Power System Analysis

Chapter 10 Semidefinite relaxations: BIM

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Outline

1. Relaxations of QCQP
2. Application to OPF
3. Exactness condition: linear separability
4. Exactness condition: small angle difference

Outline

1. Relaxations of QCQP
   - SDP relaxation
   - Partial matrices and rank-1 characterization
   - Feasible sets
   - Semidefinite relaxations and solution recovery
   - Tightness of relaxations
   - Chordal relaxation
2. Application to OPF
3. Exactness condition: linear separability
4. Exactness condition: small angle difference

Dealing with nonconvexity

OPF is nonconvex and NP-hard

- There are 3 common ways to deal with nonconvexity

1. Linear approximation
   - e.g. DC OPF is widely used for electricity market applications
2. Local algorithms, e.g., Newton-Raphson, interior-point
   - Optimality conditions studied earlier for convex problems not applicable
   - Lyapunov-like condition guarantees that, if local algorithm computes a local optimum, it is global optimum
3. Convex relaxation, e.g., semidefinite relaxation
   - Lyapunov-like condition also guarantees exactness of any convex relaxation
   - Optimality conditions studied earlier apply to convex relaxations

Unlike approximations, convex relaxation has 3 advantages

- We can easily check if a solution of relaxation is a global optimum
- If not, it provides a lower bound on optimal value
- If relaxation is infeasible, then the nonconvex problem is infeasible

QCQP

Quadratically constrained quadratic program:

$$\min_{x \in \mathbb{C}^n} \quad x^{\mathsf{H}} C_0 x$$

$$\text{s.t.} \quad x^{\mathsf{H}} C_l x \ \leq \ b_l, \qquad l = 1, \ldots, L$$

- C_l : $n \times n$ Hermitian matrix
- $b_l \ \in \ \mathbb{R}$
- Homogeneous QCQP : all monomials are of degree 2
- OPF can be formulated as (nonconvex) QCQP

QCQP

Equivalent problem

Using $x^{\mathsf{H}} C_l x = \operatorname{tr}\left(C_l x x^{\mathsf{H}}\right)$, this is equivalent to:

$$\begin{array}{lll} \min\limits_{X \in \mathbb{S}^n, x \in \mathbb{C}^n} & \operatorname{tr}\left(C_0 X\right) & \\ \text{s.t.} & \operatorname{tr}\left(C_l X\right) \leq b_l, & l = 1, \ldots, L \\ & X = x x^{\mathsf{H}} & \end{array}$$

- Any psd rank-1 matrix $X \in \mathbb{S}_+^{n \times n}$ has a spectral decomposition $X = x x^{\mathsf{H}}$ for some $x \in \mathbb{C}^n$
- x is unique up to a rotation, i.e., x satisfies $X = \left(x e^{i\theta}\right)\left(x e^{i\theta}\right)^{\mathsf{H}}$ for any $\theta \in \mathbb{R}$
- Therefore can eliminate x

QCQP

Equivalent problem

Eliminating $x \implies$ minimization over psd matrices X:

$$\begin{aligned} \min_{X \in \mathbb{S}^n} \quad & \text{tr}\left(C_0 X\right) \\ \text{s.t.} \quad & \text{tr}\left(C_l X\right) \ \leq \ b_l, \quad l = 1, \ldots, L \\ & X \ \succeq \ 0, \quad \text{rank}(X) \ = \ 1 \end{aligned}$$

- $\text{tr}\left(C_l X\right) \leq b_l$ is linear in X
- $X \succeq 0$ is convex in X
- $\text{rank}(X) = 1$ is nonconvex in X

Removing rank constraint yields SDP relaxation

SDP relaxation

SDP relaxation of QCQP

$$
\begin{aligned}
\min_{X \in \mathbb{S}^n} \quad & \operatorname{tr}\left(C_0 X\right) \\
\text{s.t.} \quad & \operatorname{tr}\left(C_l X\right) \;\le\; b_l, \quad l = 1, \ldots, L \\
& X \;\succeq\; 0
\end{aligned}
$$

- This is a standard semidefinite program which is a convex problem
- Solution strategy:
  - Solve SDP for an optimal solution X^{opt}
  - If $\operatorname{rank}\left(X^{\text{opt}}\right) = 1$, then $x^{\text{opt}} \in \mathbb{C}^n$ from spectral decomposition from $X^{\text{opt}} = x^{\text{opt}}\left(x^{\text{opt}}\right)^{\text{H}}$
- If $\operatorname{rank}\left(X^{\text{opt}}\right) > 1$, then, in general, no feasible solution of QCQP can be directly obtained

SDP relaxation

SDP relaxation of QCQP

$$\begin{aligned} \min_{X \in \mathbb{S}^n} \quad & \text{tr}\left(C_0 X\right) \\ \text{s.t.} \quad & \text{tr}\left(C_l X\right) \leq b_l, \quad l = 1, \ldots, L \\ & X \succeq 0 \end{aligned}$$

- Even though SDP is convex, for large networks, it is still computationally impractical
- How to exploit sparsity of large networks to reduce computational burden?

Ans: partial matrices and completions !

Partial matrices

A QCQP instance specified by $(C_0, C_l, b_l, l = 1,\ldots,L)$ induces graph $F := (N, E)$

- N : n nodes (where $C_l \in \mathbb{C}^{n\times n}$)
- $E \subseteq N \times N$: m links $(j,k) \in E$ iff $\exists l \in \{0,1,\ldots,L\}$ s.t. $[C_l]_{jk} = [C_l]_{kj}^{\mathsf{H}} \neq 0$

A partial matrix X_F is a set of $n + 2m$ complex numbers defined on $F = (N, E)$

$$X_F \ := \ \left\{ \ [X_F]_{jj}, [X_F]_{jk}, [X_F]_{kj} \ : \ j \in N, (j,k) \in E \ \right\}$$

- X_F can be interpreted as matrix with entries partially specified, or a partial matrix
- If F is complete graph, then X_F is a full $n \times n$ matrix

A completion X of X_F is a full $n \times n$ matrix that agrees with X_F on graph F:

$$[X]_{jj} \ = \ [X_F]_{jj}, \qquad [X]_{jk} \ = \ [X_F]_{jk}, \qquad [X]_{kj} \ = \ [X_F]_{kj}$$

Partial matrices

If q is clique (fully connected subgraph) of F with k nodes, then $X_F(q)$ is a fully specified $k \times k$ principal submatrix of X_F on q :

$$[X_F(q)]_{jj} := [X_F]_{jj}, \qquad [X_F(q)]_{jk} := [X_F]_{jk}, \qquad [X_F(q)]_{kj} := [X_F]_{kj},$$

Hermitian, psd, rank-1, trace

Partial matrix

Definition A partial matrix X_F is

- Hermitian ($X_F = X_F^{\mathsf{H}}$) if $[X_F]_{kj} = [X_F]_{jk}^{\mathsf{H}}$
- psd ($X_F \succeq 0$) if X_F is Hermitian and $X_F(q) \succeq 0$ for all cliques q of F (a set of psd constraints)
- rank-1 if rank $\left(X_F(q)\right) = 1$ for all cliques q of F (a set of psd constraints)

Hermitian, psd, rank-1, trace

Partial matrix

Definition A partial matrix X_F is

- Hermitian ($X_F = X_F^{\mathsf{H}}$) if $[X_F]_{kj} = [X_F]_{jk}^{\mathsf{H}}$
- psd ($X_F \succeq 0$) if X_F is Hermitian and $X_F(q) \succeq 0$ for all cliques q of F (a set of psd constraints)
- rank-1 if rank $\left(X_F(q)\right) = 1$ for all cliques q of F (a set of psd constraints)

- 2×2 psd if $X_F(j,k)$ is psd for all $(j,k) \in E$
- 2×2 rank-1 if $X_F(j,k)$ is rank-1 for all $(j,k) \in E$

where $X_F(j,k) \ := \ \begin{bmatrix} [X_F]_{jj} & [X_F]_{jk} \\ [X_F]_{kj} & [X_F]_{kk} \end{bmatrix}$

2×2 psd : $[X_F]_{jj} \ \geq \ 0, \quad [X_F]_{kk} \ \geq \ 0$

$$[X_X]_{jj}[X_X]_{kk} \ \geq \ \left| [X_F]_{jk} \right|^2$$

2×2 rank-1 : $[X_X]_{jj}[X_X]_{kk} \ = \ \left| [X_F]_{jk} \right|^2$

Hermitian, psd, rank-1, trace

Partial matrix

For partial matrix X_F

$$\text{tr}\left(C_l X_F\right) := \sum_{j \in N} [C_l]_{jj} [X_F]_{jj} + \sum_{j < k, (j,k) \in E} \left([C_l]_{jk} [X_F]_{kj} + [C_l]_{kj} [X_F]_{jk}\right)$$

If both C_l and X_F are Hermitian, then $\text{tr}\left(C_l X_F\right)$ is real:

$$\text{tr}\left(C_l X_F\right) = \sum_{j \in N} [C_l]_{jj} [X_F]_{jj} + 2 \sum_{j<k,(j,k)\in E} \text{Re}\left([C_l]_{jk} [X_F]_{kj}\right)$$

Chordal graph & extensions

F is a chordal graph if

- Either F has no cycles, or
- All minimal cycles (ones without chords) are of length 3

A chordal extension $c(F)$ of F is a chordal graph that contains F

- $X_{c(F)}$ is a chordal extension of X_F

Every graph has a (generally nonunique) chordal extension

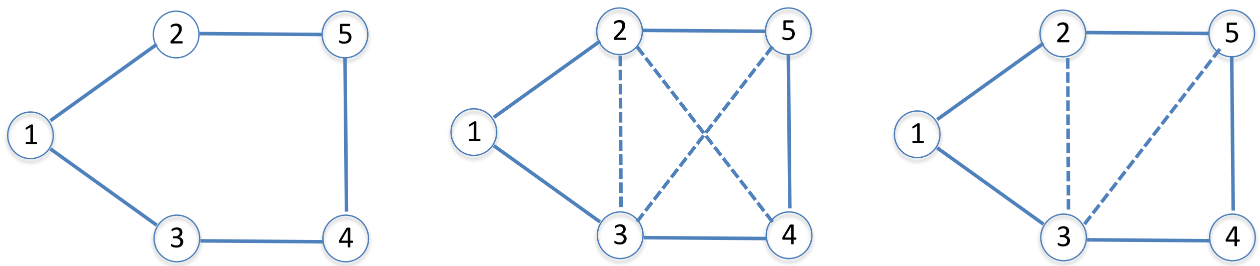
- Complete supergraph of F is a $c(F)$

Theorem [Grone et al 1984]: every psd partial matrix has a psd completion iff underlying graph is chordal

- We will extend this to psd rank-1 partial matrices

Partial matrix & chordal extensions

Example



$$W_F = \begin{bmatrix} x_{11} & x_{12} & x_{13} & & \\ x_{21} & x_{22} & & & x_{25} \\ x_{31} & & x_{33} & x_{34} & \\ & & x_{43} & x_{44} & x_{45} \\ & x_{52} & & x_{54} & x_{55} \end{bmatrix} \quad W_{c(F)} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & & \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\ & x_{42} & x_{43} & x_{44} & x_{45} \\ & x_{52} & x_{53} & x_{54} & x_{55} \end{bmatrix} \quad W_{c(F)} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & & \\ x_{21} & x_{22} & x_{23} & & x_{25} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\ & & x_{43} & x_{44} & x_{45} \\ & x_{52} & x_{53} & x_{54} & x_{55} \end{bmatrix}$$

2 cliques $W_{c(F)}(q)$

3 cliques $W_{c(F)}(q)$

Rank-1 characterization

Equivalent conditions

C1: $X \succeq 0, \quad \text{rank}(X) = 1$

C2: $X_{c(F)} \succeq 0, \quad \text{rank}(X_{c(F)}) = 1$

C3: $X_F(j,k) \succeq 0, \quad \text{rank}(X_F(j,k)) = 1, \qquad (j,k) \in E$

$$\sum_{(j,k)\in c} \angle [X_F]_{jk} = 0 \quad \text{mod } 2\pi$$ cycle condition

Theorem

Suppose F is connected and $X_{jj} > 0$, $\left[X_{c(F)}\right]_{jj} > 0$, $\left[X_F\right]_{jj} > 0$. Then C1 $\iff$ C2 $\iff$ C3.

Moreover, given X_F that satisfies C3, there is a unique completion X

Rank-1 characterization

Proof

C1: $X \succeq 0, \quad \text{rank}(X) = 1$

C2: $X_{c(F)} \succeq 0, \quad \text{rank}(X_{c(F)}) = 1$

C3: $X_F(j,k) \succeq 0, \quad \text{rank}(X_F(j,k)) = 1, \quad (j,k) \in E$

$$\sum_{(j,k)\in c} \angle[X_F]_{jk} = 0 \quad \text{mod } 2\pi$$

C1 ⇒ C2 ⇒ C3 ⇒ C1

X psd rank-1 $\implies$ all its principal submatrices are psd rank-1 : C1 ⇒ C2 ⇒ first part of C3

C2 ⇒ C3 : Suffices to prove cycle condition, by induction on size $k$of c

Induction hypothesis: for all cycles $c := (j_1, \ldots, j_k)$, $3 \le k \le n$, $\sum_{i=1}^{k} \angle\left[X_F\right]_{j_i j_{i+1}} = 0 \mod 2\pi$

Base case: $c := (n_1, n_2, n_3)$ is a clique. Hence the principal submatrix of $X_{c(F)}$

$$X_{c(F)}(n_1, n_2, n_3) := \begin{bmatrix} [X_{c(F)}]_{n_1 n_1} & [X_{c(F)}]_{n_1 n_2} & [X_{c(F)}]_{n_1 n_3} \\ [X_{c(F)}]_{n_2 n_1} & [X_{c(F)}]_{n_2 n_2} & [X_{c(F)}]_{n_2 n_3} \\ [X_{c(F)}]_{n_3 n_1} & [X_{c(F)}]_{n_3 n_2} & [X_{c(F)}]_{n_3 n_3} \end{bmatrix} \text{ is psd rank-1}$$

Rank-1 characterization

Proof

C1: $X \succeq 0, \quad \text{rank}(X) = 1$

C2: $X_{c(F)} \succeq 0, \quad \text{rank}(X_{c(F)}) = 1$

C3: $X_F(j,k) \succeq 0, \quad \text{rank}(X_F(j,k)) = 1, \quad (j,k) \in E$

$$\sum_{(j,k)\in c} \angle [X_F]_{jk} = 0 \mod 2\pi$$

C1 ⇒ C2 ⇒ C3 ⇒ C1

Induction hypothesis: for all cycles $c := (j_1, \ldots, j_k)$, $3 \le k \le n$, $\sum_{i=1}^{k} \angle \left[X_F\right]_{j_i j_{i+1}} = 0 \mod 2\pi$

Base case: $c := (n_1, n_2, n_3)$ is a clique. Hence $X_{c(F)}(n_1, n_2, n_3) = xx^{\mathsf{H}}$ and

$$\sum_{i=1}^{3} \angle \left[X_F\right]_{j_i j_{i+1}} = \angle\left(x_1 x_2^{\mathsf{H}}\right) + \angle\left(x_2 x_3^{\mathsf{H}}\right) + \angle\left(x_3 x_1^{\mathsf{H}}\right) = 0 \mod 2\pi$$

Rank-1 characterization

Proof

C1: $X \succeq 0, \quad \text{rank}(X) = 1$

C2: $X_{c(F)} \succeq 0, \quad \text{rank}(X_{c(F)}) = 1$

C3: $X_F(j,k) \succeq 0, \quad \text{rank}(X_F(j,k)) = 1, \quad (j,k) \in E$

$$\sum_{(j,k)\in c} \angle [X_F]_{jk} = 0 \mod 2\pi$$

C1 $\Rightarrow$ C2 $\Rightarrow$ C3 $\Rightarrow$ C1

Induction hypothesis: for all cycles $c := (j_1, \ldots, j_k)$, $3 \le k \le n$, $\sum_{i=1}^{k} \angle \left[X_F\right]_{j_i j_{i+1}} = 0 \mod 2\pi$

For any cycle $c := \left(j_1, \ldots, j_{k+1}\right)$. Take a chord (j_1, j_m) that breaks c into 2 cycles:

$$\sum_{i=1}^{k+1} \angle \left[X_F\right]_{j_i j_{i+1}} = \left(\sum_{i=1}^{m-1} \angle \left[X_F\right]_{j_i j_{i+1}} + \angle \left[X_F\right]_{j_m j_1} \right) + \left(\angle \left[X_F\right]_{j_1 j_m} + \sum_{i=m}^{k+1} \angle \left[X_F\right]_{j_i j_{i+1}} \right) = 0$$

Rank-1 characterization

Proof

C1: $X \succeq 0, \quad \text{rank}(X) = 1$

C2: $X_{c(F)} \succeq 0, \quad \text{rank}(X_{c(F)}) = 1$

C3: $X_F(j,k) \succeq 0, \quad \text{rank}(X_F(j,k)) = 1, \quad (j,k) \in E$

$$\sum_{(j,k)\in c} \angle [X_F]_{jk} = 0 \quad \text{mod } 2\pi$$

C1 ⇒ C2 ⇒ C3 ⇒ C1

Construct completion X from X_F by constructing vector $x \in \mathbb{C}^n$ s.t. $X = xx^{\mathsf{H}}$

Use method for solution recovery:

$$|x_j| := \sqrt{[X_F]_{jj}}, \qquad \angle x_j := \angle x_1 - \sum_{(i,k)\in \mathsf{P}_j} \angle [X_F]_{ik}$$

Cycle condition ensures any spanning tree yields the same angles $\angle x_j$

Rank-1 characterization

Proof

C1: $X \succeq 0, \quad \text{rank}(X) = 1$

C2: $X_{c(F)} \succeq 0, \quad \text{rank}(X_{c(F)}) = 1$

C3: $X_F(j,k) \succeq 0, \quad \text{rank}(X_F(j,k)) = 1, \quad (j,k) \in E$

$$\sum_{(j,k)\in c} \angle [X_F]_{jk} = 0 \quad \text{mod } 2\pi$$

C1 $\Rightarrow$ C2 $\Rightarrow$ C3 $\Rightarrow$ C1

Finally, to show X is unique, suppose $X = xx^{\mathsf{H}}$ and $\hat{X} = \hat{x}\hat{x}^{\mathsf{H}}$ are two distinct rank-1 completion of X_F
Then

$$|x_j| = \sqrt{[X_F]_{jj}} = |\hat{x}_j|, \qquad \theta_j - \theta_k = \angle [X_F]_{jk} = \hat{\theta}_j - \hat{\theta}_k$$

$\therefore C^{\mathsf{T}}(\hat{\theta} - \theta) = 0$ where C is bus-by-line incidence matrix. Cycle condition ensures there is a solution for $\hat{\theta} - \theta$ even if F is not a tree.

F connected $\Rightarrow \text{null}(C^{\mathsf{T}}) = \text{span}(\mathbf{1}) \Rightarrow \hat{\theta} = \theta + \gamma \mathbf{1} \Rightarrow \hat{x} = xe^{i\gamma}$

Hence $\hat{X} = \hat{x}\hat{x}^{\mathsf{H}} = \left(xe^{i\gamma}\right)\left(xe^{i\gamma}\right)^{\mathsf{H}} = X$

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Feasible sets

Feasible set of QCQP

$$\mathbb{V} \; := \; \{x \in \mathbb{C}^n \,|\, x^{\mathsf{H}} C_l x \leq b_l, \; l = 1,\ldots,L\}$$

psd rank-1 matrices X

$$\mathbb{X} \; := \; \{ \; X \in \mathbb{S}^n \mid X \text{ satisfies } \mathrm{tr}(C_l X) \leq b_l, \text{ C1 } \}$$

psd rank-1 chordal extensions $X_{c(F)}$

$$\mathbb{X}_{c(F)} \; := \; \{ \; X_{c(F)} \mid X_{c(F)} \text{ satisfies tr} \left(C_l X_{c(F)} \right) \leq b_l, \text{ C2 } \}$$

psd rank-1 partial matrices X_F

$$\mathbb{X}_F \; := \; \{ \; X_F \mid X_F \text{ satisfies tr} \left(C_l X_F \right) \leq b_l, \text{ C3 } \}$$

Feasible sets

Equivalence

Corollary

Fix any connected F. Any partial matrix $X_{c(F)} \in \mathbb{X}_{c(F)}$ or $X_F \in \mathbb{X}_F$ has a unique psd rank-1 completion $X \in \mathbb{X}$

Definition: Two sets A and B are equivalent ($A \equiv B$) if there is a bijection between them

Theorem

$\mathbb{V} \equiv \mathbb{X} \equiv \mathbb{X}_{c(F)} \equiv \mathbb{X}_F$

Implication: A feasible $x \in \mathbb{V}$ can be recovered from any partial matrix $X_{c(F)} \in \mathbb{X}_{c(F)}$ or $X_F \in \mathbb{X}_F$ through spectral decomposition (but there is a simpler way to compute $x \in \mathbb{V}$ than completion)

Equivalent problems

QCQP

$$\min_{x\in\mathbb{C}^n} \quad x^{\mathsf{H}} C_0 x \qquad \text{subject to} \qquad x \in \mathbb{V}$$

is equivalent to min over matrices and partial matrices:

$$\min_{X} \quad \operatorname{tr}\left(C_0 X\right) \qquad \text{subject to} \qquad X \in \hat{\mathbb{X}}$$

where $\hat{\mathbb{X}} \ \in \ \left\{\mathbb{X}, \mathbb{X}_{c(F)}, \mathbb{X}_F\right\}$

Implications:
Instead of solving for $X \in \mathbb{X}$, solve for $X_{c(F)} \in \mathbb{X}_{c(F)}$ or $X_F \in \mathbb{X}_F$ which are much smaller for large sparse networks

Equivalent problems

QCQP

$$\min_{x \in \mathbb{C}^n} \quad x^{\mathsf{H}} C_0 x \qquad \text{subject to} \qquad x \in \mathbb{V}$$

is equivalent to min over matrices and partial matrices:

$$\min_{X} \quad \text{tr}\left(C_0 X\right) \qquad \text{subject to} \qquad X \in \hat{\mathbb{X}}$$

where $\hat{\mathbb{X}} \in \left\{\mathbb{X}, \mathbb{X}_{c(F)}, \mathbb{X}_F\right\}$

Computational challenge remains:

$\mathbb{X}$, $\mathbb{X}_{c(F)}$, $\mathbb{X}_F$ are all nonconvex

Semidefinite relaxations

Convex supersets

$$\mathbb{X}^+ := \{X \in \mathbb{S}^n \mid X \text{ satisfies } \mathrm{tr}(C_l X) \leq b_l,\ X \succeq 0\}$$

$$\mathbb{X}^+_{c(F)} := \{X_{c(F)} \mid X_{c(F)} \text{ satisfies } \mathrm{tr}\left(C_l X_{c(F)}\right) \leq b_l,\ X_{c(F)} \succeq 0\}$$

$$\mathbb{X}^+_F := \{X_F \mid X_F \text{ satisfies } \mathrm{tr}\left(C_l X_F\right) \leq b_l,\ X_F(j,k) \succeq 0,\ (j,k) \in E\}$$

Semidefinite relaxations:

QCQP-sdp :	$\min_X \ C\left(X_F\right)$	s.t.	$X \in \mathbb{X}^+$	most complex
QCQP-ch :	$\min_{X_{c(F)}} \ C\left(X_F\right)$	s.t.	$X_{c(F)} \in \mathbb{X}^+_{c(F)}$	
QCQP-socp :	$\min_{X_F} \ C\left(X_F\right)$	s.t.	$X_F \in \mathbb{X}^+_F$	simplest

Semidefinite relaxations

Solution recovery

If a feasible / optimal solution of semidefinite relaxation lies in $\mathbb{X}$, $\mathbb{X}_{c(F)}$, or $\mathbb{X}_F$, then can recover feasible / optimal $x \in \mathbb{V}$ of QCQP

Recovery procedure: given $X_F \in \mathbb{X}_F$, pick an arbitrary spanning tree

1. Set $|x_1| := \sqrt{[X_F]_{11}}$ and $\angle x_1$ to arbitrary value

2. For $j = 2,\ldots,n$,

$$|x_j| := \sqrt{[X_F]_{jj}}, \qquad \angle x_j := \angle x_1 - \sum_{(i,k)\in \mathsf{P}_j} \angle [X_F]_{ik}$$

P_j : path from bus 1 to bus j in an arbitrary spanning tree rooted at bus 1

Cycle condition ensures any spanning tree yields the same angles $\angle x_j$

Tightness

Definition

1. A is an effective subset of B ($A \sqsubseteq B$) if given any $a \in A$, $\exists\ b \in B$ with same cost $C_A(a) = C_B(b)$
2. A is similar to B ($A \simeq B$) if $A \sqsubseteq B$ and $B \sqsubseteq A$

Theorem [Tightness]

1. $\mathbb{V} \sqsubseteq \mathbb{X}^+ \simeq \mathbb{X}^+_{c(F)} \sqsubseteq \mathbb{X}^+_F$
2. If F is a tree, then $\mathbb{V} \sqsubseteq \mathbb{X}^+ \simeq \mathbb{X}^+_{c(F)} \simeq \mathbb{X}^+_F$

Corollary [Optimal values]

1. $C^{\text{qcqp}} \geq C^{\text{sdp}} = C^{\text{ch}} \geq C^{\text{socp}}$
2. If F is a tree, then $C^{\text{qcqp}} \geq C^{\text{sdp}} = C^{\text{ch}} = C^{\text{socp}}$

Semidefinite relaxations

Implications

1. Radial networks: Solve QCQP-socp
   - Simplest computationally
   - Same tightness as QCQP-ch and QCQP-SDP
2. Meshed networks: Solve QCQP-ch or QCQP-socp
   - QCQP-ch strictly tighter than QCQP-socp, and same tightness as QCQP-sdp
   - QCQP-ch can be orders of magnitude simpler computationally than QCQP-sdp for large sparse networks
   - QCQP-ch is as complex as QCQP-sdp in the worst case

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Chordal relaxations

$$\mathbb{X}^+_{c(F)} := \{X_{c(F)} \mid X_{c(F)} \text{ satisfies } \operatorname{tr}\left(C_l X_{c(F)}\right) \leq b_l, \; X_{c(F)} \succeq 0\}$$

$X_{c(F)} \succeq 0$: multiple constraints, one for each maximal clique q of chordal extension $c(F)$

1. List all maximal cliques q_k of $c(F)$, $k = 1,\ldots,K$
2. Derive appropriate Hermitian matrices X_k

Then $X_{c(F)} \succeq 0$ is shorthand for: $X_k \succeq 0, \; k = 1,\ldots,K$

Explain each step next

Chordal relaxations

$$\mathbb{X}^{+}_{c(F)} := \{X_{c(F)} \mid X_{c(F)} \text{ satisfies } \operatorname{tr}\left(C_l X_{c(F)}\right) \le b_l,\ X_{c(F)} \succeq 0\}$$

1. List all maximal cliques q_k of $c(F)$, $k = 1,\ldots,K$

   Computing all maximal cliques of general graph is NP-hard, but can be done efficiently for chordal graph

2. Derive appropriate Hermitian matrices X_k

   Illustrate using example

Then $X_{c(F)} \succeq 0$ is shorthand for: $X_k \succeq 0,\ k = 1,\ldots,K$

Chordal relaxations

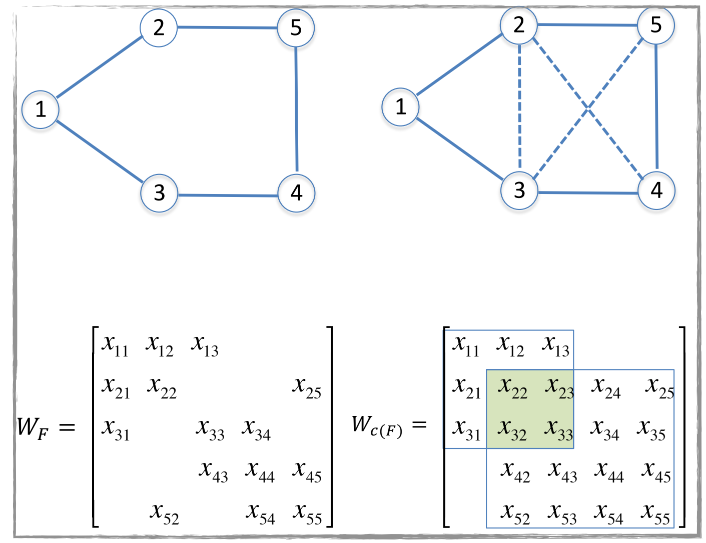
Example

1. Two cliques: $q_1 := (1,2,3)$ and $q_2 := (2,3,4,5)$
2. q_1 and q_2 share node 2 $\Longrightarrow$ principal submatrices of $X_{c(F)}$ overlap in 4 entries, requiring 4 decoupling vars and constraints:

$$X_1' \;:=\; \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & u_{22} & u_{22} \\ x_{31} & u_{32} & u_{33} \end{bmatrix}, \quad X_2 \;:=\; \begin{bmatrix} x_{22} & x_{23} & x_{24} & x_{25} \\ x_{32} & x_{33} & x_{34} & x_{35} \\ x_{42} & x_{43} & x_{44} & x_{45} \\ x_{52} & x_{53} & x_{54} & x_{55} \end{bmatrix}$$

$$u_{jk} \;= x_{jk} \quad \text{for } j,k = 2,3$$

3. Then $X_{c(F)} \succeq 0$ is: $\quad X_1' \succeq 0, \quad X_2 \succeq 0$



Chordal relaxations

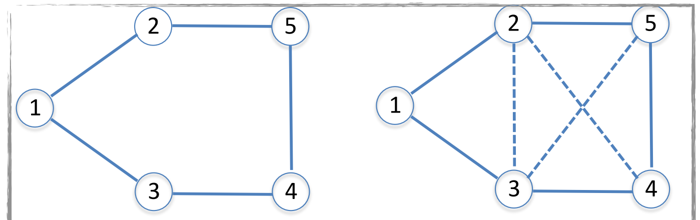
Example

Let $X' := \begin{bmatrix} X_1' & 0 \\ 0 & X_2 \end{bmatrix}$

Chordal relaxation is equivalent to SDP in standard form

$$
\begin{aligned}
\min_{X' \in \mathbb{S}^7} \quad & \mathrm{tr}(C_0' X') \\
\text{s.t.} \quad & \mathrm{tr}(C_l' X') \leq b_l, \qquad l = 1, \ldots, L \\
& \mathrm{tr}(C_r' X') = 0, \qquad r = 1,2,3,4 \\
& X' \succeq 0
\end{aligned}
$$

for appropriate C_l'



$$W_F = \begin{bmatrix} x_{11} & x_{12} & x_{13} & & \\ x_{21} & x_{22} & & & x_{25} \\ x_{31} & & x_{33} & x_{34} & \\ & & x_{43} & x_{44} & x_{45} \\ & x_{52} & & x_{54} & x_{55} \end{bmatrix} \qquad W_{c(F)} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & & \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\ & x_{42} & x_{43} & x_{44} & x_{45} \\ & x_{52} & x_{53} & x_{54} & x_{55} \end{bmatrix}$$

Outline

1. Relaxations of QCQP
2. Application to OPF
   - Semidefinte relaxation
   - Exact relaxation: definition
3. Exactness condition: linear separability
4. Exactness condition: small angle difference

OPF as QCQP

Recall

$$\begin{aligned}
\min_{V \in \mathbb{C}^{N+1}} \quad & V^{\mathsf{H}} C_0 V \\
\text{s.t.} \quad & p_j^{\min} \ \le \ \operatorname{tr}\left(\Phi_j V V^{\mathsf{H}}\right) \ \le \ p_j^{\max}, && j \in \overline{N} \\
& q_j^{\min} \ \le \ \operatorname{tr}\left(\Psi_j V V^{\mathsf{H}}\right) \ \le \ q_j^{\max}, && j \in \overline{N} \\
& v_j^{\min} \ \le \ \operatorname{tr}\left(E_j V V^{\mathsf{H}}\right) \ \le \ v_j^{\max}, && j \in \overline{N} \\
& \operatorname{tr}\left(\hat{Y}_{jk} V V^{\mathsf{H}}\right) \ \le \ \bar{I}_{jk}^{\max}, && (j,k) \in E \\
& \operatorname{tr}\left(\hat{Y}_{kj} V V^{\mathsf{H}}\right) \ \le \ \bar{I}_{kj}^{\max}, && (j,k) \in E
\end{aligned}$$

abbreviated as:
$\operatorname{tr}\left(C_l V V^{\mathsf{H}}\right) \le b_l, \ l = 1,\ldots,L$

Constraints

Given $V \in \mathbb{C}^{N+1}$, define partial matrix W_G by

$$[W_G]_{jj} := |V_j|^2, \qquad j \in \overline{N}$$

$$\left[W_G\right]_{jk} := V_j V_k^{\mathsf{H}} =: [W_G]_{kj}^{\mathsf{H}}, \qquad (j,k) \in E$$

Constraints in terms of W_G

$$p_j^{\min} \le \text{tr}\left(\Phi_j W_G\right) \le p_j^{\max}$$

$$q_j^{\min} \le \text{tr}\left(\Psi_j W_G\right) \le q_j^{\max}$$

$$v_j^{\min} \le \text{tr}\left(E_j W_G\right) \le v_j^{\max}$$

$$\text{tr}\left(\hat{Y}_{jk} W_G\right) \le I_{jk}^{\max}$$

$$\text{tr}\left(\hat{Y}_{kj} W_G\right) \le I_{kj}^{\max}$$

abbreviated as:

$\text{tr}\left(C_l W_G\right) \le b_l, \; l = 1, \ldots, L$

OPF and relaxations

OPF as QCQP

$$\min_{V} \ C_0(V) \qquad \text{s.t.} \quad \mathrm{tr}\left(C_l V V^{\mathsf{H}}\right) \le b_l, \ \ l = 1, \ldots, L$$

Semidefinite relaxations:

$$\text{OPF-sdp}: \quad \min_{W \in \mathbb{S}^{N+1}} C_0(W_G) \quad \text{s.t.} \quad \mathrm{tr}\left(C_l W\right) \le b_l, \ \ l = 1, \ldots, L, \quad W \succeq 0$$

$$\text{OPF-ch}: \quad \min_{W_{c(G)}} C_0(W_G) \quad \text{s.t.} \quad \mathrm{tr}\left(C_l W_{c(G)}\right) \le b_l, \ \ l = 1, \ldots, L, \quad W_{c(G)} \succeq 0$$

$$\text{OPF-socp}: \quad \min_{W_G} C_0(W_G) \quad \text{s.t.} \quad \mathrm{tr}\left(C_l W_G\right) \le b_l, \ \ l = 1, \ldots, L, \quad W_G(j,k) \succeq 0, \ \ (j,k) \in E$$

Exact relaxation

Definition

1. OPF-sdp is exact if every optimal solution W^{sdp} of OPF-sdp is psd rank-1
2. OPF-ch is exact if every optimal solution $W^{\text{ch}}_{c(G)}$ of OPF-ch is psd rank-1
3. OPF-socp is exact if every optimal solution W^{socp}_G of OPF-socp
    - is 2×2 psd rank-1, i.e., $W^{\text{socp}}_G(j,k)$ are psd rank-1 for all $(j,k) \in E$, and
    - satisfies cycle condition, i.e., $\sum_{(j,k)\in c} \angle [W^{\text{socp}}_G]_{jk} \; = \; 0 \mod 2\pi$

Remarks

1. Any optimal solution returned by optimization algorithm will work under this strong sense of exactness
2. Under sufficient exactness condition, optimal solution of OPF can be recovered even under weak sense of exactness (see below)

Outline

1. Relaxations of QCQP
2. Application to OPF
3. Exactness condition: linear separability
    - Sufficient condition for QCQP
    - Application to OPF
    - Proof
4. Exactness condition: small angle difference

QCQP and SOCP relaxation

QCQP:

$$\begin{aligned} \min_{x \in \mathbb{C}^n} \quad & x^{\mathsf{H}} C_0 x \\ \text{s.t.} \quad & x^{\mathsf{H}} C_l x \ \le \ b_l, \qquad l = 1, \dots, L \end{aligned}$$

SOCP relaxation:

$$\begin{aligned} \min_{X_G} \quad & \text{tr}\left(C_0 X_G\right) \\ \text{s.t.} \quad & \text{tr}\left(C_l X_G\right) \ \le \ b_l, \quad l = 1, \dots, L \\ & X_G(j,k) \ \succeq \ 0, \quad (j,k) \in E \end{aligned}$$

- $C_l : n \times n$ Hermitian matrix, $b_l \in \mathbb{R}$

Sufficient condition

C1: For every $(j, k) \in E$, $\exists \alpha_{jk}$ s.t. $\angle \left[C_l\right]_{jk} \in [\alpha_{ij}, \alpha_{ij} + \pi]$ for all $l = 0, \ldots, L$

C2: C_0 is positive definite

Theorem

Suppose G is a tree and C1 holds. Then

1. $C^{\text{opt}} = C^{\text{socp}}$
2. An optimal solution $x^{\text{opt}} \in \mathbb{C}^{N+1}$ of QCQP can be recovered from every optimal solution X_G^{socp} of its SOCP relaxation

X_G^{opt} may not be 2×2 rank-1 (i.e., SOCP may not be exact)
when optimal solutions of SOCP relaxation are nonunique

Sufficient condition

C1: For every $(j, k) \in E$, $\exists \alpha_{jk}$ s.t. $\angle \left[C_l\right]_{jk} \in [\alpha_{ij}, \alpha_{ij} + \pi]$ for all $l = 0, \ldots, L$

C2: C_0 is positive definite

Corollary

Suppose G is a tree and both C1 and C2 hold. Then SOCP relaxation is exact, i.e., every optimal solution X_G^{socp} is 2×2 psd rank-1

Cycle condition is vacuous since G is a tree

Application to OPF

Recall OPF as QCQP

$$\min_{V \in \mathbb{C}^{N+1}} \quad V^{\mathsf{H}} C_0 V$$

$$\text{s.t.} \quad p_j^{\min} \leq \operatorname{tr}\left(\Phi_j V V^{\mathsf{H}}\right) \leq p_j^{\max}, \qquad j \in \overline{N}$$

$$q_j^{\min} \leq \operatorname{tr}\left(\Psi_j V V^{\mathsf{H}}\right) \leq q_j^{\max}, \qquad j \in \overline{N}$$

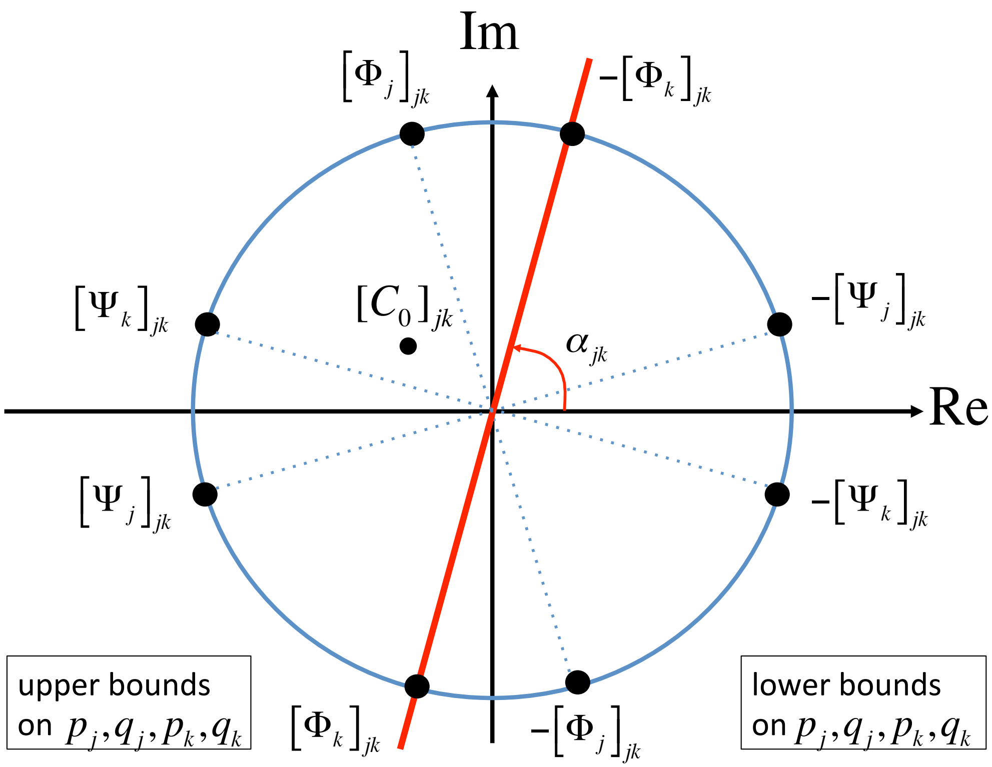
$$v_j^{\min} \leq \operatorname{tr}\left(E_j V V^{\mathsf{H}}\right) \leq v_j^{\max}, \qquad j \in \overline{N}$$

abbreviated as:

$\operatorname{tr}\left(C_l V V^{\mathsf{H}}\right) \leq b_l,\ l = 1, \ldots, L$

Application to OPF

Exactness condition



Corollary

Suppose G is a tree and both C1 and the diagram hold.

Then SOCP relaxation is exact

Sufficient condition

C1: For every $(j, k) \in E$, $\exists \alpha_{jk}$ s.t. $\angle \left[C_l\right]_{jk} \in [\alpha_{ij}, \alpha_{ij} + \pi]$ for all $l = 0, \ldots, L$

C2: C_0 is positive definite

Theorem

Suppose G is a tree and C1 holds. Then

1. $C^{\text{opt}} = C^{\text{socp}}$
2. An optimal solution $x^{\text{opt}} \in \mathbb{C}^{N+1}$ of QCQP can be recovered from every optimal solution X_G^{socp} of its SOCP relaxation

X_G^{opt} may not be 2×2 rank-1 (i.e., SOCP may not be exact)
when optimal solutions of SOCP relaxation are nonunique

Sufficient condition

Proof

Fix any partial matrix X_G that is feasible for SOCP. Suffices to construct construct $x \in \mathbb{C}^n$ s.t.

$$x^{\mathsf{H}} C_l x \;\leq\; \text{tr}\, C_l X_G, \qquad l = 0,1,\ldots,L$$

i.e., x is feasible for QCQP and has an equal or low cost

<u>*Case 1: X_G is 2×2 psd rank-1*</u>. Let $\angle x_1 := 0$ and

$$|x_j| \;:=\; \sqrt{[X_G]_{jj}}, \qquad \angle x_j \;:=\; -\sum_{(i,k)\in \mathsf{P}_j} \angle [X_G]_{ik}$$

(P_j : path from bus 1 to bus j)

Since $[X_G]_{jj}[X_G]_{kk} = |[X_G]_{jk}|^2$ we have (since X_G is Hermitian):

$$x^{\mathsf{H}} C_l x \;=\; \sum_{j,k} [C_l]_{jk}\, x_j^{\mathsf{H}} x_k \;=\; \sum_{j,k} [C_l]_{jk}\, |x_j|\,|x_k|\, e^{i\left(\angle x_k - \angle x_j\right)} \;=\; \sum_{j,k} [C_l]_{jk} \left|[X_G]_{jk}\right| e^{-i\angle [X_G]_{jk}} \;=\; \text{tr}\left(C_l X_G\right)$$

Sufficient condition

Proof

Fix any partial matrix X_G that is feasible for SOCP. Suffices to construct construct $x \in \mathbb{C}^n$ s.t.

$$x^{\mathsf{H}} C_l x \;\leq\; \text{tr } C_l X_G, \qquad l = 0,1,\ldots,L$$

i.e., x is feasible for QCQP and has an equal or low cost

Case 2: X_G is not 2×2 psd rank-1. Suppose $[X_G]_{jj}[X_G]_{kk} > |[X_G]_{jk}|^2$. We will

1. Construct $\hat{X}_G$ that is 2×2 psd rank-1
2. Show that tr $C_l \hat{X}_G \;\leq\;$ tr $C_l X_G$

Then can construct $x \in \mathbb{C}^n$ from $\hat{X}_G$ as in Case 1.

Sufficient condition

Proof

Case 2: X_G is not 2×2 psd rank-1. Suppose $[X_G]_{jj}[X_G]_{kk} > |[X_G]_{jk}|^2$. We will

1. Construct $\hat{X}_G$ that is 2×2 psd rank-1
2. Show that condition C1 implies: tr $C_l \hat{X}_G \;\leq\;$ tr $C_l X_G$

Then can construct $x \in \mathbb{C}^n$ from $\hat{X}_G$ as in Case 1.

1. Construction of $\hat{X}_G$

$$[\hat{X}_G]_{jj} := [X_G]_{jj}, \qquad\qquad [\hat{X}_G]_{jk} := [X_G]_{jk} \,+\, r_{jk} e^{-i\left(\frac{\pi}{2} - \alpha_{jk}\right)}$$

with $r_{jk} > 0$ chosen to ensure $\hat{X}_G$ is psd rank-1, i.e., to ensure

$$[\hat{X}_G]_{jj}[\hat{X}_G]_{kk} \;=\; \left| [\hat{X}_G]_{jk} \right|^2 \;=\; \left| [X_G]_{jk} + r_{jk} e^{-i\left(\frac{\pi}{2} - \alpha_{jk}\right)} \right|^2 \quad \Leftrightarrow \quad r_{jk}^2 + 2b\, r_{jk} - c \;=\; 0$$

where $b \;:=\;$ Re $\left([X_G]_{jk}\, e^{i\left(\frac{\pi}{2} - \alpha_{jk}\right)} \right)$ and $c \;:=\; [X_G]_{jj}\, [X_G]_{kk} - \left| [X_G]_{jk} \right|^2 \;>\; 0$

Sufficient condition

Proof

Case 2: X_G is not 2×2 psd rank-1. Suppose $[X_G]_{jj}[X_G]_{kk} > |[X_G]_{jk}|^2$. We will

1. Construct $\hat{X}_G$ that is 2×2 psd rank-1
2. Show that condition C1 implies: tr $C_l \hat{X}_G \;\leq\;$ tr $C_l X_G$

Then can construct $x \in \mathbb{C}^n$ from $\hat{X}_G$ as in Case 1.

1. Construction of $\hat{X}_G$

$$[\hat{X}_G]_{jj} := [X_G]_{jj}, \qquad [\hat{X}_G]_{jk} := [X_G]_{jk} + r_{jk} e^{-i\left(\frac{\pi}{2} - \alpha_{jk}\right)}$$

Therefore

$\hat{X}_G$ is psd rank-1 $\;\Leftrightarrow\;$ $[\hat{X}_G]_{jk} := [X_G]_{jk} + r_{jk} e^{-i\left(\frac{\pi}{2} - \alpha_{jk}\right)}$ with $r_{jk} := \sqrt{b^2 + c} - b > 0$

Sufficient condition

Proof

<u>*Case 2: X_G is not 2×2 psd rank-1*</u>. Suppose $[X_G]_{jj}[X_G]_{kk} > |[X_G]_{jk}|^2$. We will

1. Construct $\hat{X}_G$ that is 2×2 psd rank-1
2. Show that condition C1 implies: tr $C_l \hat{X}_G \;\leq\;$ tr $C_l X_G$

Then can construct $x \in \mathbb{C}^n$ from $\hat{X}_G$ as in Case 1.

<u>2. $\hat{X}_G$ is feasible for SOCP with lower or equal cost</u>

$$\text{tr}\left(C_l\left(\hat{X}_G - X_G\right)\right) = \sum_{(j,k)\in E} [C_l]_{jk}\left([\hat{X}_G]_{jk} - [X_G]_{jk}\right)^{\text{H}}$$

$$= 2 \sum_{j<k,(j,k)\in E} \text{Re}\left([C_l]_{jk} \cdot r_{jk}\, e^{i\left(\frac{\pi}{2} - \alpha_{jk}\right)}\right)$$

$$= 2 \sum_{\substack{j<k \\ (j,k)\in E}} \left|[C_l]_{jk}\right| \; r_{jk} \;\cos\left(\angle[C_l]_{jk} + \frac{\pi}{2} - \alpha_{jk}\right) \quad \leq \quad 0$$

Sufficient condition

Proof

Finally, if condition C2 holds as well, SOCP has a unique optimal solution X_G.

If X_G is 2×2 psd but not 2×2 rank-1, i.e., $[X_G]_{jj}[X_G]_{kk} > |[X_G]_{jk}|^2$ for some (j, k)

then proof above constructs $\hat{X}_G$ that is feasible for SOCP with lower or equal cost, contradicting uniqueness of X_G. Hence X_G must be 2×2 rank-1.

Outline

1. Relaxations of QCQP
2. Application to OPF
3. Exactness condition: linear separability
4. Exactness condition: small angle difference
   - Sufficient condition
   - 2-bus example

Assumptions

Assume

1. Series admittances are symmetric $y_{jk}^s = y_{kj}^s$ and shunt admittances are zero $y_{jk}^m = y_{kj}^m := 0$
2. Voltage magnitudes $|V_j| := 1$ pu are fixed
3. Reactive powers are ignored

Use polar form power flow equations

$\therefore$ Optimization over (s, v) reduces to optimization over (p, θ)

OPF formulation

$$\min_{p,P,\theta} \quad C(p)$$

s.t.
$$p_j^{\min} \le p_j \le p_j^{\max}, \qquad j \in \overline{N}$$

$$\theta_{jk}^{\min} \le \theta_{jk} \le \theta_{jk}^{\max}, \quad (j,k) \in E$$
constraints on line flows, line losses, or stability

$$p_j = \sum_{k:k\sim j} P_{jk}, \qquad j \in \overline{N}$$
nodal power balance

$$P_{jk} = g_{jk} - g_{jk}\cos\theta_{jk} - b_{jk}\sin\theta_{jk}, \quad (j,k) \in E$$
power flow equation (polar form)

where $V_j = |V_j| e^{i\theta_j}$ with $|V_j| := 1$ and $\theta_{jk} := \theta_j - \theta_k$

Eliminate P_{jk} and θ_{jk}

OPF formulation

Define injection region

$$\mathbb{P}_\theta := \left\{ p \in \mathbb{R}^n \;\middle|\; p_j = \sum_{k:k\sim j} \left(g_{jk} - g_{jk}\cos\theta_{jk} - b_{jk}\sin\theta_{jk} \right), \quad \theta_{jk}^{\min} \le \theta_{jk} \le \theta_{jk}^{\max} \right\}$$

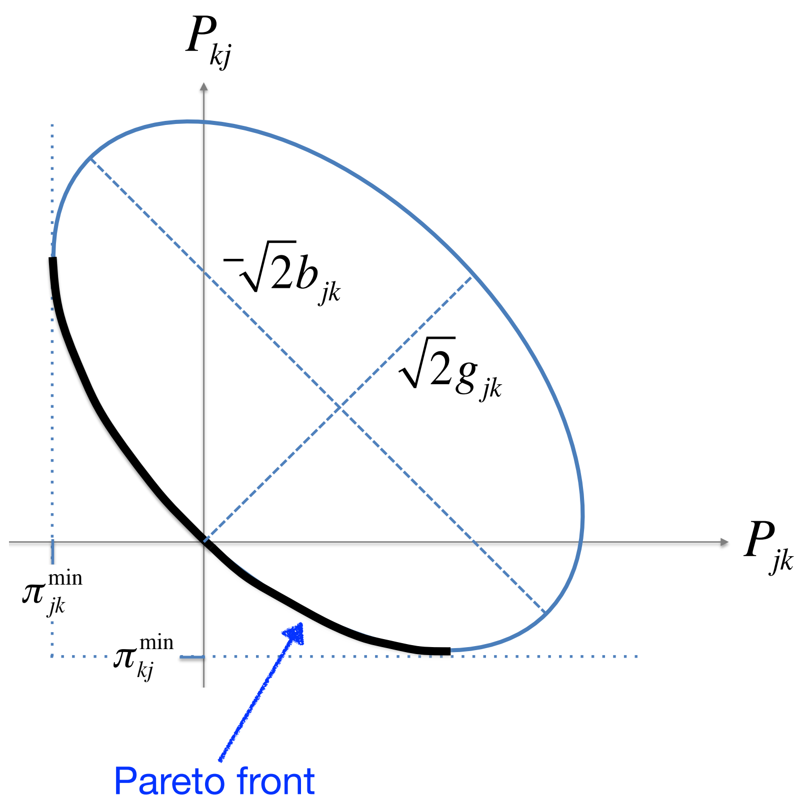
$$\mathbb{P}_p := \{ p \in \mathbb{R}^n \mid p_j^{\min} \le p_j \le p_j^{\max}, j \in N \}$$

OPF: $\min_p \quad C(p) \qquad$ s.t. $\qquad p \in \mathbb{P}_\theta \cap \mathbb{P}_p$

SOCP relaxation: $\min_p \quad C(p) \qquad$ s.t. $\qquad p \in \text{conv}\left(\mathbb{P}_\theta\right) \cap \mathbb{P}_p$

Definition: SOCP relaxation is exact if every optimal solution lies in $\mathbb{P}_\theta \cap \mathbb{P}_p$

Pareto front



Definitions

A point $x \in A \subseteq \mathbb{R}^n$ is a Pareto optimal point in A if there does not exist another $x' \in A$ such that

- $x' \leq x$, and
- $x'_j < x_j$ for at least one j

The Pareto front of A:
$\mathbb{O}(A) \quad := \quad \{$all Parento optimal points$\}$

Sufficient condition

C1: for every $(j,k) \in E$, $\tan^{-1}\frac{b_{jk}}{g_{jk}} < \theta_{jk}^{\min} \leq \theta_{jk}^{\max} < \tan^{-1}\frac{-b_{jk}}{g_{jk}}$

$b_{jk} < 0 < g_{jk}$

C2: $C(p)$ is strictly increasing in each p_j

Theorem

Suppose G is a tree and C1, C2 hold. Then

1. $\mathbb{P}_\theta \cap \mathbb{P}_p \ = \ \mathbb{O}(\text{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p)$

   feasible set is Pareto front of its relaxation

2. SOCP relaxation is exact

Geometric insight

2-bus network

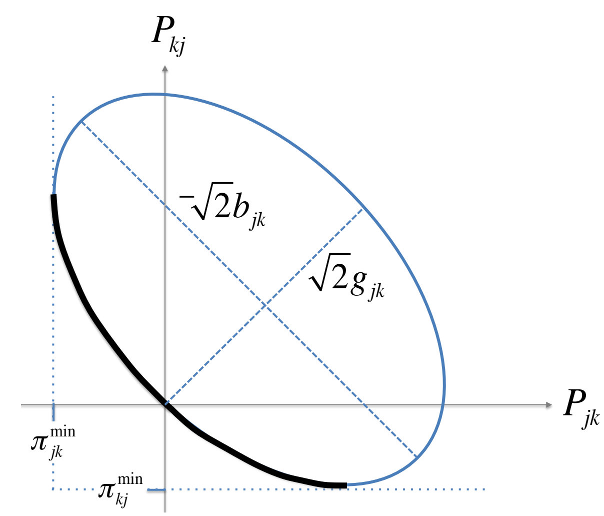
For each line $(j, k) \in E$, line flows $P := \left(P_{jk}, P_{kj}\right)$ and angle differences $\theta_{jk} := \theta_j - \theta_k$ satisfy

$$P - g_{jk}\mathbf{1} \;=\; A \begin{bmatrix} \cos\theta_{jk} \\ \sin\theta_{jk} \end{bmatrix} \qquad \text{where} \quad A \;:=\; \begin{bmatrix} -g_{jk} & -b_{jk} \\ -g_{jk} & b_{jk} \end{bmatrix}$$

1. P traces out an ellipse in $\mathbb{R}^2$ as θ_{jk} ranges over $[-\pi, \pi]$.

   Hence feasible set (subset of ellipse) is noncovex.

2. C1 restricts $\mathbb{P}_\theta$ to lower half of ellipse



Geometric insight

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For each line $(j, k) \in E$, line flows $P := \left(P_{jk}, P_{kj}\right)$ and angle differences $\theta_{jk} := \theta_j - \theta_k$ satisfy

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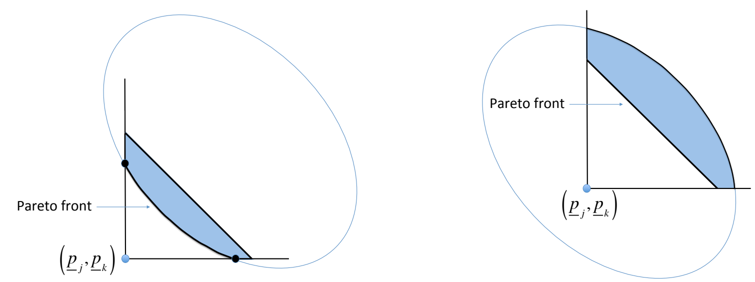
1. P traces out an ellipse in $\mathbb{R}^2$ as θ_{jk} ranges over $[-\pi, \pi]$.
   Hence feasible set (subset of ellipse) is noncovex.
2. C1 restricts $\mathbb{P}_\theta$ to lower half of ellipse
3. C2 implies Pareto front of relaxed feasible set coincides with feasible set, i.e., relaxation is exact



(a) Exact relaxation with constraint

(b) Inexact relaxation with constraint