Power System Analysis

Chapter 11 Semidefinite relaxations: BFM

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Outline

1. SOCP relaxation
2. Exactness condition: inactive injection lower bounds
3. Exactness condition: inactive voltage upper bounds

Outline

1. SOCP relaxation
   - SOCP relaxation
   - Equivalence
2. Exactness condition: inactive injection lower bounds
3. Exactness condition: inactive voltage upper bounds

Radial network

Assumptions: DistFlow model

Radial network

- BFM most useful for modeling distribution systems which are mostly radial (and unbalanced)

$z^s_{jk} = z^s_{kj}$ or equivalently $y^s_{jk} = y^s_{kj}$

- Does not apply to 3-phase transformers in ΔY or $Y\Delta$ configuration or their per-phase equivalent with complex gains

$$y^m_{jk} = y^m_{kj} = 0$$

- Reasonable assumption for distribution line where $|y^m_{jk}|, |y^m_{kj}| \ll |y^s_{jk}|$

Includes only voltage sources and power sources

- Optimization variables are voltages (squared magnitudes) v_j and power injections s_j respectively
- Can include current sources or an impedances with additional vars and constraints.

DistFlow model

Power flow equations

- All lines point away from bus 0 (root)

$$\sum_{k:j\to k} S_{jk} = S_{ij} - z^s_{ij}\ell_{ij} + s_j, \qquad j \in \overline{N}$$

$$v_j - v_k = 2\,\mathrm{Re}\left(\bar{z}^s_{jk} S_{jk}\right) - |z^s_{jk}|^2 \ell_{jk}, \qquad j \to k \in E$$

$$v_j \ell_{jk} = |S_{jk}|^2, \qquad j \to k \in E$$

nonconvex constraint
(other constraints are linear in x)

Operational constraints

$$s_j^{\min} \leq s_j \leq s_j^{\max}$$

$$v_j^{\min} \leq v_j \leq v_j^{\max}$$

$$\ell_{jk} \leq \ell_{jk}^{\max}$$

Single-phase OPF

DistFlow model

Feasible set

$$\mathbb{X}_{\text{df}} := \left\{ x := (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies PF equations \& operational constraints} \right\}$$

OPF in BFM

$$\min_{x} \quad C(x) \qquad \text{s.t.} \qquad x \in \mathbb{X}_{\text{df}}$$

Single-phase OPF

Equivalence

Recall for BIM:

- Feasible set: $\mathbb{V} := \left\{ V \in \mathbb{C}^{N+1} \mid V \text{ satisfies operational constraints} \right\}$
- OPF: $\min_{V \in \mathbb{V}} \ C(V)$

OPF in BFM is equivalent to OPF in BIM:

- Feasible sets $\mathbb{X}_{\text{df}}$ and $\mathbb{V}$ are equivalent (Ch 5)
- … provided cost functions $C(x)$ and $C(V)$ are the same

SOCP relaxation

Power flow equations

- All lines point towards bus 0 (root)

$$S_{jk} = \sum_{i:i\to j} \left(S_{ij} - z_{ij}\ell_{ij} \right) + s_j, \qquad j \in \overline{N}$$

$$v_j - v_k = 2\,\mathrm{Re}\left(z_{jk}^{\mathsf{H}} S_{jk} \right) - |z_{jk}|^2 \ell_{jk}, \qquad j \to k \in E$$

$$v_j \ell_{jk} \geq |S_{jk}|^2, \qquad j \to k \in E$$

second-order cone

Operational constraints

$$s_j^{\min} \leq s_j \leq s_j^{\max}$$

$$v_j^{\min} \leq v_j \leq v_j^{\max}$$

$$\ell_{jk} \leq \ell_{jk}^{\max}$$

SOCP relaxation

Feasible set

$$\mathbb{X}_{\text{df}}^{+} := \left\{ x := (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies } v_j \ell_{jk} \geq |S_{jk}|^2 \text{ \& other constraints} \right\}$$

SOCP relaxation in BFM

$$\min_{x} \quad C(x) \qquad \text{s.t.} \qquad x \in \mathbb{X}_{\text{df}}^{+}$$

Exactness

Definition (Strong exactness)

SOCP relaxation is exact if every optimal solution x^{opt} of SOCP relaxation attains equality:

$$v_j^{\text{opt}} \ell_{jk}^{\text{opt}} = \left| S_{jk}^{\text{opt}} \right|^2, \qquad j \to k \in E$$

- Convenient because any algorithm that solves an exact relaxation produces an optimal solution for original OFP
- Not necessary: under sufficient conditions for radial networks, can always recover an optimal solution of OPF from any solution of SOCP relaxation, even when SOCP relaxation is not exact

Exactness implies uniqueness

Theorem

Suppose network graph G is tree and the cost function $C(x)$ is convex. If SOCP relaxation in DistFlow model is exact, then its optimal solution is unique

Outline

1. SOCP relaxation
   - SOCP relaxation
   - Equivalence
2. Exactness condition: inactive injection lower bounds
3. Exactness condition: inactive voltage upper bounds

OPF in BIM

Recall

$$
\begin{aligned}
\min_{V \in \mathbb{C}^{N+1}} \quad & V^{\mathsf{H}} C_0 V \\
\text{s.t.} \quad & p_j^{\min} \le \operatorname{tr}\left(\Phi_j V V^{\mathsf{H}}\right) \le p_j^{\max}, & j \in \overline{N} \\
& q_j^{\min} \le \operatorname{tr}\left(\Psi_j V V^{\mathsf{H}}\right) \le q_j^{\max}, & j \in \overline{N} \\
& v_j^{\min} \le \operatorname{tr}\left(E_j V V^{\mathsf{H}}\right) \le v_j^{\max}, & j \in \overline{N} \\
& \operatorname{tr}\left(\hat{Y}_{jk} V V^{\mathsf{H}}\right) \le \bar{I}_{jk}^{\max}, & (j,k) \in E \\
& \operatorname{tr}\left(\hat{Y}_{kj} V V^{\mathsf{H}}\right) \le \bar{I}_{kj}^{\max}, & (j,k) \in E
\end{aligned}
$$

abbreviated as:
$\operatorname{tr}\left(C_l V V^{\mathsf{H}}\right) \le b_l,\ l = 1, \ldots, L$

SOCP relaxation in BIM

Given $V \in \mathbb{C}^{N+1}$, define partial matrix W_G by

$$[W_G]_{jj} := |V_j|^2, \qquad j \in \overline{N}$$

$$\left[W_G\right]_{jk} := V_j V_k^{\mathsf{H}} =: [W_G]_{kj}^{\mathsf{H}}, \qquad (j,k) \in E$$

Constraints in terms of W_G

$$s_j^{\min} \le \sum_{k:j\sim k} \bar{y}_{jk}^s \left([W_G]_{jj} - [W_G]_{jk}\right) \le s_j^{\max}, \qquad j \in \overline{N}$$

$$v_j^{\min} \le [W_G]_{jj} \le v_j^{\max}, \qquad j \in \overline{N}$$

$$\left|y_{jk}^s\right|^2 \left([W_G]_{jj} + [W_G]_{kk} - [W_G]_{jk} - [W_G]_{kj}\right) \le \ell_{jk}^{\max}, \qquad j \to k \in E$$

abbreviated as: $\mathrm{tr}\left(C_l W_G\right) \le b_l,\ l = 1,\ldots,L$

SOCP relaxation in BIM

OPF as QCQP

$$\min_{V} \; C(V) \qquad \text{s.t.} \quad \mathrm{tr}\left(C_l V V^{\mathsf{H}}\right) \le b_l, \;\; l = 1, \ldots, L$$

SOCP relaxations

$$\text{OPF-socp}: \qquad \min_{W_G} \; C(W_G) \qquad \text{s.t.} \qquad \begin{aligned} \mathrm{tr}\left(C_l W_G\right) \;&\le\; b_l, \;\; l \;=\; 1, \ldots, L, \\ W_G(j,k) \;&\succeq\; 0, \;\; (j,k) \;\in E \end{aligned}$$

Equivalence

SOCP relaxation in BFM

$$\min_{x} \quad C(x) \quad \text{s.t.} \quad x \in \mathbb{X}_{\text{df}}^{+} := \left\{ x \mid x \text{ satisfies } v_j \ell_{jk} \geq |S_{jk}|^2 \text{ \& other constraints} \right\}$$

SOCP relaxation in BIM

$$\min_{W_G} \quad C(W_G) \quad \text{s.t.} \quad W_G \in \mathbb{W}_G^{+} := \left\{ W_G \mid W_G \text{ satisfies constraints} \right\}$$

Theorem

$$\mathbb{X}_{\text{df}}^{+} \equiv \mathbb{W}_G^{+}$$

Implication: The two problems are equivalent in the sense that ∃ bijection $g : \mathbb{W}_G^{+} \longrightarrow \mathbb{X}_{\text{df}}^{+}$ s.t. W_G^{opt} is optimal in BIM iff $x^{\text{opt}} := g\left(W_G^{\text{opt}}\right)$ is optimal in BFM

Equivalence

SOCP relaxation in BFM

$$\min_{x} \quad C(x) \quad \text{s.t.} \quad x \in \mathbb{X}_{\text{df}}^{+} := \left\{ x \mid x \text{ satisfies } v_j \ell_{jk} \geq |S_{jk}|^2 \text{ \& other constraints} \right\}$$

SOCP relaxation in BIM

$$\min_{W_G} \quad C(W_G) \quad \text{s.t.} \quad W_G \in \mathbb{W}_G^{+} := \left\{ W_G \mid W_G \text{ satisfies constraints} \right\}$$

Theorem

$$\mathbb{X}_{\text{df}}^{+} \equiv \mathbb{W}_G^{+}$$

Generalization:
Extends to general radial networks without assuming
$z_{jk}^{s} = z_{kj}^{s},\ y_{jk}^{m} = y_{kj}^{m} = 0$

Proof

Motivated by $W = VV^{\mathsf{H}}$ of psd rank-1 completion W of psd rank-1 W_G, define $g : \mathbb{W}_G^+ \to \mathbb{X}_{\mathsf{df}}^+$:

$$\begin{aligned}
s_j &:= \sum_{i:i\to j} \bar{y}_{ij}^s \Big([W_G]_{jj} - [W_G]_{ji} \Big) \;+\; \sum_{k:j\to k} \bar{y}_{jk}^s \Big([W_G]_{jj} - [W_G]_{jk} \Big), && j \in \overline{N} \\
v_j &:= [W_G]_{jj}, && j \in \overline{N} \\
\ell_{jk} &:= |y_{jk}^s|^2 \Big([W_G]_{jj} + [W_G]_{kk} - [W_G]_{jk} - [W_G]_{kj} \Big), && j \to k \in E \\
S_{jk} &:= \bar{y}_{jk}^s \Big([W_G]_{jj} - [W_G]_{jk} \Big), && j \to k \in E
\end{aligned}$$

and $g^{-1} : \mathbb{X}_{\mathsf{df}}^+ \to \mathbb{W}_G^+$:

$$\begin{aligned}
[W_G]_{jj} &:= v_j, && j \in \overline{N} \\
\big[W_G\big]_{jk} &:= v_j - \bar{z}_{jk}^s S_{jk} \;=\; \big[W_G\big]_{kj}^{\mathsf{H}}, && j \to k \in E
\end{aligned}$$

Proof

Will prove

1. $x := g(W_G) \in \mathbb{X}^+_{\text{df}}$ for every $W_G \in \mathbb{W}^+_G$
2. $W_G := g^{-1}(x) \in \mathbb{W}^+_G$ for every $x \in \mathbb{X}^+_{\text{df}}$
3. g and g^{-1} are indeed inverses of each other

Proof

Will prove

1. $x := g(W_G) \in \mathbb{X}_{\text{df}}^+$ for every $W_G \in \mathbb{W}_G^+$
2. $W_G := g^{-1}(x) \in \mathbb{W}_G^+$ for every $x \in \mathbb{X}_{\text{df}}^+$
3. g and g^{-1} are indeed inverses of each other

Step 1:

Clearly x satisfies operational constraints since W_G does

For power balance:

$$\begin{aligned}\sum_{i:i\to j}\left(S_{ij} - z_{ij}\ell_{ij}\right) \; + \; s_j &= \sum_{i:i\to j}\left(\bar{y}_{ij}\left([W_G]_{ii} - [W_G]_{ij}\right) - \bar{y}_{ij}\left([W_G]_{ii} + [W_G]_{jj} - [W_G]_{ij} - [W_G]_{ji}\right)\right) \; + \; s_j \\ &= \sum_{i:i\to j}\left(-\bar{y}_{ij}\left([W_G]_{jj} - [W_G]_{ji}\right)\right) \; + \; \sum_{i:i\to j}\bar{y}_{ji}\left([W_G]_{jj} - [W_G]_{ji}\right) \; + \; \sum_{k:j\to k}\bar{y}_{jk}\left([W_G]_{jj} - [W_G]_{jk}\right) \\ &= \sum_{k:j\to k} S_{jk}\end{aligned}$$

Proof

Will prove

1. $x := g(W_G) \in \mathbb{X}^+_{\text{df}}$ for every $W_G \in \mathbb{W}^+_G$
2. $W_G := g^{-1}(x) \in \mathbb{W}^+_G$ for every $x \in \mathbb{X}^+_{\text{df}}$
3. g and g^{-1} are indeed inverses of each other

Step 1:

For voltage equation:

$$
\begin{aligned}
2\,\text{Re}\left(\bar{z}_{jk} S_{jk}\right) - |z_{jk}|^2 \ell_{jk} &= 2\,\text{Re}\left([W_G]_{jj} - [W_G]_{jk}\right) - \left([W_G]_{jj} + [W_G]_{kk} - [W_G]_{jk} - [W_G]_{kj}\right) \\
&= \left([W_G]_{jj} - [W_G]_{kk}\right) - [W_G]^{\mathsf{H}}_{jk} + [W_G]_{kj} \\
&= v_j - v_k
\end{aligned}
$$

Proof

Will prove

1. $x := g(W_G) \in \mathbb{X}^+_{\text{df}}$ for every $W_G \in \mathbb{W}^+_G$
2. $W_G := g^{-1}(x) \in \mathbb{W}^+_G$ for every $x \in \mathbb{X}^+_{\text{df}}$
3. g and g^{-1} are indeed inverses of each other

Step 1:

For SOC constraint: $[W_G]_{jj}[W_G]_{kk} \geq |[W_G]_{jk}|^2$ implies

$$\begin{aligned} v_j \ell_{jk} &= \left|y_{jk}\right|^2 \; [W_G]_{jj}\Big([W_G]_{jj} + [W_G]_{kk} - [W_G]_{jk} - [W_G]_{kj}\Big) \\ &\geq \left|y_{jk}\right|^2 \left([W_G]^2_{jj} + \left|[W_G]_{jk}\right|^2 - [W_G]_{jj}[W_G]_{jk} - [W_G]_{jj}[W_G]^{\mathsf{H}}_{jk}\right) \\ &= |S_{jk}|^2 \end{aligned}$$

Proof

Will prove

1. $x := g(W_G) \in \mathbb{X}^+_{\text{df}}$ for every $W_G \in \mathbb{W}^+_G$
2. $W_G := g^{-1}(x) \in \mathbb{W}^+_G$ for every $x \in \mathbb{X}^+_{\text{df}}$
3. g and g^{-1} are indeed inverses of each other

Step 2:

Need to prove:

$$\begin{aligned}
s_j^{\min} \quad \leq \quad \sum_{k:j\sim k} \bar{y}_{jk}^s \left([W_G]_{jj} - [W_G]_{jk} \right) \quad &\leq \quad s_j^{\max}, & j \in \overline{N} \\
v_j^{\min} \quad \leq \quad [W_G]_{jj} \quad &\leq \quad v_j^{\max}, & j \in \overline{N} \\
\left| y_{jk}^s \right|^2 \left([W_G]_{jj} + [W_G]_{kk} - [W_G]_{jk} - [W_G]_{kj} \right) \quad &\leq \quad \ell_{jk}^{\max}, & j \to k \in E \\
W_G(j,k) \quad &\succeq \quad 0, & (j,k) \in E
\end{aligned}$$

Proof

Will prove

1. $x := g(W_G) \in \mathbb{X}^+_{\text{df}}$ for every $W_G \in \mathbb{W}^+_G$
2. $W_G := g^{-1}(x) \in \mathbb{W}^+_G$ for every $x \in \mathbb{X}^+_{\text{df}}$
3. g and g^{-1} are indeed inverses of each other

Step 2:

For injection limits:

$$\sum_{k:(j,k)\in E} \bar{y}^s_{jk} \left([W_G]_{jj} - [W_G]_{jk}\right) = \sum_{i:i\to j} \bar{y}^s_{ji} \left([W_G]_{jj} - [W_G]_{ji}\right) + \sum_{k:j\to k} \bar{y}^s_{jk} \left([W_G]_{jj} - [W_G]_{jk}\right)$$

$$= \sum_{i:i\to j} \bar{y}^s_{ij} \left(v_j - \left(v_i - \bar{z}^s_{ij} S_{ij}\right)^{\mathsf{H}}\right) + \sum_{k:j\to k} \bar{y}^s_{jk} \left(v_j - \left(v_j - \bar{z}^s_{jk} S_{jk}\right)\right)$$

$$= \sum_{k:j\to k} S_{jk} - \sum_{i:i\to j} \bar{y}^s_{ij} \left(v_i - v_j - z^s_{ij} S^{\mathsf{H}}_{ij}\right)$$

$$= \sum_{k:j\to k} S_{jk} - \sum_{i:i\to j} \bar{y}^s_{ij} \left(2\,\text{Re}(\bar{z}^s_{ij} S_{ij}) - \left|z^s_{ij}\right|^2 \ell_{ij} - z^s_{ij} S^{\mathsf{H}}_{ij}\right)$$

Proof

Will prove

1. $x := g(W_G) \in \mathbb{X}_{\mathsf{df}}^+$ for every $W_G \in \mathbb{W}_G^+$
2. $W_G := g^{-1}(x) \in \mathbb{W}_G^+$ for every $x \in \mathbb{X}_{\mathsf{df}}^+$
3. g and g^{-1} are indeed inverses of each other

Step 2:

But

$$\left(2\,\mathrm{Re}(\bar{z}_{ij}^s S_{ij}) - z_{ij}^s S_{ij}^{\mathsf{H}}\right) = \left(\bar{z}_{ij}^s S_{ij} + z_{ij}^s S_{ij}^{\mathsf{H}}\right) - z_{ij}^s S_{ij}^{\mathsf{H}} = \bar{z}_{ij}^s S_{ij}$$

Hence

$$\sum_{k:(j,k)\in E} \bar{y}_{jk}^s \left([W_G]_{jj} - [W_G]_{jk}\right) = \sum_{k:j\to k} S_{jk} - \sum_{i:i\to j}\left(S_{ij} - z_{ij}^s \ell_{ij}\right) = s_j$$

Proof

Will prove

1. $x := g(W_G) \in \mathbb{X}_{\text{df}}^+$ for every $W_G \in \mathbb{W}_G^+$
2. $W_G := g^{-1}(x) \in \mathbb{W}_G^+$ for every $x \in \mathbb{X}_{\text{df}}^+$
3. g and g^{-1} are indeed inverses of each other

Step 2:

The voltage limits are clearly satisfied.

For line limits:

$$|y_{jk}|^2 \left([W_G]_{jj} + [W_G]_{kk} - [W_G]_{jk} - [W_G]_{kj} \right) = |y_{jk}|^2 \left(v_j + v_k - \left(v_j - \bar{z}_{jk}^s S_{jk} \right) - \left(v_j - \bar{z}_{jk}^s S_{jk} \right)^{\mathsf{H}} \right)$$

$$= |y_{jk}|^2 \left(-v_j + v_k + \bar{z}_{jk}^s S_{jk} + z_{jk}^s S_{jk}^{\mathsf{H}} \right) = \ell_{jk}$$

Hence line limits on ℓ_{jk} implies the limit limits on W_G

Proof

Will prove

1. $x := g(W_G) \in \mathbb{X}^+_{\text{df}}$ for every $W_G \in \mathbb{W}^+_G$
2. $W_G := g^{-1}(x) \in \mathbb{W}^+_G$ for every $x \in \mathbb{X}^+_{\text{df}}$
3. g and g^{-1} are indeed inverses of each other

Step 2:

For psd constraints: we have $[W_G]_{jk} = [W_G]^{\mathsf{H}}_{kj}$, $[W_G]_{jj} > 0$, $[W_G]_{kk} > 0$ and

$$[W_G]_{jj}[W_G]_{kk} - \left|[W_G]_{jk}\right|^2 = v_j v_k - \left|v_j - \bar{z}^s_{jk} S_{jk}\right|^2 = v_j v_k - \left(v_j^2 + \left|z^s_{jk}\right|^2 \left|S_{jk}\right|^2 - 2v_j \,\text{Re}\left(\bar{z}^s_{jk} S_{jk}\right)\right)$$

$$= v_j \left(v_k - v_j + 2\,\text{Re}\left(\bar{z}^s_{jk} S_{jk}\right)\right) - \left|z^s_{jk}\right|^2 \left|S_{jk}\right|^2$$

$$= \left|z^s_{jk}\right|^2 \left(v_j \ell_{jk} - \left|S_{jk}\right|^2\right) \geq 0$$

Proof

Will prove

1. $x := g(W_G) \in \mathbb{X}^+_{\text{df}}$ for every $W_G \in \mathbb{W}^+_G$
2. $W_G := g^{-1}(x) \in \mathbb{W}^+_G$ for every $x \in \mathbb{X}^+_{\text{df}}$
3. g and g^{-1} are indeed inverses of each other

Step 3:

Similar argument shows $g(g^{-1}(x)) = x$ and $g^{-1}(g(W_G)) = W_G$

Outline

1. SOCP relaxation
2. Exactness condition: inactive injection lower bounds
3. Exactness condition: inactive voltage upper bounds

DistFlow model

OPF and SOCP relaxation

DistFlow model

- Radial network
- $z_{jk}^s = z_{kj}^s$ or equivalently $y_{jk}^s = y_{kj}^s$
- $y_{jk}^m = y_{kj}^m = 0$
- Does not apply to 3-phase transformers in ΔY or $Y\Delta$ configuration or their per-phase equivalents

OPF in BFM

$$\min_x \quad C(x) \qquad \text{s.t.} \qquad x \in \mathbb{X}_{\text{df}} \; := \; \left\{ x \in \mathbb{R}^{6N+3} \; : \; x \text{ satisfies DF equations \& constraints} \right\}$$

SOCP relaxation in BFM

$$\min_x \quad C(x) \qquad \text{s.t.} \qquad x \in \mathbb{X}_{\text{df}}^+ \; := \; \left\{ x \in \mathbb{R}^{6N+3} \; : \; x \text{ satisfies } v_j \ell_{jk} \geq |S_{jk}|^2 \text{ \& constraints} \right\}$$

Exactness: injection lower bounds

Assume:

1. Cost function $C(x) := C(p, q, v, \ell)$ is independent of branch flow $S = (P, Q)$ and nondecreasing in (p, q, ℓ). Moreover it is strictly increasing in every component of $\ell := (\ell_{jk}, j \to k \in E)$, or in every component of $p := (p_j, j \in \overline{N})$, or in every component of $q := (q_j, j \in \overline{N})$
2. No injection lower bounds: $s_j^{\min} = -\infty - i\infty$

Theorem

Suppose network graph G is tree and conditions 1 and 2 hold. Then SOCP relaxation is exact, i.e., every optimal solution x^{opt} of SOCP relaxation is optimal for OPF

Remark: When $C(x)$ is only nondecreasing in (p, q, ℓ), the SOCP relaxation may not be exact, but an optimal solution of OPF can always be recovered from any solution of SOCP relaxation

Exactness: injection lower bounds

Assume:

1. Cost function $C(x) := C(p, q, v, \ell)$ is independent of branch flow $S = (P, Q)$ and nondecreasing in (p, q, ℓ). Moreover it is strictly increasing in every component of $\ell := (\ell_{jk}, j \to k \in E)$, or in every component of $p := (p_j, j \in \overline{N})$, or in every component of $q := (q_j, j \in \overline{N})$
2. No injection lower bounds: $s_j^{\min} = -\infty - i\infty$

Theorem

Suppose network graph G is tree and conditions 1 and 2 hold. Then SOCP relaxation is exact, i.e., every optimal solution x^{opt} of SOCP relaxation is optimal for OPF

Remark: Theorem can be extended to general radial network without assuming $z_{jk}^s = z_{kj}^s$ nor $y_{jk}^m = y_{kj}^m = 0$ (see PSA book)

Proof

Fix any optimal solution $x := (s, v, \ell, S)$ of SOCP relaxation of OPF. Since G is a tree, we only need to show that $v_j \ell_{jk} \geq |S_{jk}|^2$ holds with equality for every line $j \to k \in E$

Suppose $v_j \ell_{jk} > |S_{jk}|^2$ on line $j \to k \in E$. Will construct $\tilde{x}$ that is feasible for SOCP relaxation and attains a strictly lower cost, contradicting optimality of x

For an $\epsilon > 0$ to be determined, obtain $\tilde{x}$ by modifying only (ℓ_{jk}, S_{jk}) on the single line $j \to k$ and (s_j, s_k) at two ends of $j \to k$ (all other vars remain unchanged):

$$\begin{aligned} \tilde{\ell}_{jk} &:= \ell_{jk} - \epsilon \\ \tilde{S}_{jk} &:= S_{jk} - z_{jk}\epsilon/2 \\ \tilde{s}_j &:= s_j - z_{jk}\epsilon/2 \\ \tilde{s}_k &:= s_k - z_{kj}\epsilon/2 \end{aligned}$$

Assumption 1 implies that $\tilde{x}$ has a strictly lower cost than x. It hence suffices to show that $\exists \epsilon > 0$ s.t. $\tilde{x}$ is feasible for SOCP relaxation

Proof

Assumption 2 implies operational constraints are satisfied since $z_{jk} > 0$ and $\epsilon > 0$. We only need to show that $\tilde{x}$ satisfies: power balance at buses j, k, voltage and SOC constraints at line $j \rightarrow k$

For power balance at j:

$$\tilde{S}_{jk} \;=\; S_{jk} - z_{jk}\frac{\epsilon}{2} \;=\; \sum_{i:i\rightarrow j}\left(S_{ij} - z_{ij}\ell_{ij}\right) + s_j \;-\; z_{jk}\frac{\epsilon}{2} \;=\; \sum_{i:i\rightarrow j}\left(\tilde{S}_{ij} - z_{ij}\tilde{\ell}_{ij}\right) + \tilde{s}_j$$

For power balance at k:

$$\tilde{S}_{kl} = S_{kl} \;=\; \left(S_{jk} - z_{jk}\ell_{jk}\right) + \sum_{i\neq j:i\rightarrow k}\left(S_{ik} - z_{ik}\ell_{ik}\right) + s_k$$

$$= \left(\tilde{S}_{jk} - z_{jk}\tilde{\ell}_{jk} - z_{jk}\frac{\epsilon}{2}\right) + \sum_{i\neq j:i\rightarrow k}\left(\tilde{S}_{ik} - z_{ik}\tilde{\ell}_{ik}\right) + s_k \;=\; \sum_{i:i\rightarrow k}\left(\tilde{S}_{ik} - z_{ik}\tilde{\ell}_{ik}\right) + \tilde{s}_k$$

Proof

Assumption 2 implies operational constraints are satisfied since $z_{jk} > 0$ and $\epsilon > 0$. We only need to show that $\tilde{x}$ satisfies: power balance at buses j, k, voltage and SOC constraints at line $j \to k$

For voltage equation at $j \to k$:

$$\tilde{v}_j - \tilde{v}_k = v_j - v_k = 2\,\mathrm{Re}\left(z_{jk}^{\mathsf{H}} S_{jk}\right) - |z_{jk}|^2 \ell_{jk} = 2\,\mathrm{Re}\left(z_{jk}^{\mathsf{H}} \tilde{S}_{jk}\right) - |z_{jk}|^2 \tilde{\ell}_{jk}$$

For SOC constraint at $j \to k$:

$$\tilde{v}_j \tilde{\ell}_{jk} - \left|\tilde{S}_{jk}\right|^2 = -\frac{\left|z_{jk}\right|^2}{4}\epsilon^2 - \left(v_j - \mathrm{Re}\left(z_{jk}^{\mathsf{H}} S_{jk}\right)\right)\epsilon + \left(v_j \ell_{jk} - \left|S_{jk}\right|^2\right)$$

Hence $v_j \ell_{jk} > |S_{jk}|^2$ implies that $\exists \epsilon > 0$ s.t. $\tilde{v}_j \tilde{\ell}_{jk} = \left|\tilde{S}_{jk}\right|^2$

Outline

1. SOCP relaxation
2. Exactness condition: inactive injection lower bounds
3. Exactness condition: inactive voltage upper bounds

Exactness: voltage upper bounds

Geometric insight: 2-bus example

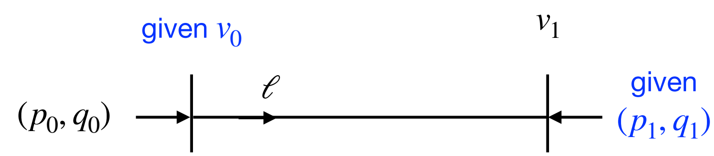
Power flow solution $x := (p_0, q_0, v_1, \ell)$ satisfies:

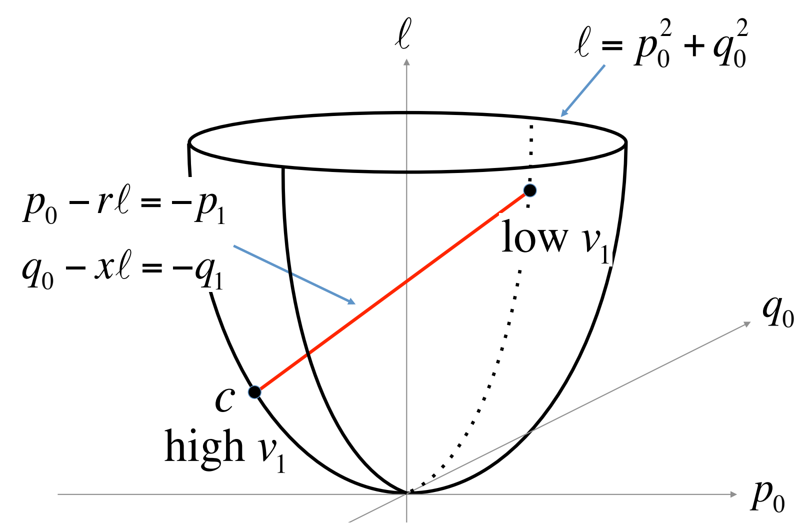
$$p_0 - r\ell = -p_1$$

$$q_0 - x\ell = -q_1$$

$$p_0^2 + q_0^2 = \ell$$

$$v_1 - v_0 = 2\left(rp_1 + xq_1\right) - (r^2 + x^2)\ell$$





power flow solutions (feasible set) : { 2 intersection points }

Exactness: voltage upper bounds

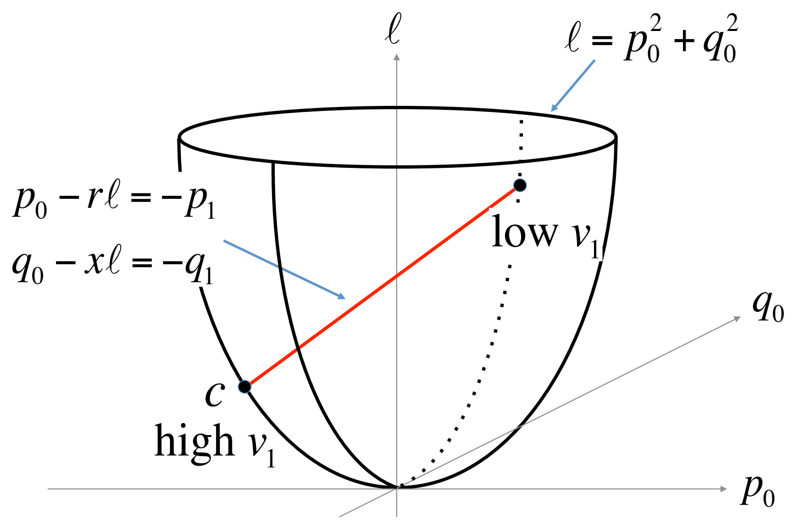
Geometric insight: 2-bus example

Feasible set (without voltage constraints)

- OPF : { 2 intersection points }, nonconvex
- SOCP relaxation : line segment, convex

Cost function $C(x)$ increasing in ℓ

- Optimal solution x^{opt} has high v_1
- SOCP relaxation is exact

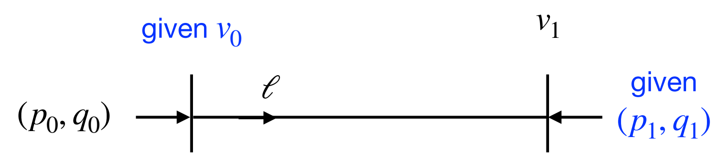


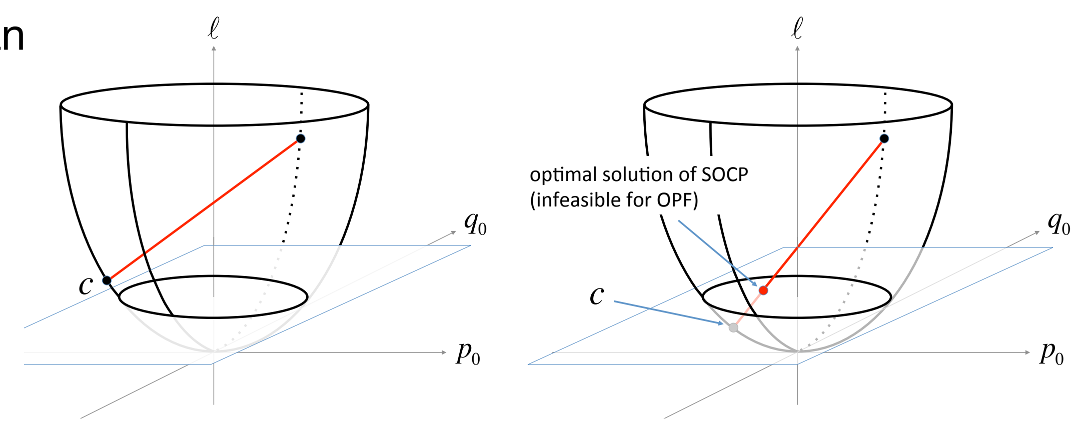
Exactness: voltage upper bounds

Example: geometric insight

Voltage constraints

- $\frac{1}{|z|^2}\left(a - v_1^{\max}\right) \;\leq\; \ell \;\leq\; \frac{1}{|z|^2}\left(a - v_1^{\min}\right)$
- $\therefore\; v_1^{\min}$ leads to upper bound on ℓ and will not affect exactness
- $v_1^{\max}$ leads to lower bound on ℓ and can affect exactness when it binds





(a) Voltage constraint not binding (b) Voltage constraint binding

Exactness: voltage upper bounds

Assume:

3. Cost function $C(x) := \sum_j C_j(p_j)$ with $C_0(p_0)$ strictly increasing in p_0. There is no constraint on s_0
4. $\hat{v}_j^{\text{lin}}(s) \leq v_j^{\max}, \quad j \in N$
5. Technical condition: small change in a line power affects all upstream line powers in the same direction

Theorem

Suppose network graph G is tree and Assumptions 3-5 hold. Then SOCP relaxation is exact, i.e., every optimal solution x^{opt} of SOCP relaxation is optimal for OPF

Remark: If $C_0(p_0)$ is nondecreasing in p_0, the SOCP relaxation may not be exact, but an optimal solution of OPF can always be recovered from any solution of SOCP relaxation