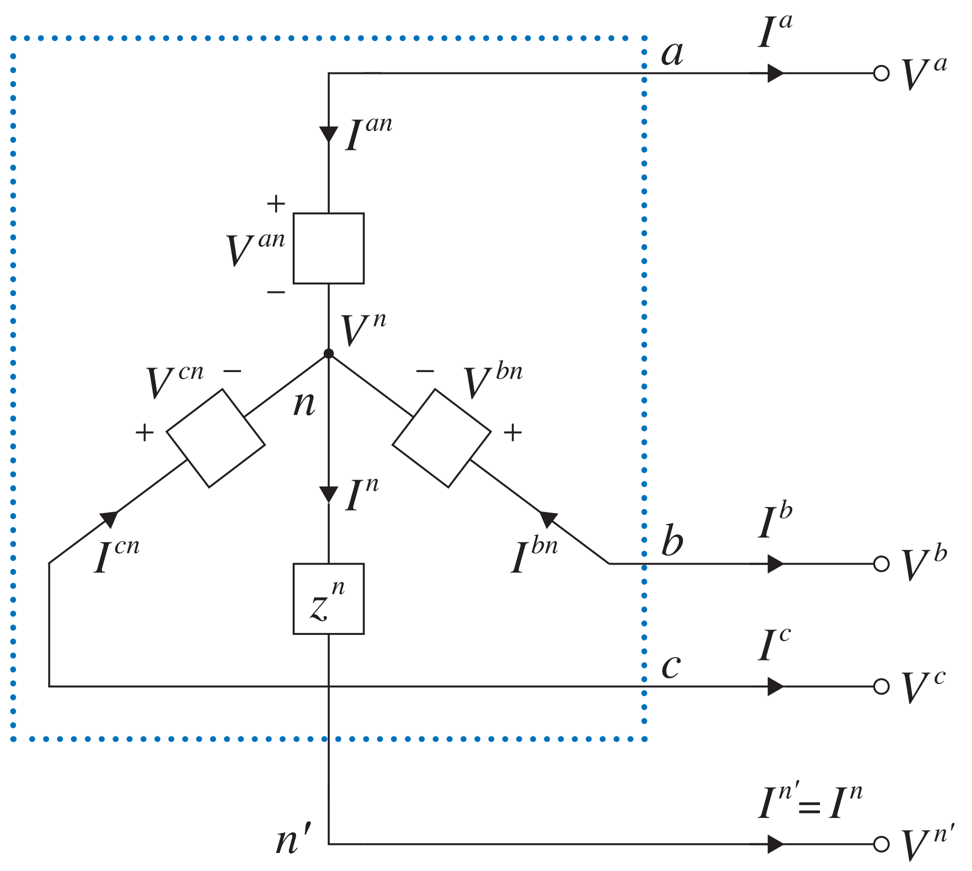


Conversion rules

Y-configured device

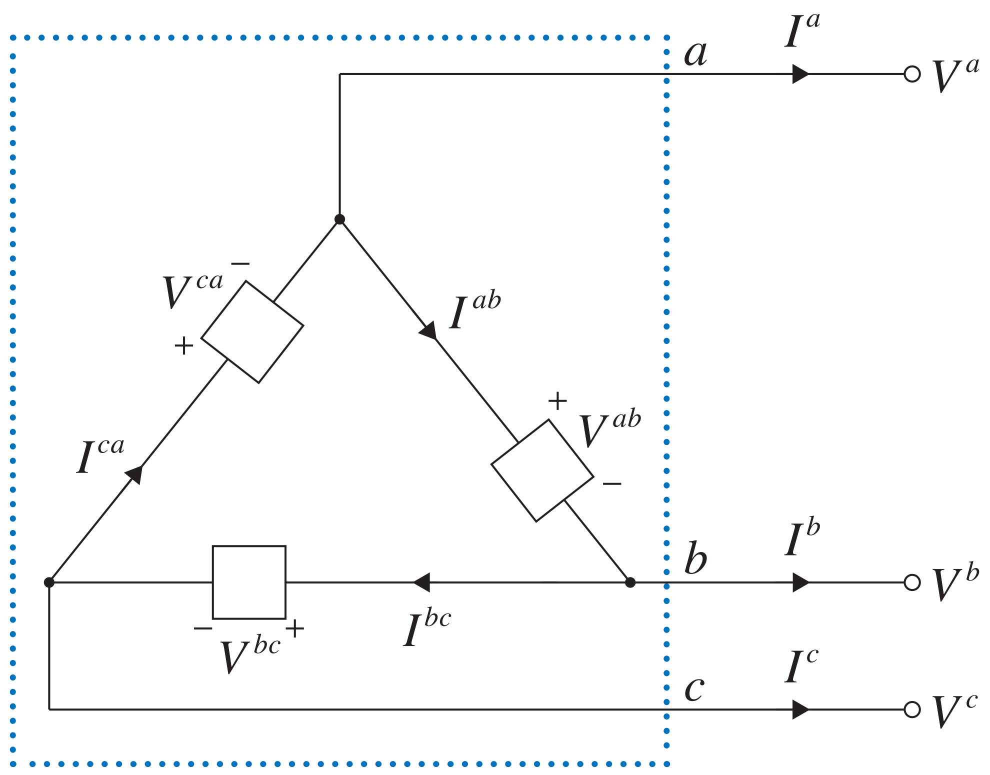
$$V = V^Y + V^n \mathbf{1}, \qquad I = -I^Y$$

- $V = V^Y$ if $V^n = 0$, i.e., if neutral is directly grounded and ground is voltage reference



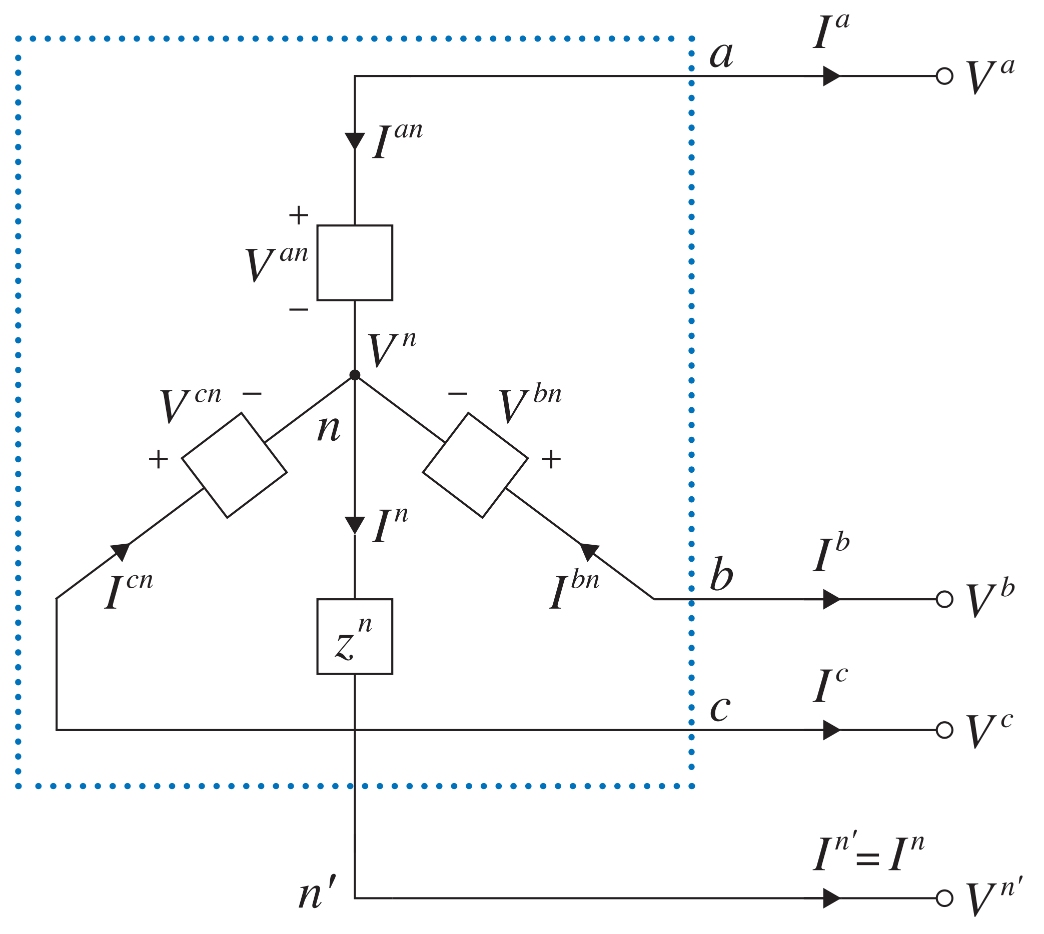
Δ-configured device

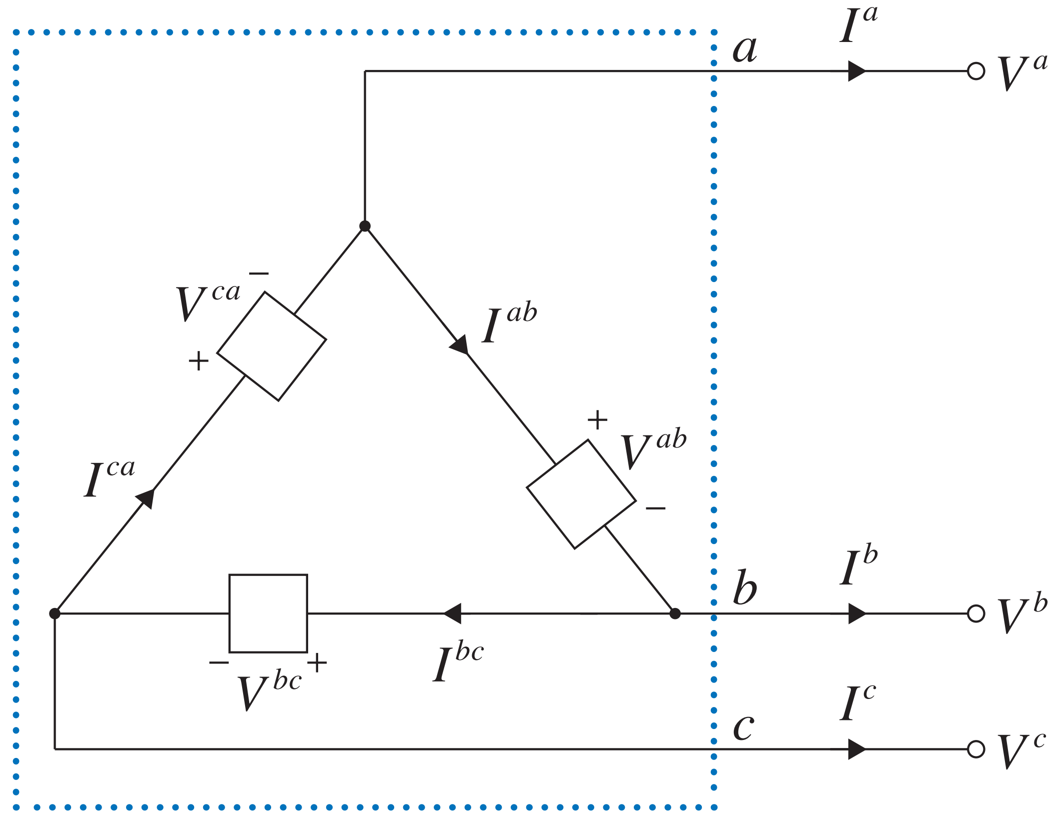
$$\begin{bmatrix} V^{ab} \\ V^{bc} \\ V^{ca} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}}_{\Gamma} \begin{bmatrix} V^a \\ V^b \\ V^c \end{bmatrix}, \qquad \begin{bmatrix} I^a \\ I^b \\ I^c \end{bmatrix} = -\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{\Gamma^{\mathsf{T}}} \begin{bmatrix} I^{ab} \\ I^{bc} \\ I^{ca} \end{bmatrix}$$



Internal vs external model

1. Internal model depends only on type of single-phase devices
   - Internal model: relation between $\left(V^{Y/\Delta}, I^{Y/\Delta}, s^{Y/\Delta}\right)$
   - Voltage/current/power source, impedance
   - Independent of Y or Δ configuration
2. Conversion rule depends only on type of configuration
   - Converts between internal and terminal variables
   - Depends only on Y or Δ configuration
   - Independent of type of single-phase devices
3. External model = Internal model + Conversion rule
   - External model: relation between (V, I, s)
   - Devices interact over network only through their terminal variables

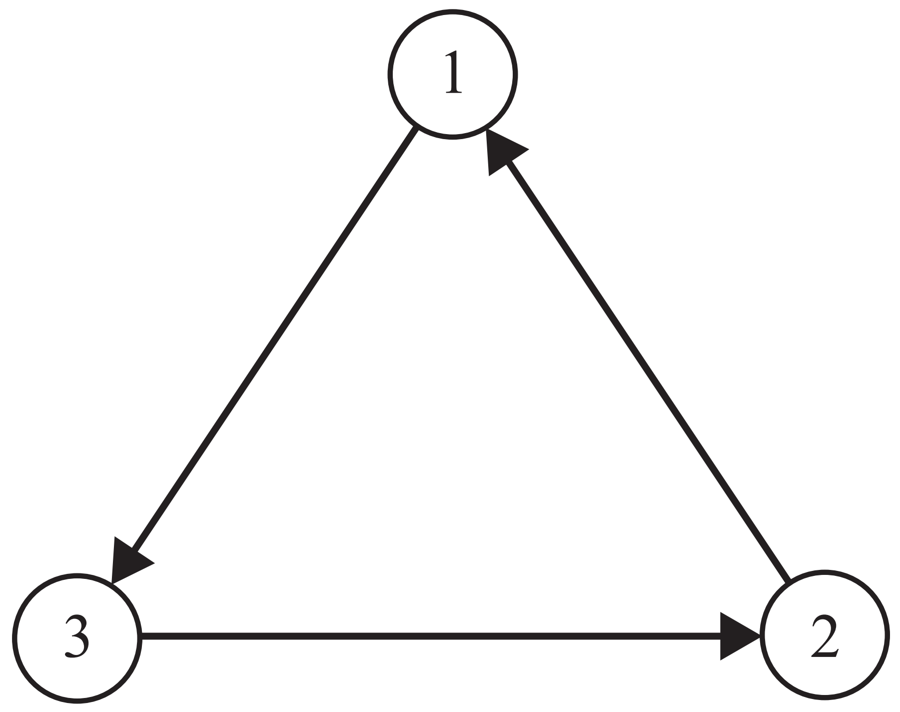




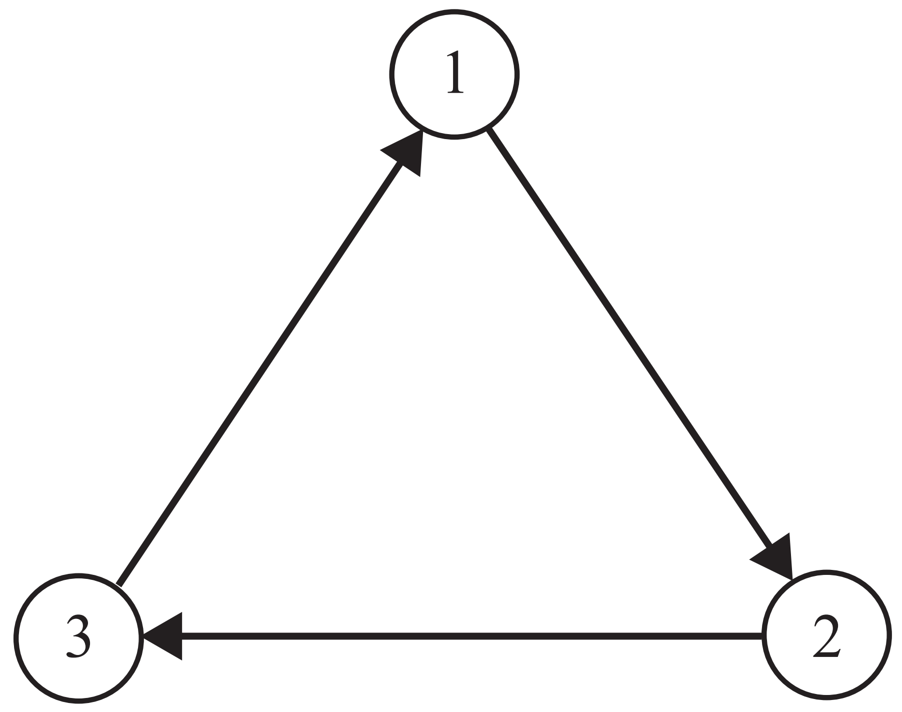
Conversion matrices

$$\Gamma := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \qquad \Gamma^{\mathsf{T}} := \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Incidence matrices for:



(a) Γ



(b) Γ^{T}

Conversion matrices

Convert between internal vars and terminal vars

$$\begin{bmatrix} V^{ab} \\ V^{bc} \\ V^{ca} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}}_{\Gamma} \begin{bmatrix} V^{a} \\ V^{b} \\ V^{c} \end{bmatrix}, \qquad \begin{bmatrix} I^{a} \\ I^{b} \\ I^{c} \end{bmatrix} = - \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{\Gamma^{\mathsf{T}}} \begin{bmatrix} I^{ab} \\ I^{bc} \\ I^{ca} \end{bmatrix}$$

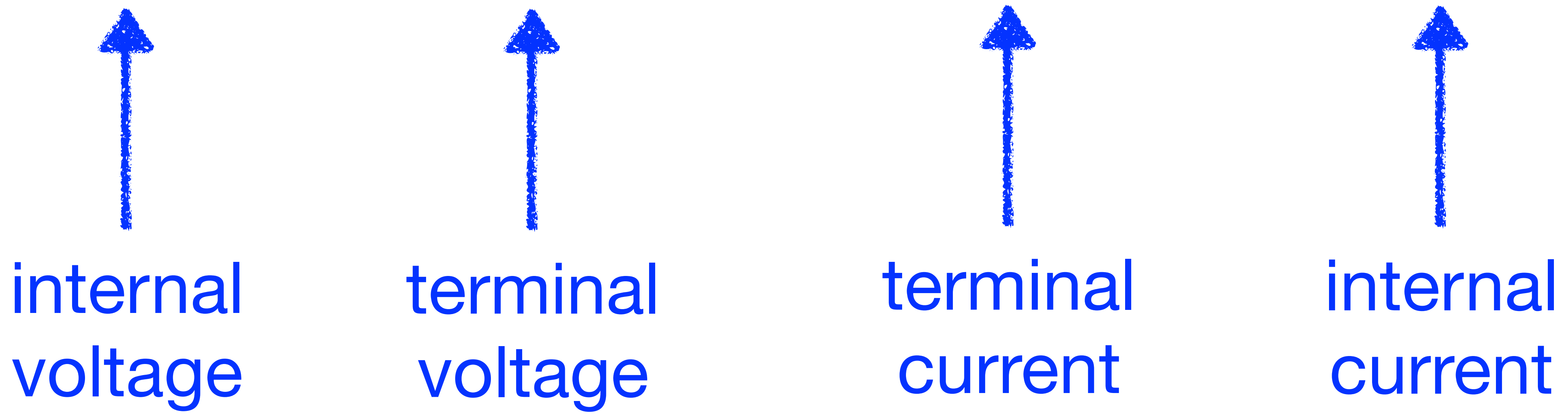
Conversion matrices

Convert between internal vars and terminal vars

$$\begin{bmatrix} V^{ab} \\ V^{bc} \\ V^{ca} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}}_{\Gamma} \begin{bmatrix} V^{a} \\ V^{b} \\ V^{c} \end{bmatrix}, \qquad \begin{bmatrix} I^{a} \\ I^{b} \\ I^{c} \end{bmatrix} = -\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{\Gamma^{\mathsf{T}}} \begin{bmatrix} I^{ab} \\ I^{bc} \\ I^{ca} \end{bmatrix}$$

In vector form

$$V^{\Delta} = \Gamma V, \qquad I = -\Gamma^{\mathsf{T}} I^{\Delta}$$



Balanced vector

Definition

A vector $x := \left(x_1, x_2, x_3\right)$ with $x_j = |x_j| e^{i\theta_j} \in \mathbb{C}$ is called balanced if

- $|x_1| \;=\; |x_2| \;=\; |x_3|$
- Either $\theta_2 - \theta_1 = -\frac{2\pi}{3}$ and $\theta_3 - \theta_1 = \frac{2\pi}{3}$ (positive sequence)

  or $\theta_2 - \theta_1 = \frac{2\pi}{3}$ and $\theta_3 - \theta_1 = -\frac{2\pi}{3}$ (negative sequence)

Spectral properties of $\Gamma, \Gamma^{\mathsf{T}}$

Let

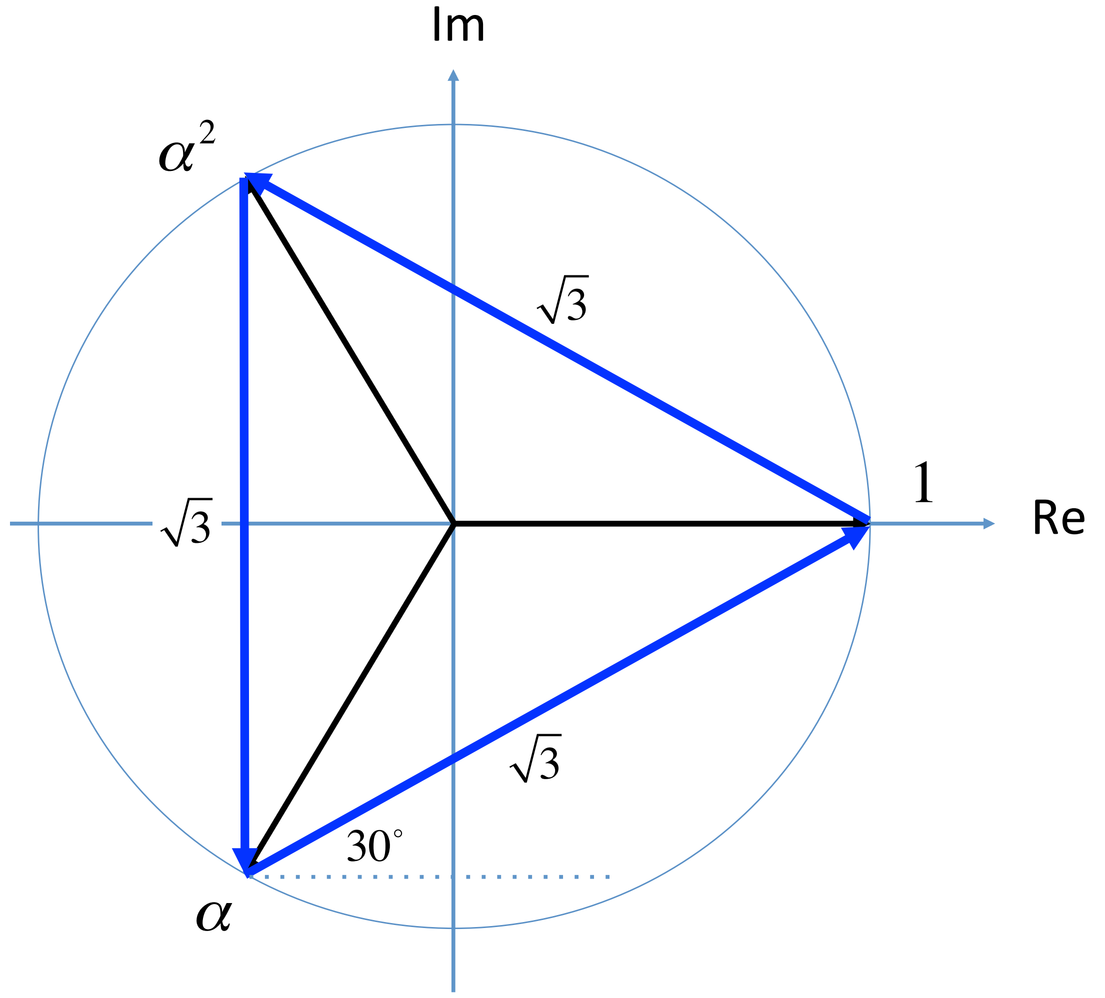
1. $\alpha := e^{-\mathrm{i}2\pi/3}$

2. Positive and negative sequence vectors:

$$\alpha_+ := \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \end{bmatrix}, \qquad \alpha_- := \begin{bmatrix} 1 \\ \alpha^2 \\ \alpha \end{bmatrix}$$

Balanced vectors in positive and negative seq: $x = a\alpha_+$, $y = b\alpha_-$ for $a, b \in \mathbb{C}$

Balanced system: all voltages and currents are in span(α_+) WLOG



3. Fortescue matrix

$$F := \frac{1}{\sqrt{3}} \begin{bmatrix} \mathbf{1} & \alpha_+ & \alpha_- \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix}$$

Symmetrical components are similarity transformation using matrix F

Spectral decomposition of $\Gamma, \Gamma^{\mathsf{T}}$

Theorem

1. $F^{-1} = F^{\mathsf{H}} = \bar{F} = \frac{1}{\sqrt{3}} \begin{bmatrix} \mathbf{1} & \alpha_- & \alpha_+ \end{bmatrix}$

2. Eigenvectors and eigenvalues are given by the spectral decomposition:

$$\Gamma = F \begin{bmatrix} 0 & & \\ & 1-\alpha & \\ & & 1-\alpha^2 \end{bmatrix} \bar{F}, \qquad \Gamma^{\mathsf{T}} = \bar{F} \begin{bmatrix} 0 & & \\ & 1-\alpha & \\ & & 1-\alpha^2 \end{bmatrix} F$$

where $1-\alpha = \sqrt{3}e^{\mathrm{i}\pi/6}$ and $1-\alpha^2 = \sqrt{3}e^{-\mathrm{i}\pi/6}$

Transformation of balanced vectors by $(\Gamma, \Gamma^{\mathsf{T}})$

$$\Gamma\mathbf{1} = 0, \qquad \Gamma\alpha_+ = (1-\alpha)\alpha_+, \qquad \Gamma\alpha_- = (1-\alpha^2)\alpha_-$$

$$\Gamma^{\mathsf{T}}\mathbf{1} = 0, \qquad \Gamma^{\mathsf{T}}\alpha_- = (1-\alpha)\alpha_-, \qquad \Gamma^{\mathsf{T}}\alpha_+ = (1-\alpha^2)\alpha_+$$

Spectral decomposition of $\Gamma, \Gamma^\mathsf{T}$

Corollary

For any balanced vector (positive seq) $x \in \text{span}(\alpha_+)$ and $\gamma \in \mathbb{C}$

1. $\Gamma(x + \gamma\mathbf{1}) \;=\; (1 - \alpha)x$
2. $\Gamma^\mathsf{T}(x + \gamma\mathbf{1}) \;=\; (1 - \alpha^2)x$
3. $\Gamma\Gamma^\mathsf{T}(x + \gamma\mathbf{1}) \;=\; \Gamma^\mathsf{T}\Gamma(x + \gamma\mathbf{1}) \;=\; 3x$

For balanced 3-phase systems, this result is enough.
For unbalanced systems, we need to study pseudo-inverses of $(\Gamma, \Gamma^\mathsf{T})$

Spectral properties of $\Gamma, \Gamma^\mathsf{T}$

Almost all properties of balanced 3-phase systems originate from properties of (α_+, α_-) and their transformation under $\left(\Gamma, \Gamma^\mathsf{T}\right)$

1. Transformation by $\left(\Gamma, \Gamma^\mathsf{T}\right)$ preserve balanced nature of a vector, ensuring that all voltages and currents are balanced in a symmetric network driven by balanced sources

2. This is because balanced sources are in span(α_+) (or span(α_-) for negative-seq systems), and (α_+, α_-) are eigenvectors of $\left(\Gamma, \Gamma^\mathsf{T}\right)$; see Theorem

3. For unbalanced systems, sources have a mix of components in span(α_+), span(α_-), span$(\mathbf{1})$ and hence transformation by $\left(\Gamma, \Gamma^\mathsf{T}\right)$ maintains the mix

Balanced systems

Implications

1. Informally, a balanced system is one in which all voltages and currents are in span(α_+) (WLOG)
2. Balanced voltage and current sources are in span(α_+)
3. Voltages and currents at every point in a network can be written as linear combination of transformed source voltages and source currents, transformed by $(\Gamma, \Gamma^\mathsf{T})$
4. But α_+ are eigenvectors of $(\Gamma, \Gamma^\mathsf{T}) \implies$ transformation by $(\Gamma, \Gamma^\mathsf{T})$ reduces to scaling by $1 - \alpha$ and $1 - \alpha^2$ respectively (provided impedances & lines are balanced)
5. $\implies$ all voltages and currents remain in span(α_+)

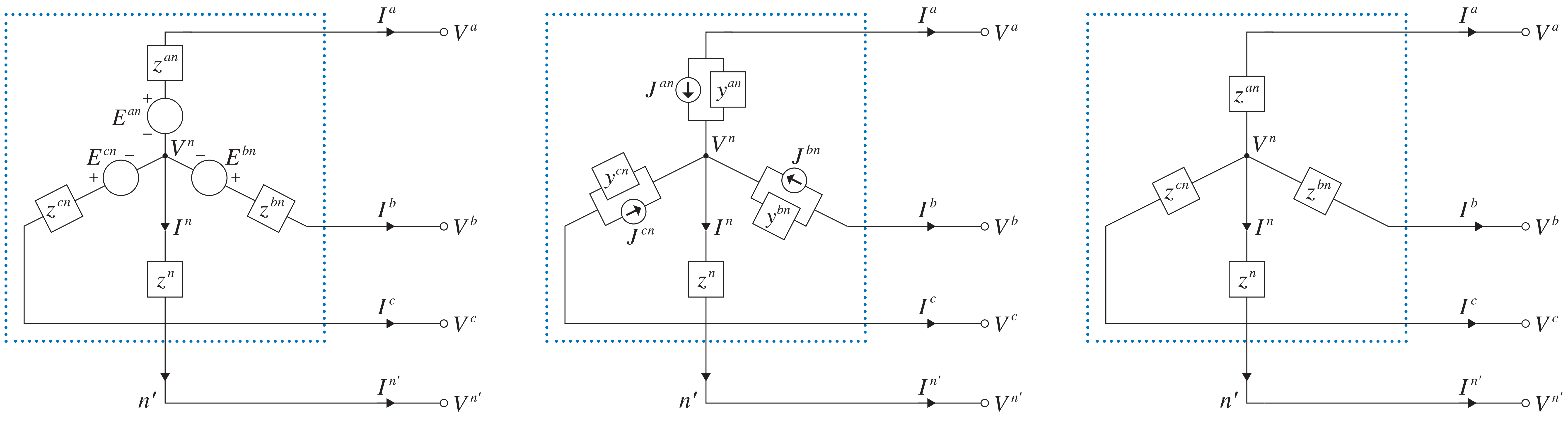
Formal statement and proof need to wait till Part III where we study unbalanced systems

Outline

1. Single-phase systems
2. Balanced three-phase systems
    - Internal and terminal vars
    - Balanced vectors and conversion matrices $\Gamma, \Gamma^{\mathsf{T}}$
    - Balanced systems in Y configurations
    - Balanced systems in Δ configurations
    - Per-phase analysis
3. Complex power

Balanced 3-phase systems

Y configuration



(a) Voltage source E^Y (b) Current source J^Y (c) Impedance z^Y

Balanced voltage source if internal voltage E^Y is a balanced vector and $z^Y := z^{an}$

- positive sequence: $E^{an} = 1\angle\theta, \quad E^{bn} = 1\angle\theta - 120^\circ, \quad E^{cn} = 1\angle\theta + 120^\circ$ $E^Y \in \text{span}(\alpha_+)$
- negative sequence: $E^{an} = 1\angle\theta, \quad E^{bn} = 1\angle\theta + 120^\circ, \quad E^{cn} = 1\angle\theta - 120^\circ$ $E^Y \in \text{span}(\alpha_-)$

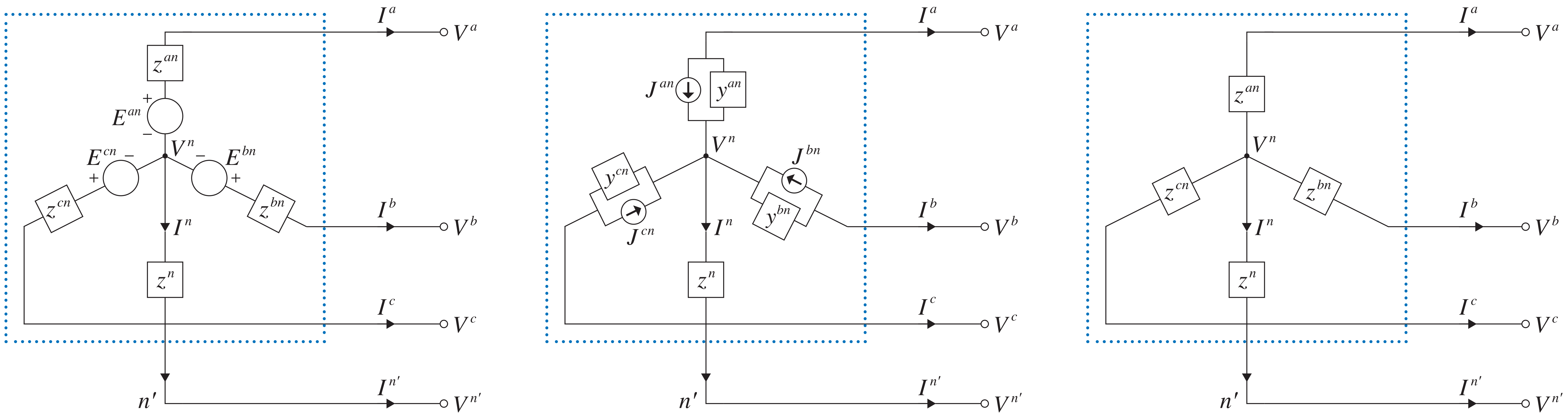
Balanced current source if $J^Y := (J^{an}, J^{bn}, J^{cn}) \in \text{apan}(\alpha_+)$ and $y^Y := y^{an}$

Ideal sources: $z = 0, \ y = 0$

Balanced impedance if impedances are identical, i.e., $z^Y := z^{an}$

Balanced 3-phase systems

Y configuration



(a) Voltage source E^Y (b) Current source J^Y (c) Impedance z^Y

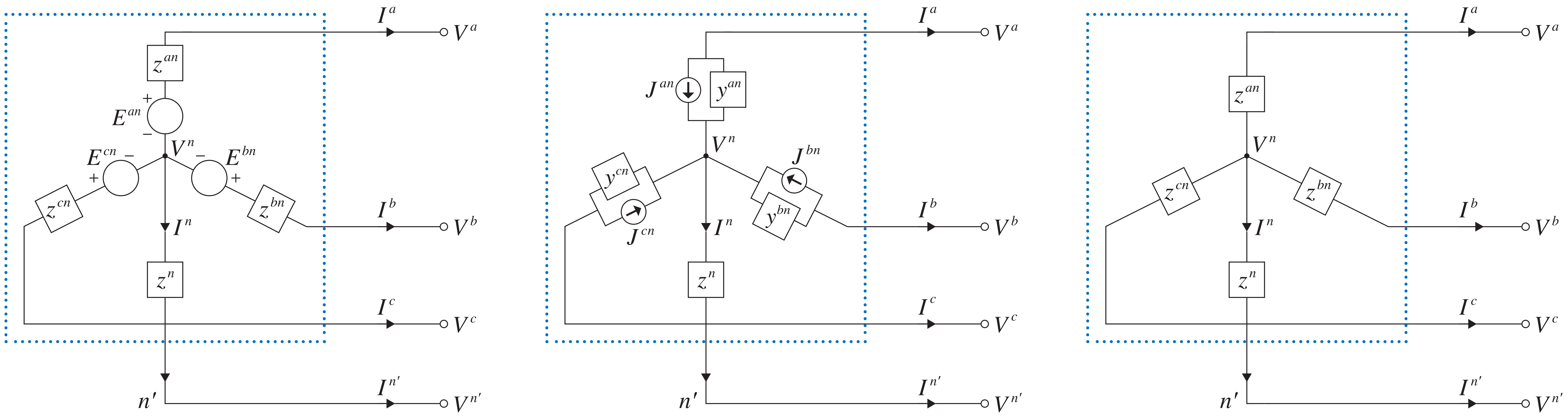
Corollary implies:

1. Sum to zero: $E^{an} + E^{bn} + E^{cn} = 0$ and $J^{an} + J^{bn} + J^{cn} = 0$

   becasuse $E^Y = E^{an}\alpha_+ \;\Rightarrow\; \mathbf{1}^\mathsf{T} E^Y = E^{an}\left(\mathbf{1}^\mathsf{T}\alpha_+\right) = 0$

Balanced 3-phase systems

Y configuration



(a) Voltage source E^Y (b) Current source J^Y (c) Impedance z^Y

Corollary implies:

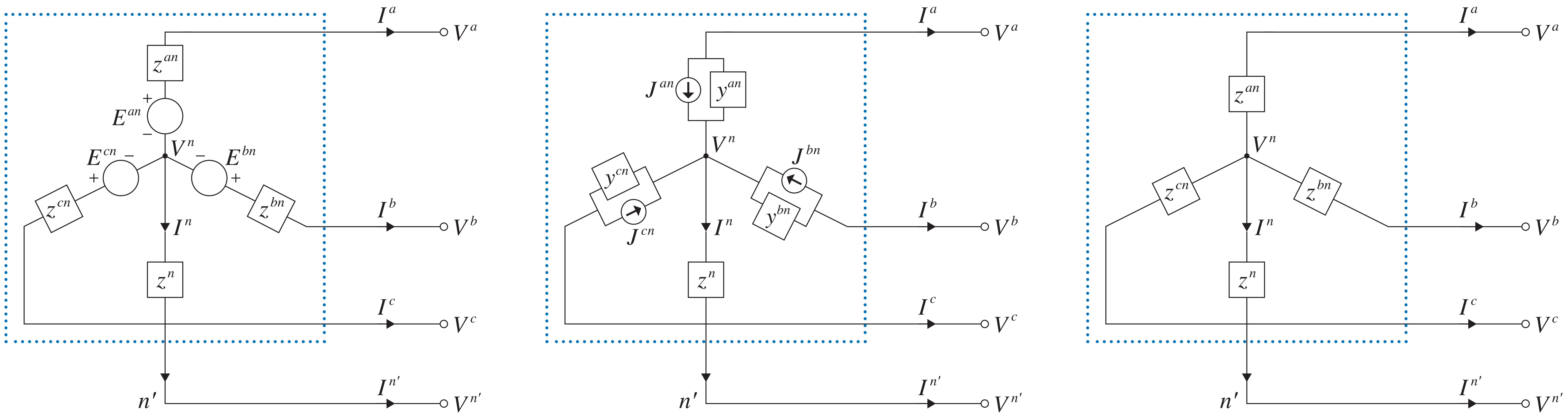
2. Ideal voltage source ($z^Y := 0$): line voltage V^{line} is balanced:

$$V = E^Y + V^n \mathbf{1} = E^{an}\alpha_+ + V^n \mathbf{1} \quad \Rightarrow \quad V^{\text{line}} := \Gamma V = E^{an}(\Gamma \alpha_+) \;=\; (1-\alpha)E^Y$$

phases are decoupled

Balanced 3-phase systems

Y configuration



(a) Voltage source E^Y (b) Current source J^Y (c) Impedance z^Y

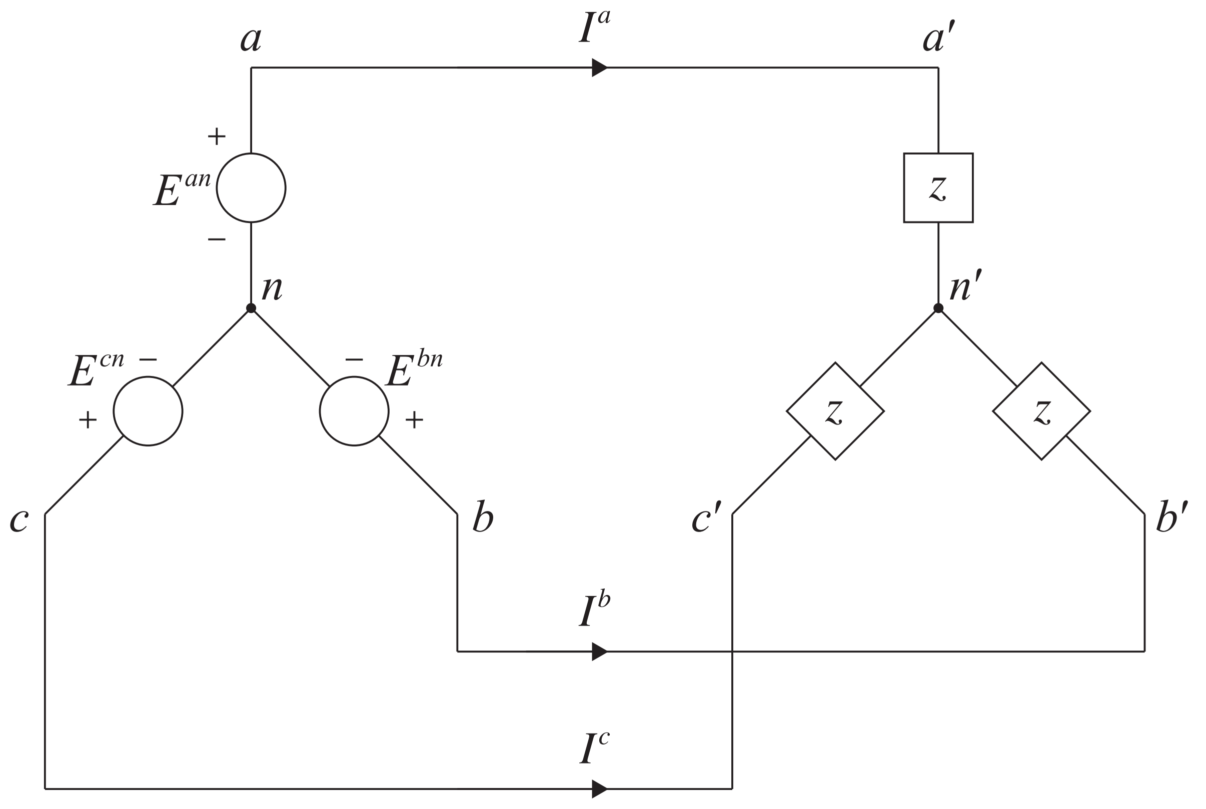
Corollary implies:

3. Ideal current source ($y^Y := 0$): terminal current I is balanced:

$$I = -J^Y$$

phases are decoupled

Phase decoupling
Example



(a) Balanced three-phase system

Given balanced ideal voltage source and impedance in Y configuration, show that

1. Neutral-to-neutral voltage $V_{nn'} = 0$
2. Internal voltages and currents across impedances are balanced
3. Passes are decoupled

Solution

Internal vars $E^Y := \left(E^{an}, E^{bn}, E^{cn}\right)$, $V'^Y := \left(V^{a'n'}, V^{b'n'}, V^{c'n'}\right)$, $I'^Y := \left(I^{a'n'}, I^{b'n'}, I^{c'n'}\right)$, $V^n, V^{n'}$ neutral voltages

Terminal voltages $V := \left(V^a, V^b, V^c\right)$

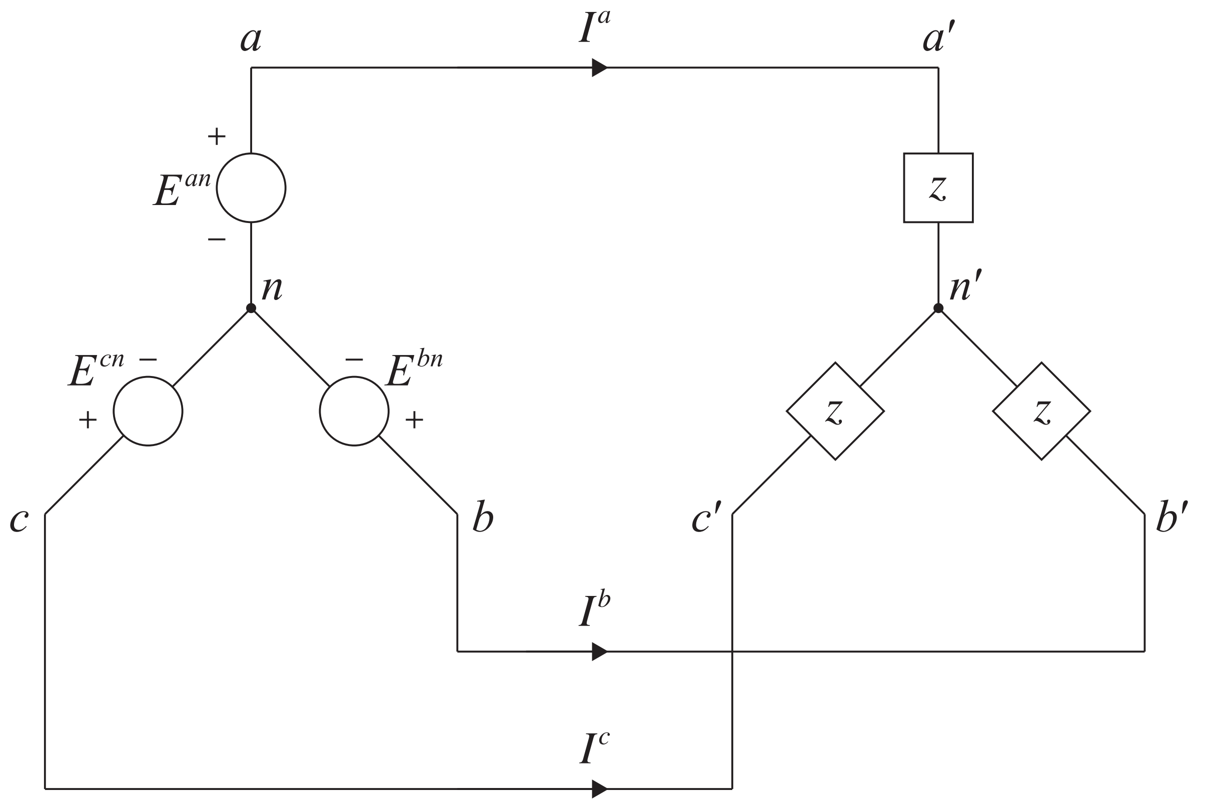
KVL, KCL, Ohm's law: $E^Y = V - V^n\mathbf{1}, \quad V'^Y = V - V^{n'}\mathbf{1}, \quad V'^Y = zI'^Y, \quad \mathbf{1}^\mathsf{T} I'^Y = 0$

Hence $E^Y - V'^Y = \left(V^{n'} - V^n\right)\mathbf{1}$ and $\mathbf{1}^\mathsf{T}\left(E^Y - V'^Y\right) = 3\left(V^{n'} - V^n\right) \implies 3\left(V^{n'} - V^n\right) = -z\left(\mathbf{1}^\mathsf{T} I'^Y\right) = 0$

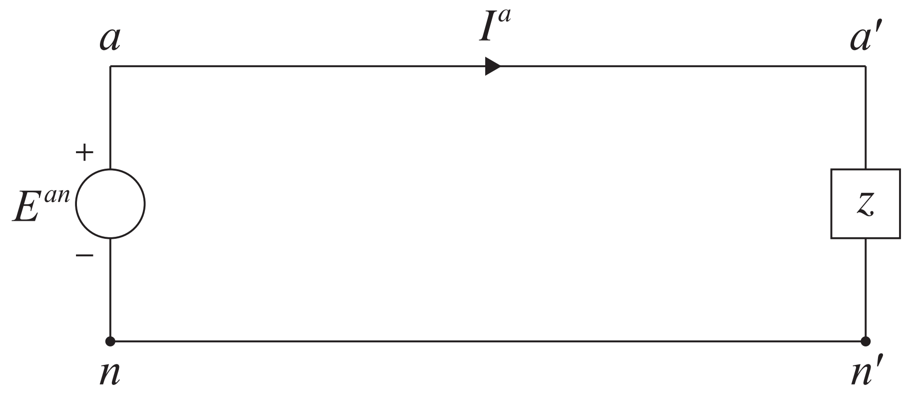
i.e., $V_{nn'} = 0$

$\implies V'^Y = E^Y + \left(V^n - V^{n'}\right)\mathbf{1} = E^Y, \quad I'^Y = yV'^Y = yE^Y$, i.e., V'^Y and I'^Y are balanced and phase-decoupled

Phase decoupling
Example



(a) Balanced three-phase system



(b) Equivalent per-phase system

Given balanced ideal voltage source and impedance in Y configuration, show that

1. Neutral-to-neutral voltage $V_{nn'} = 0$
2. Internal voltages and currents across impedances are balanced

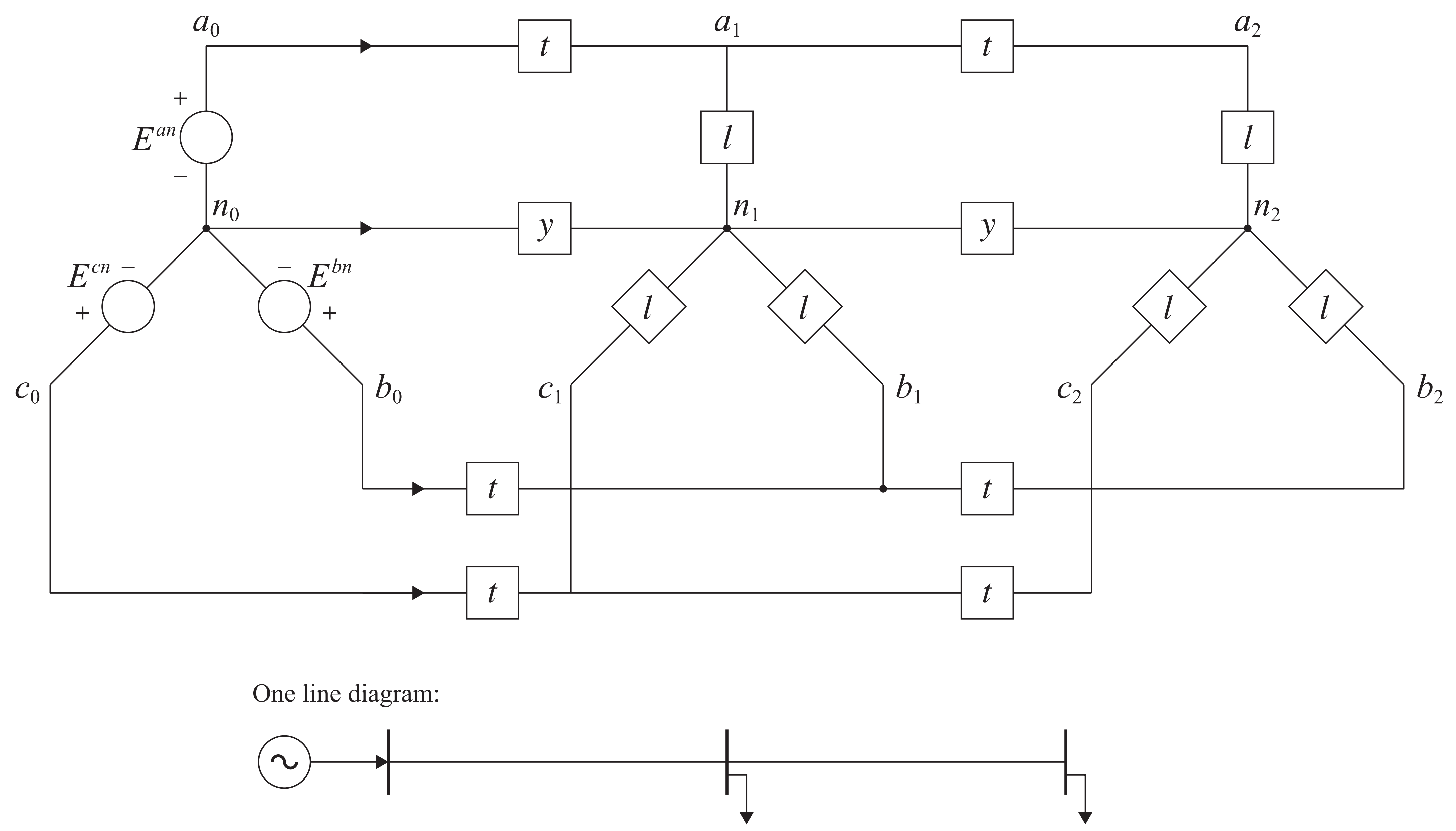
Solution [per-phase analysis]

Balanced and phase-decoupled voltages and currents lead to equivalent per-phase system and per-phase analysis

- Since $V_{nn'} = 0$, can assume a neutral line between n and n' (same potential)
- Analyze phase a equivalent circuit
- Variables in phases b and c are obtained from phase-a variables and rotating by 120°

Phase decoupling

Example



Show:

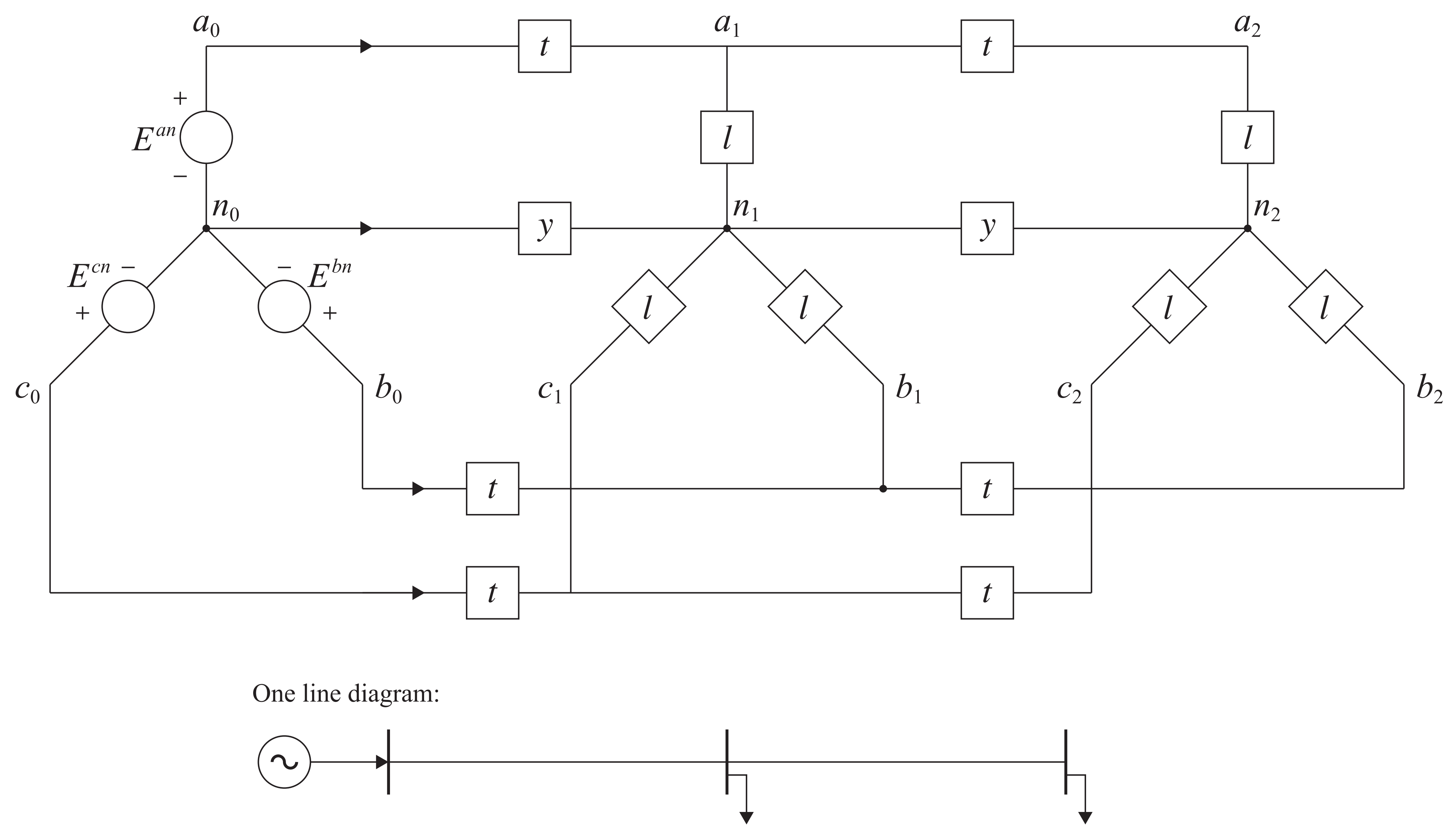
1. $V^{n_0 n_1} = V^{n_1 n_2} = 0$
2. All currents and voltages are balanced positive sequence sets
3. Phases are decoupled, i.e.,

$$E_0^{an} = V^{a_0 a_1} + V_1^{an}$$
$$V_1^{an} = V^{a_1 a_2} + V_2^{an}$$

Solution: see PSA Ch 1

Phase decoupling

Example



Show:

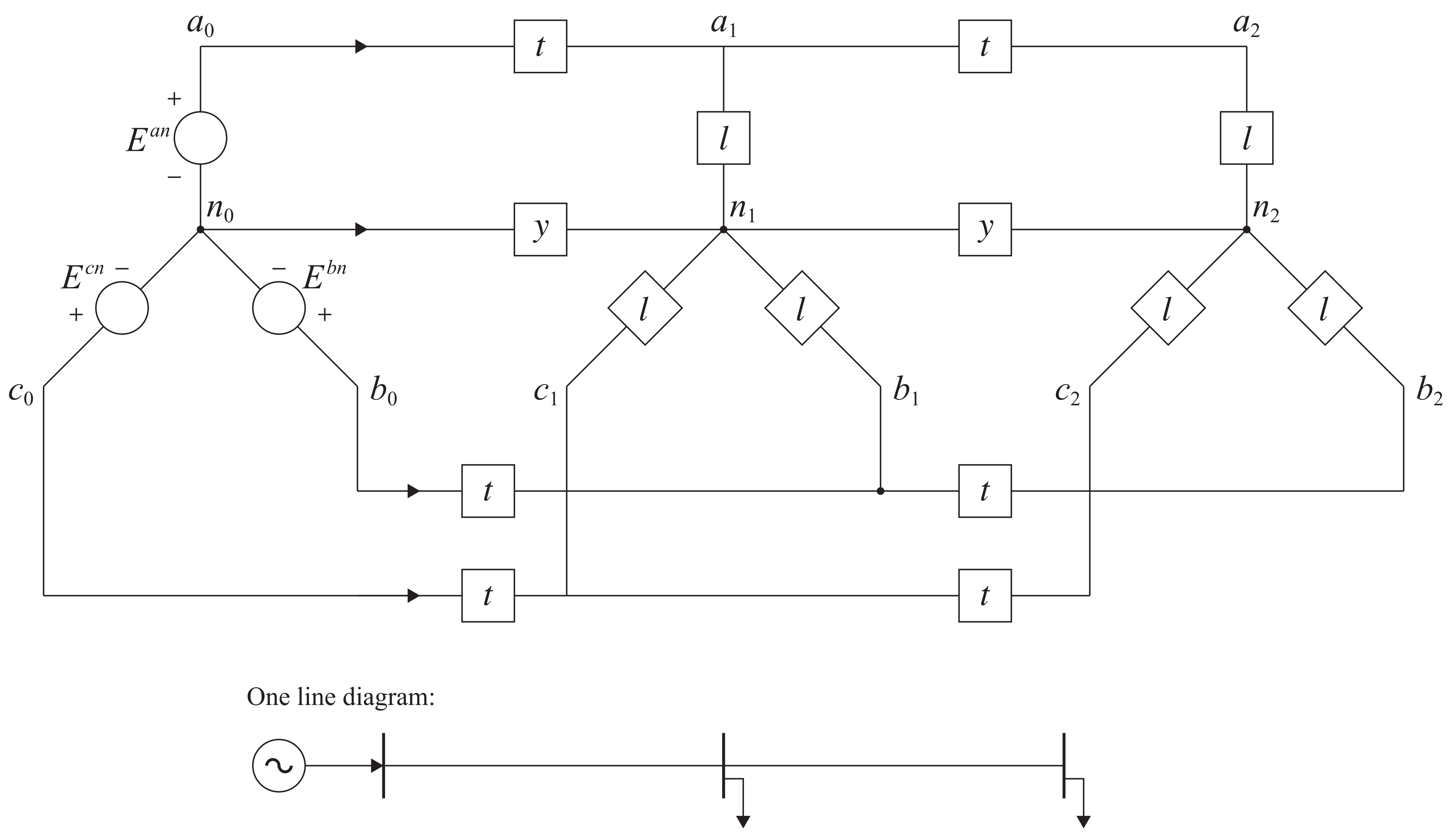
1. $V^{n_0 n_1} = V^{n_1 n_2} = 0$

Implications:

- Zero currents on neutral lines even if present ⇒ can assume neutrals are connected or not for analysis
- No physical wires necessary for return currents, saving materials

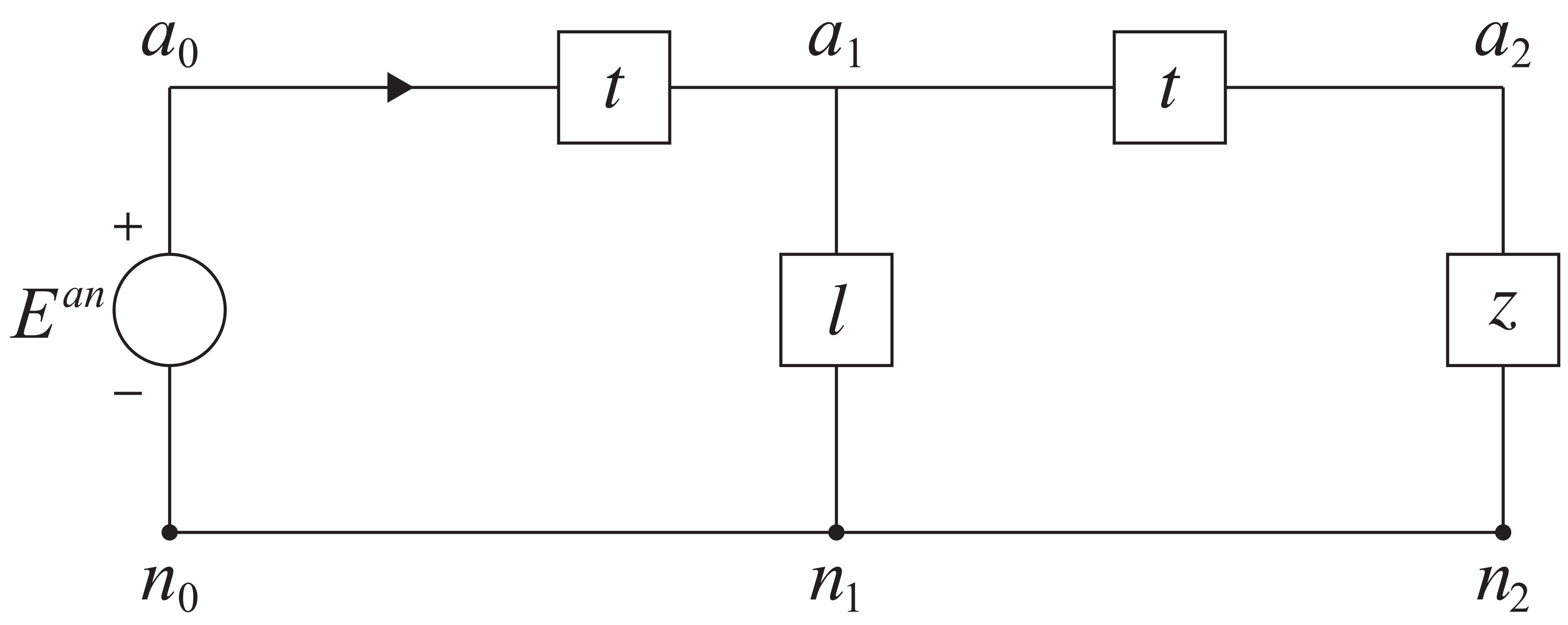
Phase decoupling

Example



Phase a equivalent circuit:

1. $V^{n_0 n_1} = V^{n_1 n_2} = 0$

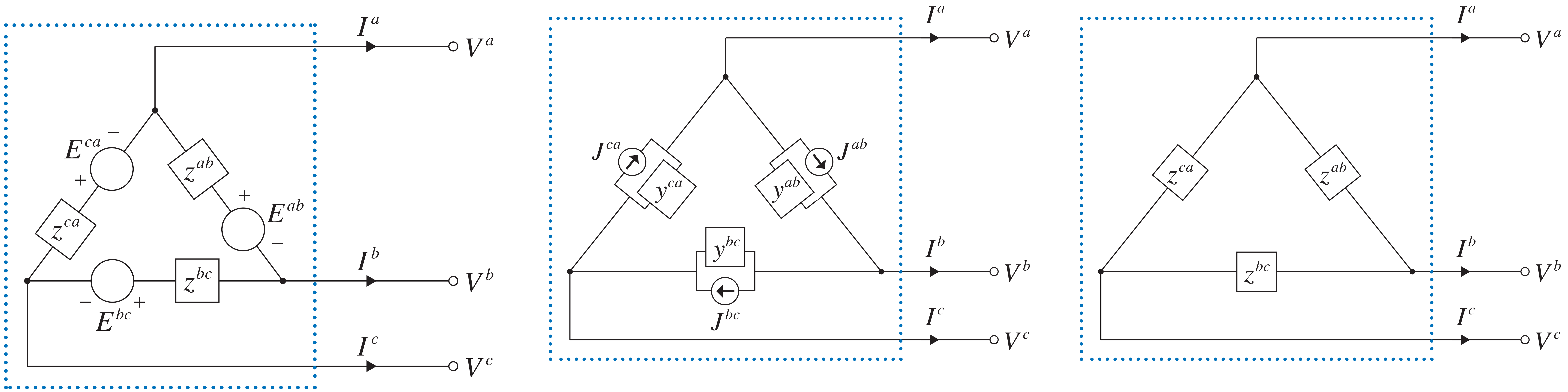


Outline

1. Single-phase systems
2. Balanced three-phase systems
    - Internal and terminal vars
    - Balanced vectors and conversion matrices $\Gamma, \Gamma^\mathsf{T}$
    - Balanced systems in Y configurations
    - Balanced systems in Δ configurations
    - Per-phase analysis
3. Complex power

Balanced 3-phase systems

Δ configuration



(a) Voltage source E^Δ (b) Current source J^Δ (c) Impedance z^Δ

Balanced voltage source if internal voltage E^Δ is a balanced vector and $z^\Delta := z^{ab}$▯

- positive sequence: $E^{ab} = 1\angle\theta, \quad E^{bc} = 1\angle\theta - 120^\circ, \quad E^{ca} = 1\angle\theta + 120^\circ$ $E^\Delta \in \text{span}(\alpha_+)$
- negative sequence: $E^{ab} = 1\angle\theta, \quad E^{bc} = 1\angle\theta + 120^\circ, \quad E^{ca} = 1\angle\theta - 120^\circ$ $E^\Delta \in \text{span}(\alpha\_)$

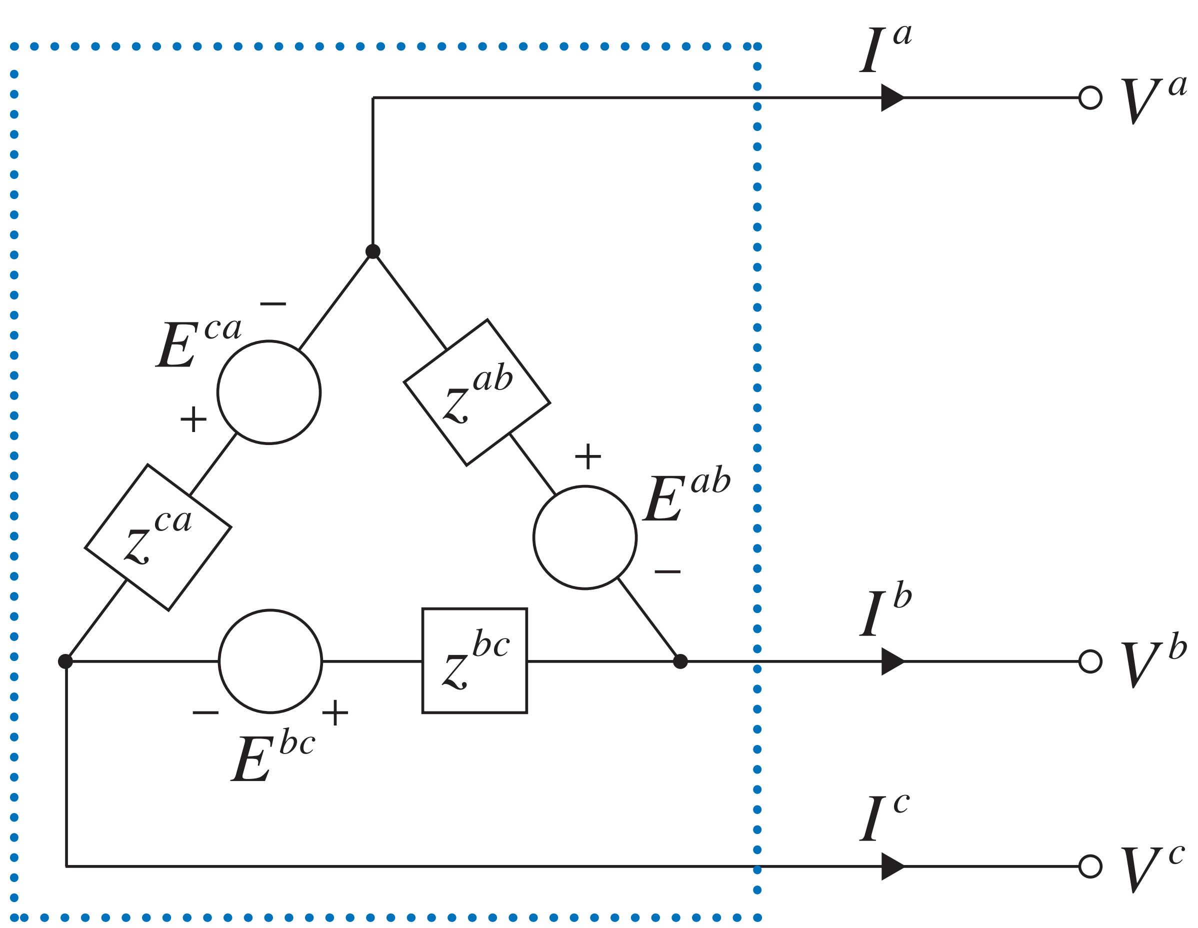
Balanced current source if $J^\Delta := (J^{ab}, J^{bc}, J^{ca}) \in \text{apan}(\alpha_+)$ and $y^\Delta := y^{ab}$▯

Ideal sources: $z = 0, \ y = 0$

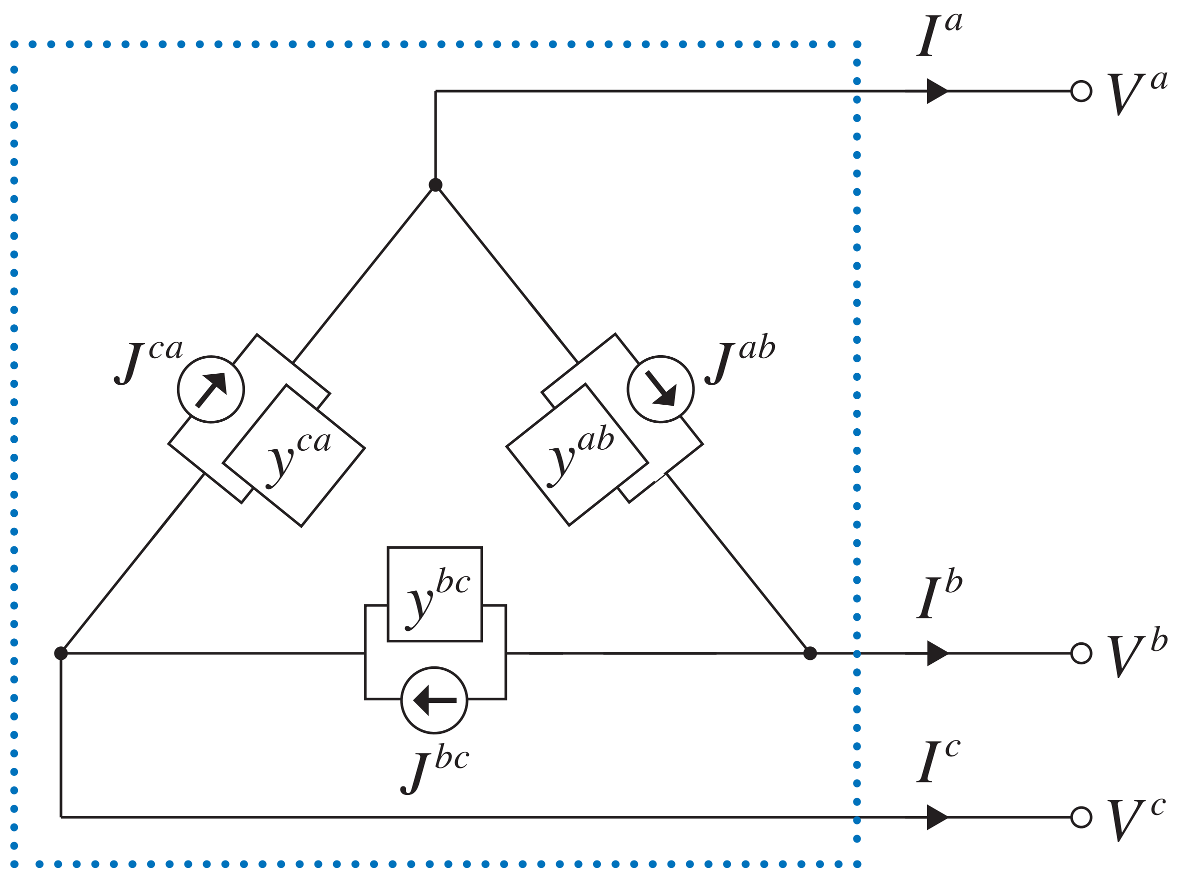
Balanced impedance if impedances are identical, i.e., $z^\Delta := z^{ab}$▯

Balanced 3-phase systems

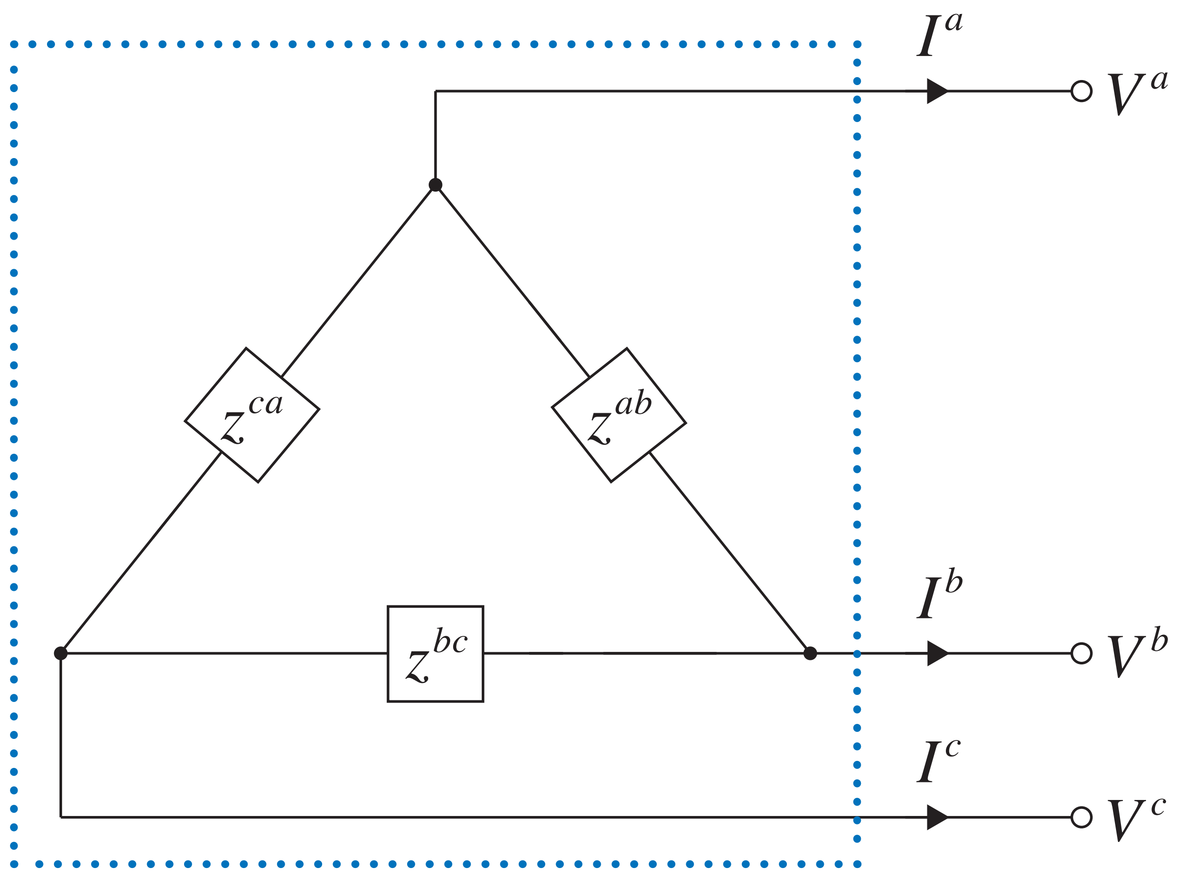
Δ configuration



(a) Voltage source E^Δ



(b) Current source J^Δ



(c) Impedance z^Δ

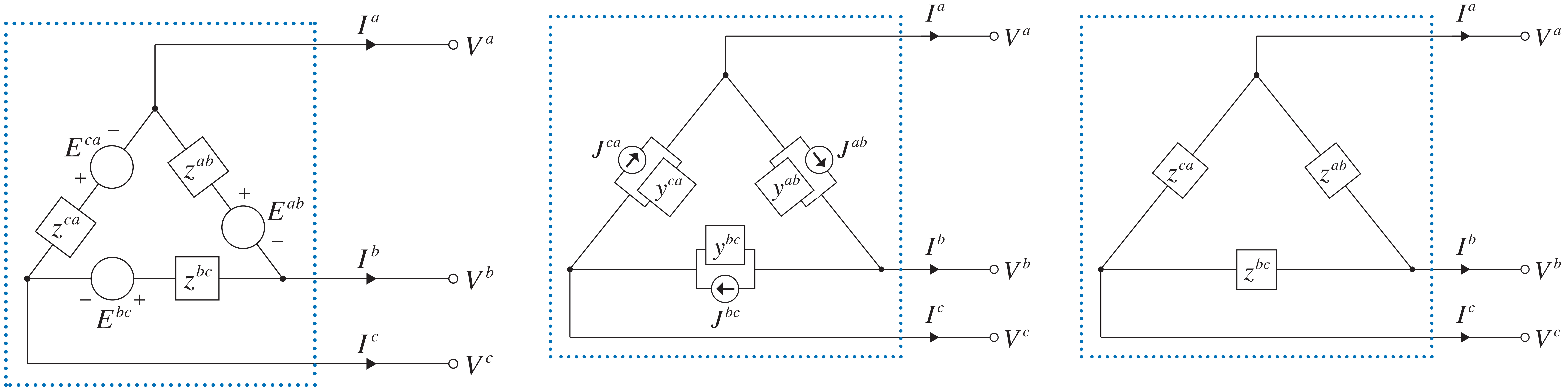
Corollary implies:

1. Sum to zero: $\mathbf{1}^\mathsf{T} E^\Delta = 0$ and $\mathbf{1}^\mathsf{T} I = 0$

   because $E^\Delta = E^{ab}\alpha_+$ and $I = -\Gamma^\mathsf{T} I^\Delta \;\Rightarrow\; \mathbf{1}^\mathsf{T} I = -\mathbf{1}^\mathsf{T}\Gamma^\mathsf{T} I^\Delta = 0$

Balanced 3-phase systems

Δ configuration



(a) Voltage source E^Δ (b) Current source J^Δ (c) Impedance z^Δ

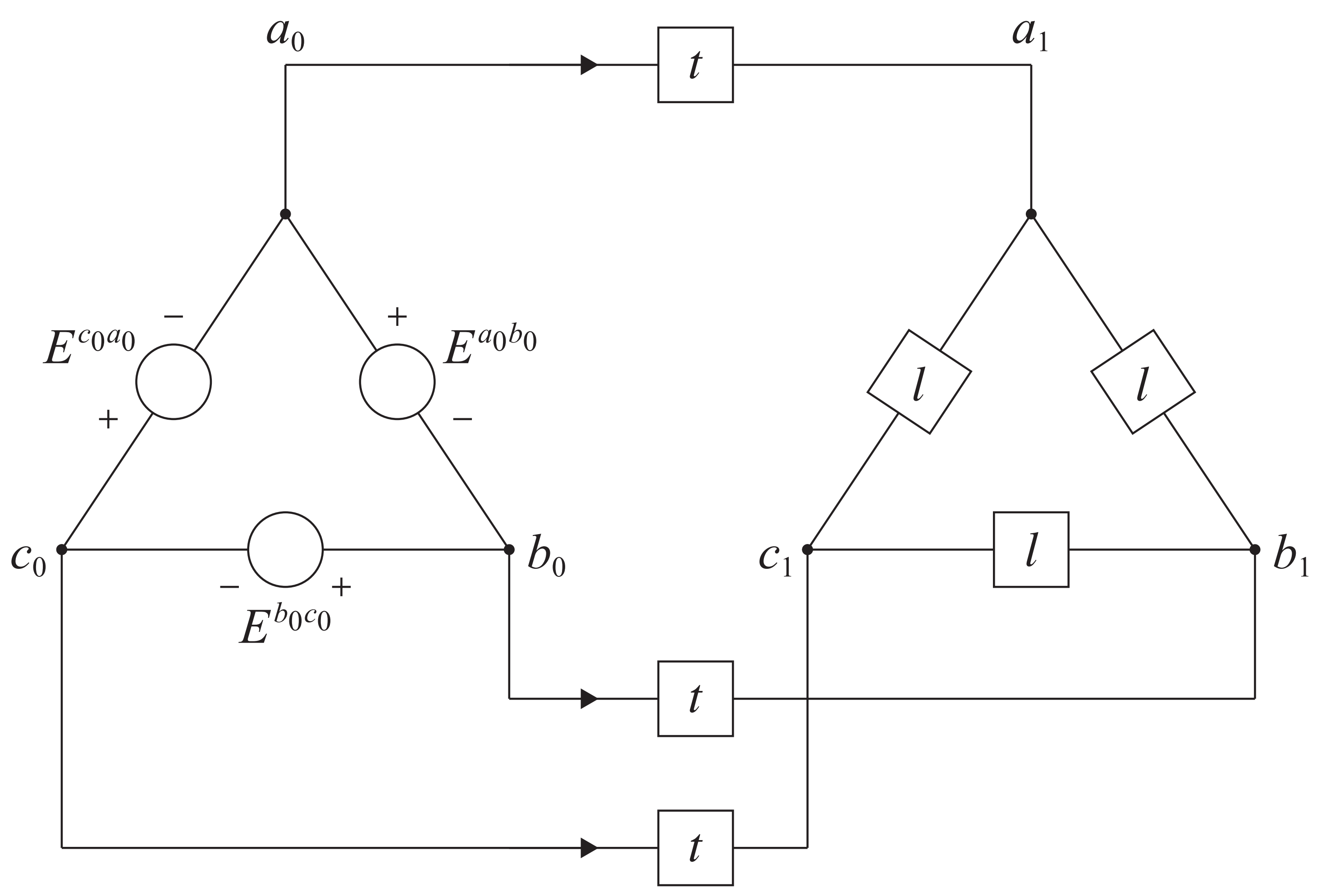
Corollary implies:

2. Ideal voltage source ($z^\Delta := 0$): line voltage $V^{\text{line}} = V^\Delta$ is balanced
3. Ideal current source ($y^\Delta := 0$): terminal current $I = -\Gamma^\mathsf{T} J^\Delta = -(1-\alpha^2)J^\Delta$ since $J^\Delta \in$ apan(α_+)

Phases are decoupled, but what is a per-phase equivalent circuit ?

Balanced 3-phase systems

Example



Analysis shows: if $E^{\Delta} \in \text{span}(\alpha_+)$ then

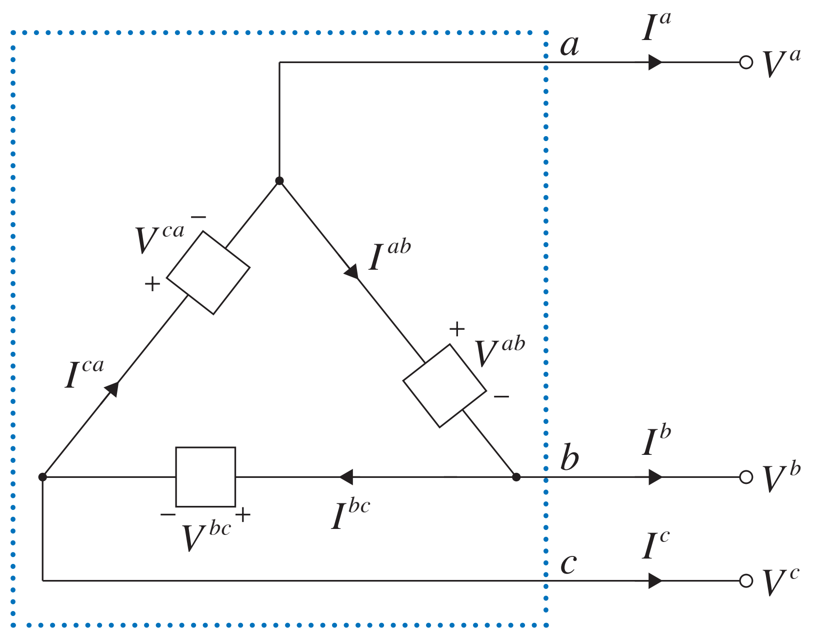
- Terminal current $I := (I^{a_0a_1}, I^{b_0b_1}, I^{c_0c_1})$, voltage drop across line $V := (V^{a_0a_1}, V^{b_0b_1}, V^{c_0c_1})$, and load voltage $U^{\Delta} := (V^{a_1b_1}, V^{b_1c_1}, V^{c_1a_1})$ are balanced

Phases are decoupled, but what is a per-phase equivalent circuit ?

$\Delta \to Y$ transformation
Voltage and current sources

Given: Δ device (e.g., voltage source) with internal voltage and current (V^{Δ}, I^{Δ})

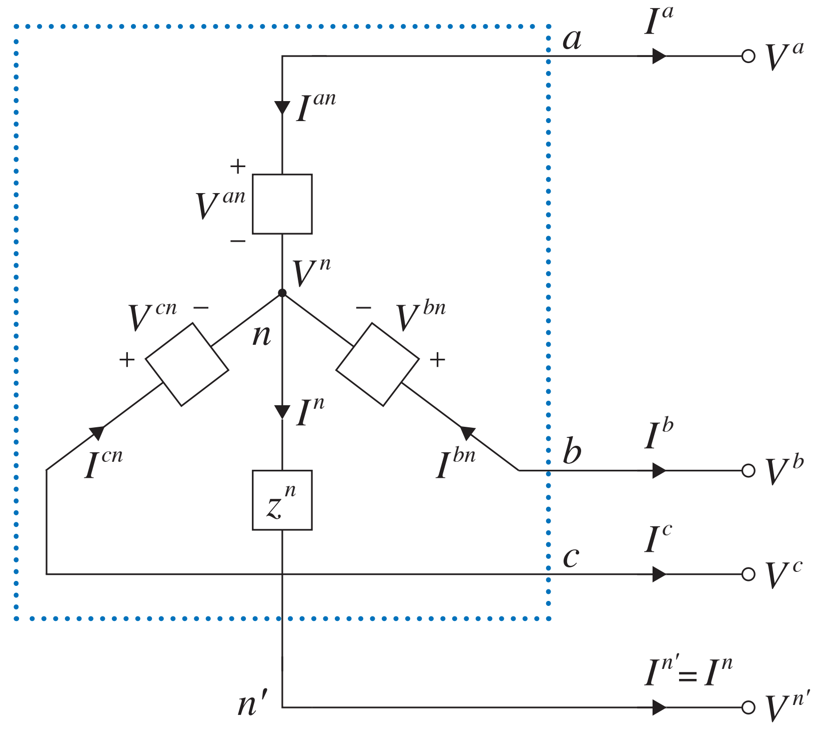
Equivalent Y-configured device: one with internal voltage and current $\left(V^{Y}_{\text{eq}}, I^{Y}_{\text{eq}}\right)$ that has an equivalent external behavior:



identical line-to-line voltage: $\Gamma V^{Y}_{\text{eq}} = V^{\Delta}$

identical line current: $I^{Y}_{\text{eq}} = \Gamma^{\mathsf{T}} I^{\Delta}$

zero neutral voltage $V^{n}_{\text{eq}} := 0$ s.t. $I^{n}_{\text{eq}} = \mathbf{1}^{\mathsf{T}} I^{Y}_{\text{eq}} = 0$



Balanced system: all voltages and currents are in span(α_+)

Hence

$$V^{Y}_{\text{eq}} = \frac{V^{\Delta}}{1-\alpha}, \qquad I^{Y}_{\text{eq}} = (1-\alpha^2) I^{\Delta}, \qquad V^{n}_{\text{eq}} := 0$$

$\Delta \to Y$ transformation

Voltage and current sources

Example

1. Voltage source (ideal) E^Δ : Y-equivalent is $E^Y \ := \ (1-\alpha)^{-1}E^\Delta \ = \ \dfrac{E^\Delta}{\sqrt{3}e^{i\pi/6}}$
2. Current source (ideal) J^Δ : Y-equivalent is $J^Y \ := \ (1-\alpha^2)J^\Delta \ = \ \sqrt{3}e^{-i\pi/6}J^\Delta$

Balanced system: all voltages and currents are in span(α_+)

Hence

$$V^Y_{\text{eq}} = \frac{V^\Delta}{1-\alpha}, \qquad I^Y_{\text{eq}} \ = \ (1-\alpha^2)I^\Delta, \qquad V^n_{\text{eq}} \ := \ 0$$

$\Delta \to Y$ transformation

Impedance

Given: balanced impedance $Z^{\Delta} :=$ Diag $\left(z^{\Delta}, z^{\Delta}, z^{\Delta}\right)$

componentwise division

Y-eq: balanced impedance $Z^{Y} :=$ Diag $\left(z^{Y}, z^{Y}, z^{Y}\right)$ with equivalent external behavior $V^{\text{line}} \% I$:

$\Delta \to Y$ transformation

Impedance

Given: balanced impedance $Z^\Delta := \text{Diag}\left(z^\Delta, z^\Delta, z^\Delta\right)$

componentwise division

Y-eq: balanced impedance $Z^Y := \text{Diag}\left(z^Y, z^Y, z^Y\right)$ with equivalent external behavior $V^{\text{line}} \% I$:

	Z^Δ : **balanced imp**	Z^Y : Y**-equilent**
internal model	$V^\Delta = Z^\Delta I^\Delta$	$V^Y = Z^Y I^Y$
line-to-line voltage V^{line}		
terminal current I		
$V^{\text{line}} \% I$		

$\Delta \rightarrow Y$ transformation

Impedance

componentwise division

Given: balanced impedance $Z^\Delta :=$ Diag $\left(z^\Delta, z^\Delta, z^\Delta\right)$

Y-eq: balanced impedance $Z^Y :=$ Diag $\left(z^Y, z^Y, z^Y\right)$ with equivalent external behavior $V^{\text{line}} \% I$:

	Z^Δ : **balanced imp**	Z^Y : Y**-equilent**
internal model	$V^\Delta = Z^\Delta I^\Delta$	$V^Y = Z^Y I^Y$
line-to-line voltage V^{line}	$V^\Delta = Z^\Delta I^\Delta$	
terminal current I	$-\Gamma^\mathsf{T} I^\Delta = -(1-\alpha^2) I^\Delta$	
$V^{\text{line}} \% I$	phase-$a = -z^\Delta/(1-\alpha^2)$	

$\Delta \rightarrow Y$ transformation

Impedance

componentwise division

Given: balanced impedance $Z^\Delta :=$ Diag $\left(z^\Delta, z^\Delta, z^\Delta\right)$

Y-eq: balanced impedance $Z^Y :=$ Diag $\left(z^Y, z^Y, z^Y\right)$ with equivalent external behavior $V^{\text{line}} \,\%\, I$:

	Z^Δ : **balanced imp**	Z^Y : Y**-equilent**
internal model	$V^\Delta = Z^\Delta I^\Delta$	$V^Y = Z^Y I^Y$
line-to-line voltage V^{line}	$V^\Delta = Z^\Delta I^\Delta$	$\Gamma V^Y = (1-\alpha) Z^Y I^Y$
terminal current I	$-\Gamma^{\mathsf{T}} I^\Delta = -(1-\alpha^2) I^\Delta$	$I = -I^Y$
$V^{\text{line}} \,\%\, I$	phase-$a = -z^\Delta/(1-\alpha^2)$	phase-$a = -(1-\alpha) z^Y$

$\Delta \to Y$ transformation

Impedance

Given: balanced impedance $Z^\Delta := \text{Diag}\left(z^\Delta, z^\Delta, z^\Delta\right)$

componentwise division

Y-eq: balanced impedance $Z^Y := \text{Diag}\left(z^Y, z^Y, z^Y\right)$ with equivalent external behavior $V^{\text{line}} \% I$:

	Z^Δ : **balanced imp**	Z^Y : Y**-equilent**
internal model	$V^\Delta = Z^\Delta I^\Delta$	$V^Y = Z^Y I^Y$
line-to-line voltage V^{line}	$V^\Delta = Z^\Delta I^\Delta$	$\Gamma V^Y = (1-\alpha) Z^Y I^Y$
terminal current I	$-\Gamma^\mathsf{T} I^\Delta = -(1-\alpha^2) I^\Delta$	$I = -I^Y$
$V^{\text{line}} \% I$	phase-$a = -z^\Delta/(1-\alpha^2)$	phase-$a = -(1-\alpha) z^Y$

Equivalent external behavior $V^{\text{line}} \% I \implies z^Y = \dfrac{z^\Delta}{(1-\alpha)(1-\alpha^2)} = \dfrac{z^\Delta}{3}$

Y-equivalent admittance $y^Y = 3y^\Delta$

Outline

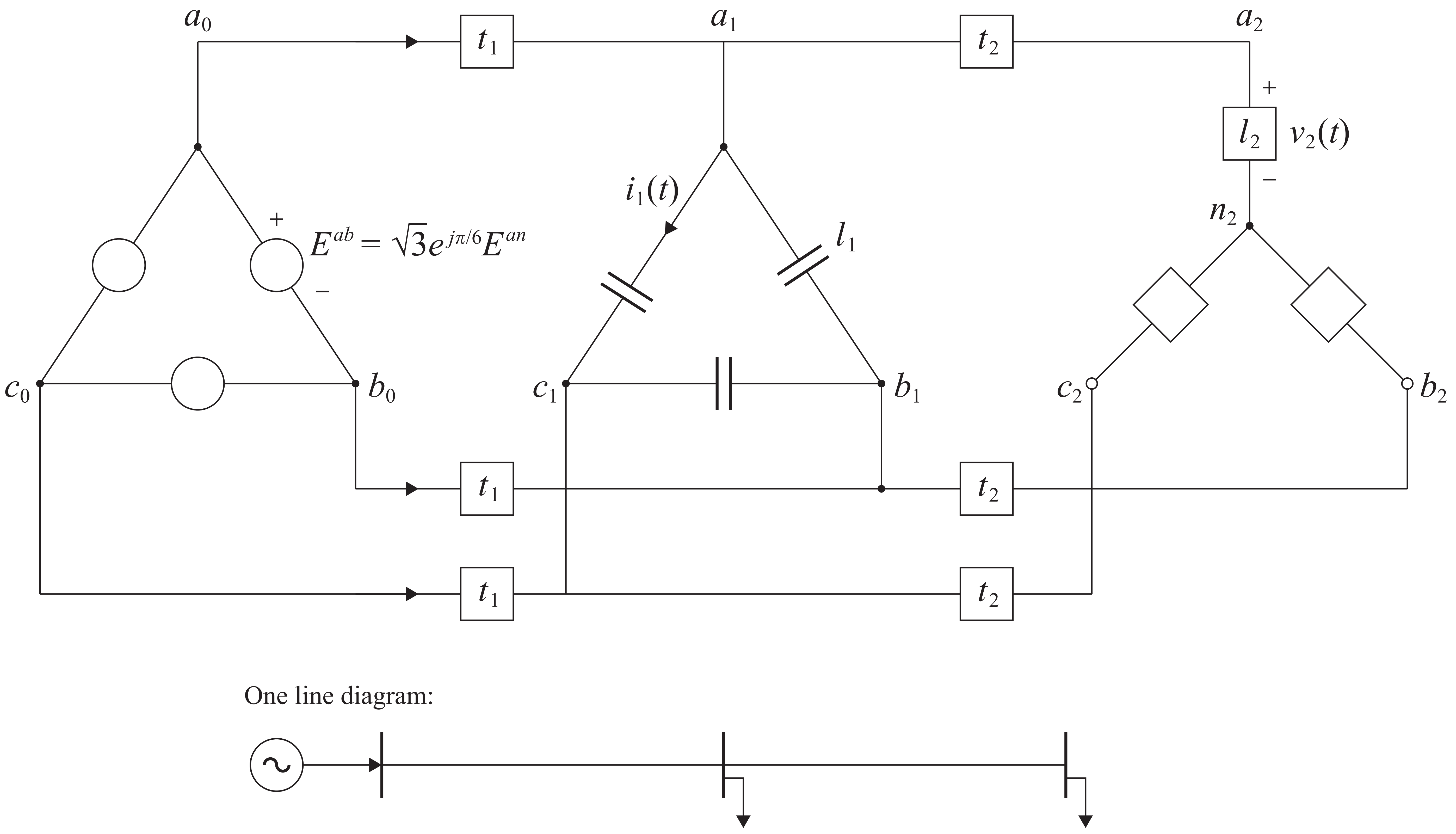
1. Single-phase systems
2. Balanced three-phase systems
    - Internal and terminal vars
    - Balanced vectors and conversion matrices $\Gamma, \Gamma^{\mathsf{T}}$
    - Balanced systems in Y configurations
    - Balanced systems in Δ configurations
    - Per-phase analysis
3. Complex power

Per-phase analysis

1. Convert all voltage sources, current sources, impedances in Δ configuration into their Y -equivalents
2. Solve for phase a vars using equivalent phase a circuit with all neutrals directly connected
3. If all sources are in positive-sequence sets, phase b and c vars are determined by subtracting $120°$ and $240°$ respectively from corresponding phase a vars. (If all sources are negative-sequence, add $120°$ and $240°$ instead.)
4. If vars in the internal of a Δ configuration are desired, drive them from original circuit.

Per-phase analysis

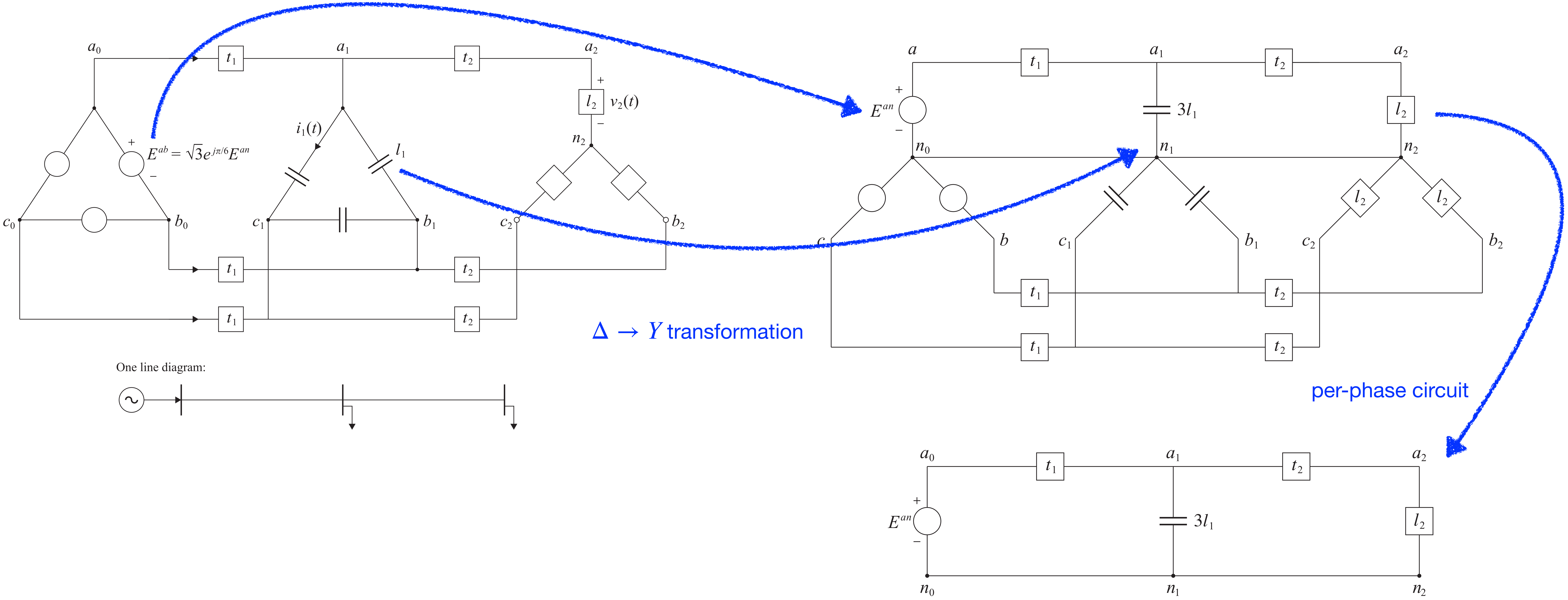
Example



Find: $i_1(t)$ and $v_2(t)$

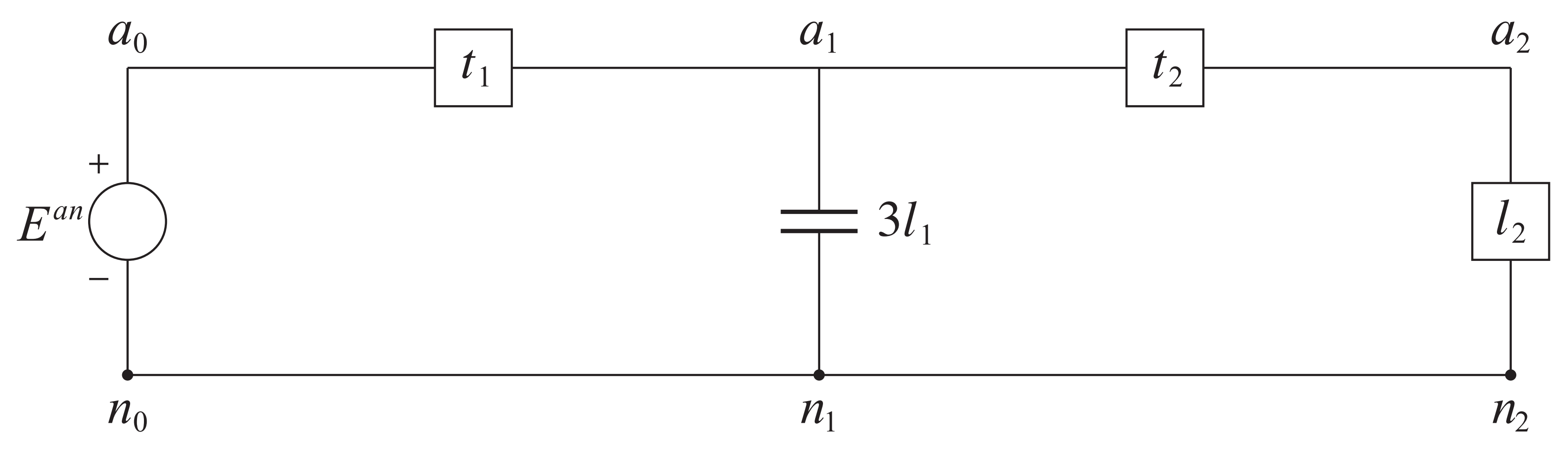
Per-phase analysis

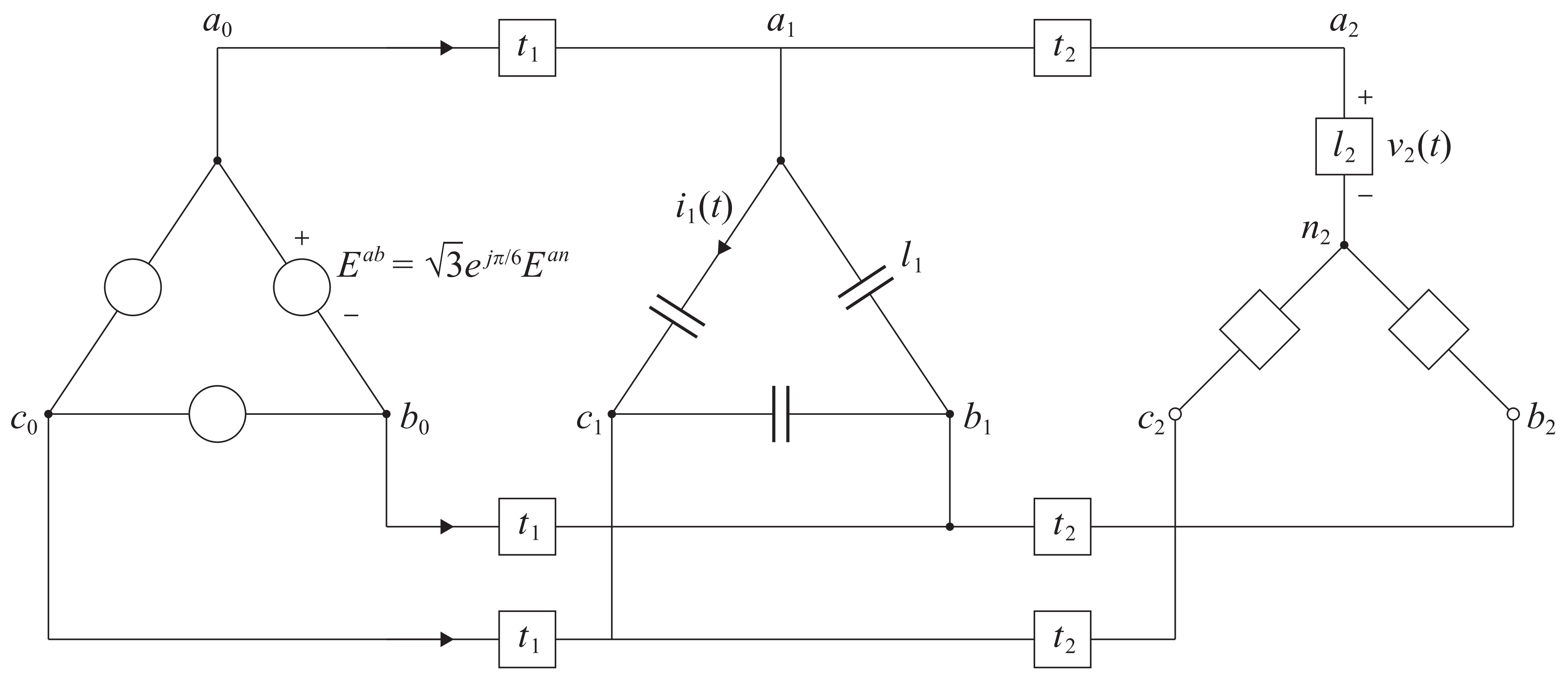
Example



Per-phase analysis

Example





Find: $i_1(t)$ and $v_2(t)$

Solution:

1. Using per-phase circuit, solve for $V^{a_1n_1}$ and $V^{a_2n_2}$
2. $v_2(t) \;=\; \sqrt{2}\,\left|V^{a_2n_2}\right| \cos\left(\omega t + \angle V^{a_2n_2}\right)$
3. $i_1(t) \;=\; \sqrt{2}\,\left|I^{a_1c_1}\right| \cos\left(\omega t + \angle I^{a_1c_1}\right)$
4. To calculate $I^{a_1c_1}$, obtain $V^{a_1b_1} \;=\; \sqrt{3}e^{i\pi/6}V^{a_1n_1}$
5. Obtain $I^{a_1b_1} \;=\; l_1V^{a_1b_1} \;=\; \sqrt{3}l_1e^{i\pi/6}V^{a_1n_1}$
6. Obtain $I^{a_1c_1} \;=\; -\,I^{a_1b_1}e^{i2\pi/3} \;=\; 3\sqrt{3}e^{-i\pi/6}l_1V^{a_1n_1}$

Per-phase analysis

1. Convert all voltage sources, current sources, impedances in Δ configuration into their Y -equivalents
2. Solve for phase a vars using equivalent phase a circuit with all neutrals directly connected
3. If all sources are in positive-sequence sets, phase b and c vars are determined by subtracting 120° and 240° respectively from corresponding phase a vars. (If all sources are negative-sequence, add 120° and 240° instead.)
4. If vars in the internal of a Δ configuration are desired, drive them from original circuit.

Can this approach be formally justified for general networks ?

Yes, see Part III on Unbalanced three-phase networks

Outline

1. Single-phase systems
2. Balanced three-phase systems
3. Complex power
    - Single-phase power
    - Three-phase power
    - Advantages of 3ϕ systems

Single-phase power

Instantaneous power:

$$p(t) := v(t)i(t)$$
$$= \frac{V_{\text{max}} I_{\text{max}}}{2} \left(\cos(\theta_V - \theta_I) + \cos(2\omega t + \theta_V + \theta_I) \right)$$

Average power:

$$\frac{1}{T}\int_0^T p(t)dt = \frac{V_{\text{max}} I_{\text{max}}}{2} \cos(\theta_V - \theta_I)$$

$\phi := \theta_V - \theta_I$: power factor angle

Single-phase power

Complex power:

$$S \ := \ V\bar{I} \ = \ \frac{V_{\text{max}} I_{\text{max}}}{2} \ e^{i(\theta_V - \theta_I)} \ = \ |V||I| e^{i\phi}$$

Active and reactive power:

$$P \ := \ |V||I| \cos\phi \quad \text{kW} \qquad Q := |V||I| \sin\phi \quad \text{var}$$

Apparent power:

$$|S| \ = \ |V||I| \ = \ \sqrt{P^2 + Q^2} \quad \text{VA}$$

Instantaneous and complex power

Relationship:

$$p(t) \;=\; P \;+\; P\cos 2(\omega t + \theta_I) - Q\sin 2(\omega t + \theta_I)$$

Average power:

$$P \;=\; \frac{1}{T}\int_0^T p(t)dt$$

Power delivered to impedance

Voltage and current across impedance are related

$$V = zI$$

Complex power

$$S = |z|\,|I|^2 e^{i\phi}, \qquad \phi := \angle z = \theta_V - \theta_I$$

	$\lvert z \rvert$	$\phi = \angle z$	P	Q	
Resistor $z = r$	r	0	$r\lvert I\rvert^2$	0	purely real power
Inductor $z = \mathbf{i}\omega l$	ωl	$\pi/2$	0	$\omega l\lvert I\rvert^2$	purely reactive power
Capacitor $z = (\mathbf{i}\omega c)^{-1}$	$(\omega c)^{-1}$	$-\pi/2$	0	$-(\omega c)^{-1}\lvert I\rvert^2$	purely reactive power

Table 1.2 Power delivered to RLC elements.

Power delivered to impedance

Instantaneous power delivered to

resistor r : $\quad p(t) = P\left(1 + \cos 2\left(\omega t + \theta_I\right)\right)$

inductor $i\omega l$: $\quad p(t) = -Q \sin 2\left(\omega t + \theta_I\right)$

capacitor $(i\omega c)^{-1}$: $\quad p(t) = Q \sin 2\left(\omega t + \theta_V\right)$

Three-phase power

Complex power

Per-phase power: $S := V_{an} \bar{I}_{an}$

Three-phase power: $S_{3\phi} := V_{an}\bar{I}_{an} + V_{bn}\bar{I}_{bn} + V_{cn}\bar{I}_{cn} = 3S$

because $V^Y := V^{an}\alpha_+$ and $I^Y := I^{an}\alpha_+$, and hence

$$S_{3\phi} = I^{Y\mathsf{H}} V^Y = V^{an}\bar{I}^{an}(\alpha_+^{\mathsf{H}}\alpha_+) = 3S$$

$(\alpha_+^{\mathsf{H}}\alpha_+ = 3)$

Three-phase power

Instantaneous power

Instantaneous 3ϕ power is constant

$$p_{3\phi}(t) := v_a(t)i_a(t) + v_b(t)i_b(t) + v_c(t)i_c(t) \quad = \quad 3P$$

Implications: 3ϕ motor receives constant torque

More generally, a balanced K-based system has a total instantaneous power

$$p_{K\phi}(t) = KP \text{ for } K \geq 3$$

In constrast, instantaneous 1ϕ power is sinusoidal

$$p(t) \;=\; P \;+\; P\cos 2(\omega t + \theta_I) - Q\sin 2(\omega t + \theta_I)$$

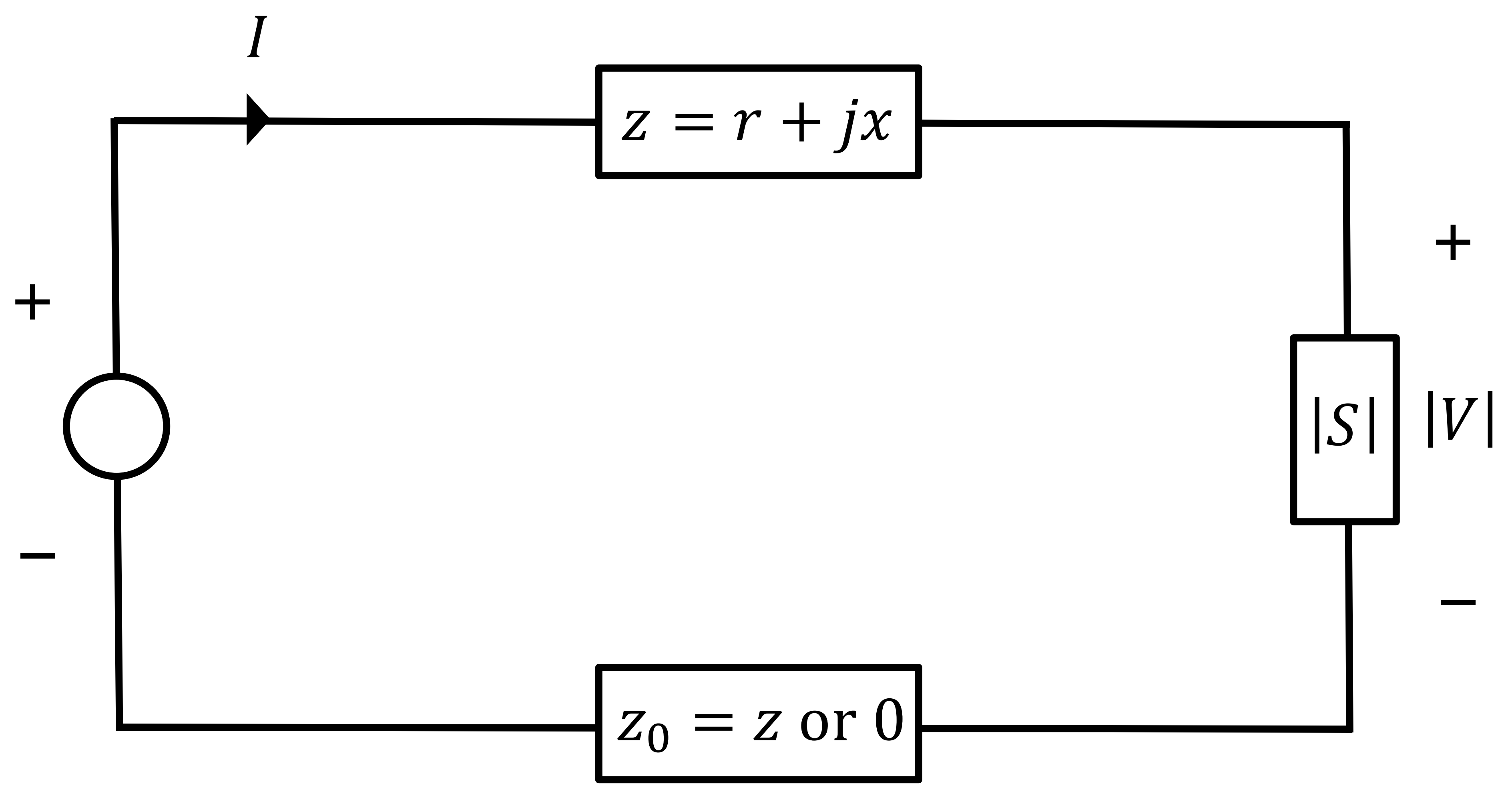
Three-phase power
Instantaneous power

Instantaneous 3ϕ power is constant

$$
\begin{aligned}
p_{3\phi}(t) \;:=\; & v_a(t)i_a(t) + v_b(t)i_b(t) + v_c(t)i_c(t) \\
= \; & |V_a||I_a|\left(\cos\phi \;+\; \cos(2\omega t + \theta_V + \theta_I)\right) \\
& + \; |V_a||I_a|\left(\cos\phi \;+\; \cos(2\omega t + (\theta_V - 2\pi/3) + (\theta_I - 2\pi/3))\right) \\
& + \; |V_a||I_a|\left(\cos\phi + \cos(2\omega t + (\theta_V + 2\pi/3) + (\theta_I + 2\pi/3))\right) \\
= \; & 3|V_a||I_a|\cos\phi \;+\; |V_a||I_a|\underbrace{\left(\cos\theta(t) \;+\; \cos(\theta(t) - 4\pi/3) \;+\; \cos(\theta(t) + 4\pi/3)\right)}_{=0} \\
= \; & 3P
\end{aligned}
$$

Savings from 3ϕ system

Example



Spec:

Supply load with power $|S|$ at voltage $|V|$

Distance between generator & load: d

Line impedance $z = r + ix$ ohm/meter

Resistance / unit length $r = \frac{\rho}{\text{area}}$

Line current $\leq \delta$ area

Savings:

Material required: $m_{3\phi} = \frac{1}{2} m_{1\phi}$

Active power loss: $l_{3\phi} = \frac{1}{2} l_{1\phi}$

Summary

1. Single-phase systems
   - Steady-state behavior of power systems can be described by voltage and current phasors
   - Component models: single-phase devices (PSA Ch1), line (PSA Ch 2), transformer (PSA Ch 3)
   - Phasors satisfy Kirchhoff's and Ohm's laws, as do corresponding time-domain quantities
   - A one-line diagram is defined by its equivalent circuit
2. Three-phase systems
   - A three-phase device can be in Y or Δ configuration
   - In a balanced system, all Δ-configured devices have Y-equivalents
   - All voltages and currents in a balanced three-phase system are in span(α_+) and phase-decoupled
   - This enables per-phase analysis using an equivalent per-phase circuit
3. Complex power
   - Single-phase complex power is $S_{1\phi} := V^a \bar{I}^a$; instantaneous power is $p^a(t) := v^a(t) i^a(t)$.
   - Three-phase complex power is $S_{3\phi} := \mathbf{1}^{\mathsf{T}} V I^{\mathsf{H}} = 3 S_{1\phi}$
   - Three-phase instantaneous power $p_{3\phi}(t) := \sum_{\phi} p^{\phi}(t) = 3P$