# Power System Analysis

## Chapter 5  Branch flow models: radial networks

Steven Low    EE/CS/EST 135    Caltech    (Oct 31, 2024)

# Outline

1. Radial networks
2. Equivalence
3. Backward forward sweep
4. Linear power flow model

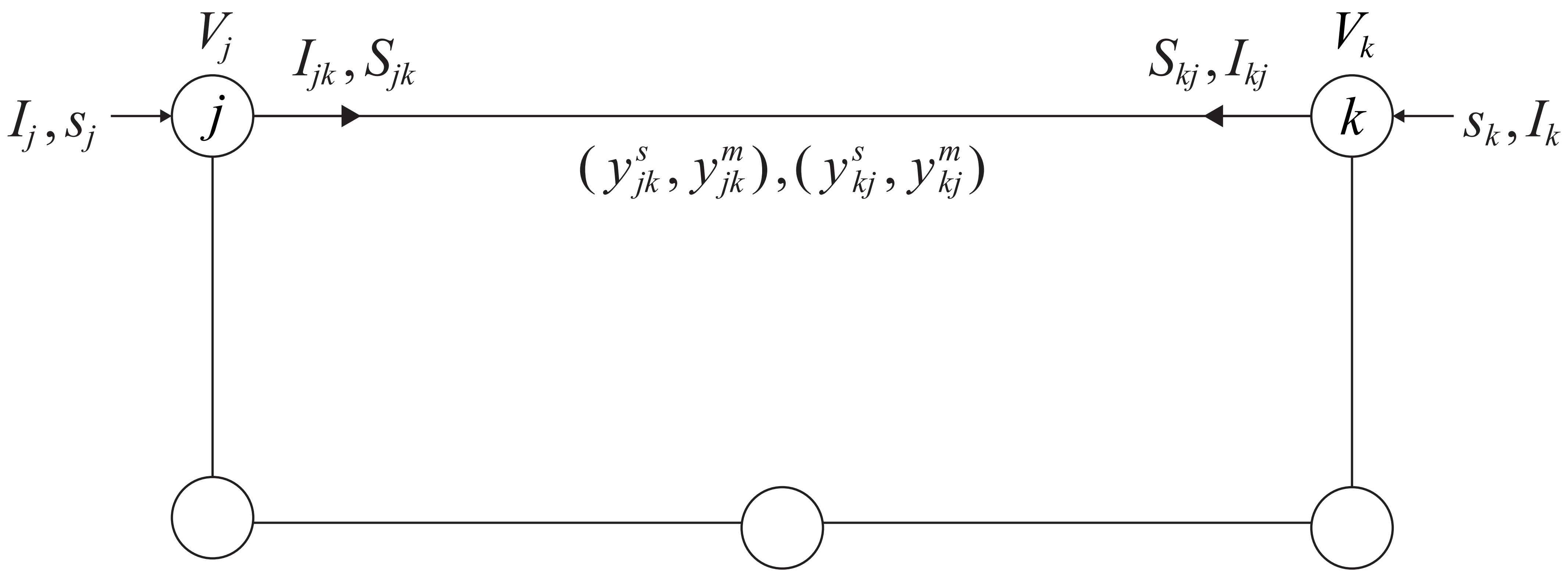
# Outline

1. Radial network
    - Line model
    - With shunt admittances
    - Without shunt admittances
    - Power flow solution
2. Equivalence
3. Backward forward sweep
4. Linear power flow model

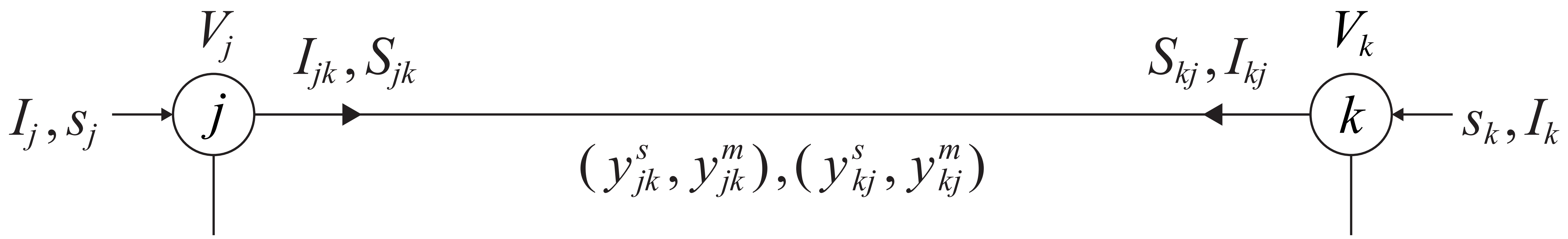
# Line model

1. Network $G := (\overline{N}, E)$
   - $\overline{N} := \{0\} \cup N := \{0\} \cup \{1,\ldots,N\}$ : buses/nodes/terminals
   - $E \subseteq \overline{N} \times \overline{N}$ : lines/branches/links/edges

2. Each line $(j, k)$ is parameterized by $\left(y^s_{jk}, y^m_{jk}\right)$ and $\left(y^s_{kj}, y^m_{kj}\right)$
   - $(y^s_{jk}, y^m_{jk})$ : series and shunt admittances from $j$ to $k$
   - $(y^s_{kj}, y^m_{kj})$ : series and shunt admittances from $k$ to $j$
   - Models transmission or distribution lines, single-phase transformers



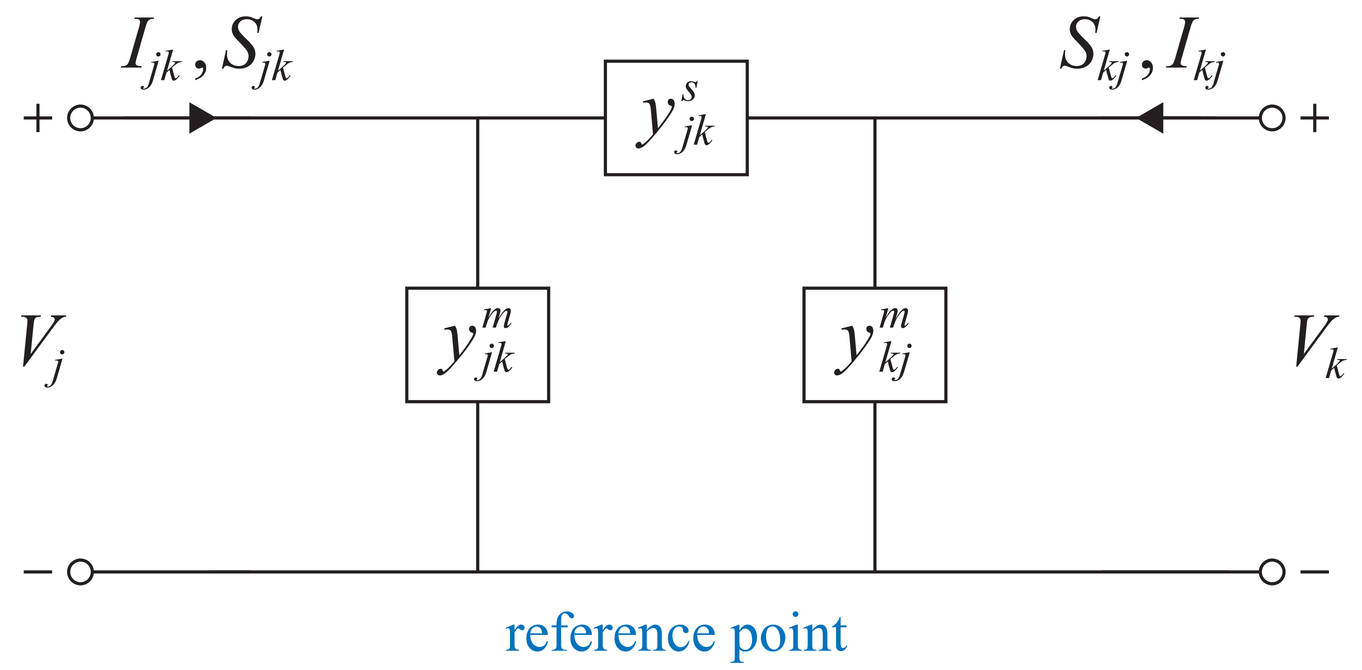
# Line model



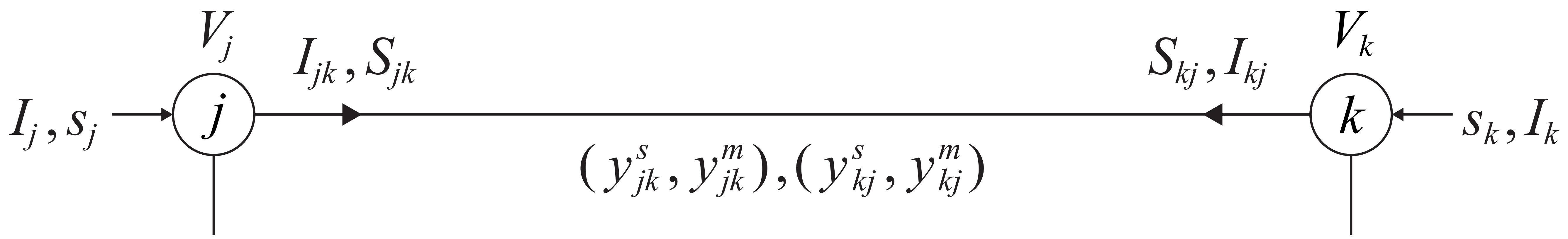

Sending-end currents

$$I_{jk} = y^s_{jk}(V_j - V_k) + y^m_{jk} V_j, \qquad I_{kj} = y^s_{kj}(V_k - V_j) + y^m_{kj} V_k,$$

If $y^s_{jk} = y^s_{kj}$ : same relation but equivalent to $\Pi$ circuit:



# Line model



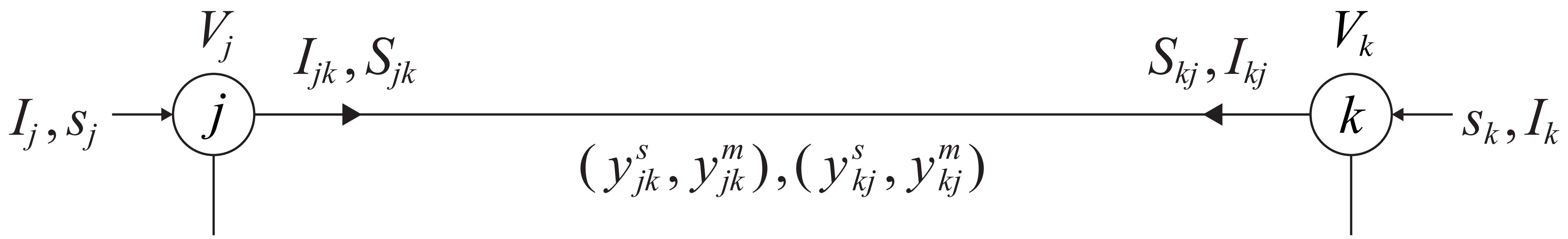

Sending-end currents

$$I_{jk} \;=\; y^s_{jk}(V_j - V_k) \;+\; y^m_{jk}\, V_j, \qquad\qquad I_{kj} \;=\; y^s_{kj}(V_k - V_j) \;+\; y^m_{kj}\, V_k,$$

Recall: bus injection models relate nodal variables $(s, V)$ and are suitable for general networks

$$s_j = \sum_{k: j\sim k} \left(y^s_{jk}\right)^H \left( |V_j|^2 - V_j V_k^H \right) \;+\; \left(y^m_{jj}\right)^H |V_j|^2$$

# Line model




Sending-end currents

$$I_{jk} = y^s_{jk}(V_j - V_k) + y^m_{jk} V_j, \qquad I_{kj} = y^s_{kj}(V_k - V_j) + y^m_{kj} V_k,$$

Branch flow models: key features

- Involve branch variables as well
- Particularly suitable for radial networks
- Variables contain no voltage/current phase angles (only magnitudes)
- Can recover voltage/current angles due to tree topology
- Equivalent to bus injection model

# Radial network

## With shunt admittances: variables

For each bus $j$

- $s_j := (p_j, q_j)$ or $s_j := p_j + iq_j$ : power injection
- $v_j$ : squared voltage magnitude

For each branch $(j, k)$

- $\left(\ell_{jk}, \ell_{kj}\right)$ : squared magnitude of sending-end current $j \to k$, and $k \to j$
- $S_{jk} := \left(P_{jk}, Q_{jk}\right)$ or $S_{jk} := P_{jk} + iQ_{jk}$ : sending-end power $j \to k$; also $S_{kj}$ from $k \to j$

The variables $v_j$ and $\left(\ell_{jk}, \ell_{kj}\right)$ contain no angle information

Angles must be recovered from a power flow solution $x := (s, v, \ell, S) \in \mathbb{R}^{3(N+1)+6M}$

- This is easy for radial networks; trickier for meshed networks

# Radial network

## With shunt admittances

For each line $(j, k)$ let:

$$\alpha_{jk} := 1 + z^s_{jk}\, y^m_{jk}, \qquad \alpha_{kj} := 1 + z^s_{kj}\, y^m_{kj}$$

$$z^s_{jk} := \left(y^s_{jk}\right)^{-1}, \quad z^s_{kj} := \left(y^s_{kj}\right)^{-1}$$

$\alpha_{jk} = \alpha_{kj}$ if and only if $y^m_{jk} = y^m_{kj}$

$\alpha_{jk} = \alpha_{kj} = 1$ if and only if $y^m_{jk} = y^m_{kj} = 0$

# Radial network

## With shunt admittances

$$s_j = \sum_{k:j\sim k} S_{jk}$$

power balance

# Radial network

## With shunt admittances

$$s_j = \sum_{k:j\sim k} S_{jk}$$

power balance

$$\left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

branch power magnitude

The complex notation is only shorthand for real equations

$$p_j = \sum_k P_{jk}, \qquad q_j = \sum_k Q_{jk}$$

$$v_j \ell_{jk} = P_{jk}^2 + Q_{jk}^2, \qquad v_k \ell_{kj} = P_{kj}^2 + Q_{kj}^2$$

# Radial network

## With shunt admittances

$$s_j = \sum_{k:j\sim k} S_{jk}$$

power balance

$$\left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

branch power magnitude

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

Ohm’s law, KCL (magnitude)

# Radial network

## With shunt admittances

$$s_j = \sum_{k:j\sim k} S_{jk}$$

power balance

$$\left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

branch power magnitude

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

Ohm's law, KCL (magnitude)

$$\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk} = \left(\alpha^{\mathsf{H}}_{kj} v_k - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

cycle condition:
$V_j \bar{V}_k = (V_k \bar{V}_j)^{\mathsf{H}}$

$2(N+1) + 6M$ real equations in $3(N+1) + 6M$ real vars $x := (s, v, \ell, S) \in \mathbb{R}^{3(N+1)+6M}$

# Radial network

## With shunt admittances

$$s_j = \sum_{k:j\sim k} S_{jk}$$ power balance

$$\left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$ branch power magnitude

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$ Ohm's law, KCL (magnitude)

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk} = \left(\alpha^{\mathsf{H}}_{kj} v_k - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$ cycle condition: $V_j \bar{V}_k = (V_k \bar{V}_j)^{\mathsf{H}}$

Any $x := (s, v, \ell, S) \in \mathbb{R}^{3(N+1)+6M}$ that satisfies these equations with $(v, \ell) \geq 0$ is a **power flow solution**

# Radial network

## With shunt admittances

All equations are linear in $x$, except the quadratic equalities

$$\left|S_{jk}\right|^2 = v_j \ell_{jk}, \quad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

There may be 0, 1, or $>1$ power flow solutions

This can be relaxed to second-order cone constraint in OPF (later)

# Example

## 2-bus network

Buses $j$ and $k$ connected by a transformer characterized by $(K, \tilde{y}^s, \tilde{y}^m)$ (voltage gain $K$ may be complex)

Line parameters are:

$$y_{jk}^s := \frac{\tilde{y}^s}{K}, \quad y_{jk}^m := \left(1 - \frac{1}{K}\right)\tilde{y}^s, \quad y_{kj}^s := \frac{\tilde{y}^s}{\bar{K}} \quad y_{kj}^m := \frac{1}{|K|^2}\left((1-K)\tilde{y}^s + \tilde{y}^m\right)$$

BFM:

$$v_j - v_k \Big/ \left|K\right|^2 = 2\,\mathrm{Re}\left(\left(\tilde{z}^s\right)^{\mathsf{H}} s_j\right) - \left|\tilde{z}^s\right|^2 \ell_{jk}$$

$$\left|\tilde{\alpha}/K\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\tilde{\alpha}\left(\tilde{z}^s\right)^{\mathsf{H}} s_k\right) - \left|K\tilde{z}^s\right|^2 \ell_{kj}$$

$$\left|s_j\right|^2 = v_j \ell_{jk}, \qquad \left|s_k\right|^2 = v_k \ell_{kj}$$

$$v_j - \left(\tilde{z}^s\right)^{\mathsf{H}} s_j = \left(\tilde{\alpha}/|K|^2\right) v_k - \tilde{z}^s \bar{s}_k$$

$$\tilde{z}^s := (\tilde{y}^s)^{-1}$$

$$\tilde{\alpha} := (1 + \tilde{z}^s \tilde{y}^m)$$

# Radial network

## Without shunt admittances

**Assume:** $y_{jk}^s = y_{kj}^s$ and $y_{jk}^m = y_{kj}^m = 0$

Then

1. $\alpha_{jk} = \alpha_{kj} = 1$
2. $\ell_{kj} = \ell_{jk}$ and $S_{kj} + S_{jk} = z_{jk}^s \ell_{jk}$

Can use directed graph with vars $\left(\ell_{jk}, S_{jk}\right)$ defined only in direction of lines $j \to k \in E$

Substitute $\left(\ell_{kj}, S_{kj}\right)$ in terms of $\left(\ell_{jk}, S_{jk}\right)$ into previous power flow equations yields original DistFlow equations of [Baran-Wu 1989]

# Radial network

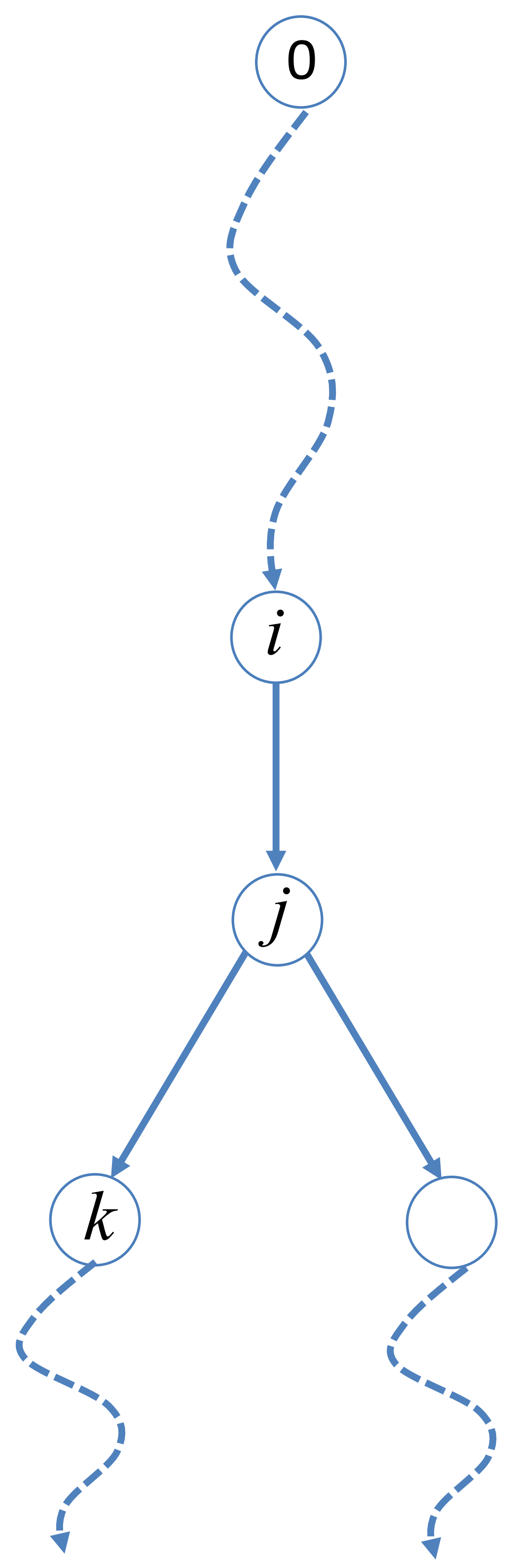
## Without shunt admittances

DistFlow equations [Baran-Wu 1989]:

$$\sum_{k:j\to k} S_{jk} = \sum_{i:i\to j} \left( S_{ij} - z_{ij}^s \ell_{ij} \right) + s_j$$ power balance

$$v_j - v_k = 2\,\mathrm{Re}\left( z_{jk}^{s\mathsf{H}} S_{jk} \right) - |z_{jk}^s|^2 \ell_{jk}$$ Ohm's law (magnitude)

$$v_j \ell_{jk} = |S_{jk}|^2$$ branch power magnitude




- Cycle condition becomes vacuous (because $S_{kj} := z_{jk}^s \ell_{jk} - S_{jk}$)
- $2(N+1) + 2M$ real equations in $3(N+1) + 3M$ real vars
- e.g. given $(v_0, s_j, j \in N)$, there are $4N+2$ equations in $4N+2$ vars $(s_0, v_j, j \in N, \ell, S)$

# Radial network

## Without shunt admittances

All equations are linear in $x$, except the quadratic equalities

$$v_j \ell_j \; = \; \left| S_{jk} \right|^2$$

There may be 0, 1, or $>1$ power flow solutions

This can be relaxed to second-order cone constraint in OPF (later)

# Angle recovery

Given power flow solution $x := (s, v, \ell, S)$, define nonlinear functions

$$\beta_{jk}(x) := \angle\left(\alpha_{jk}^H v_j - \left(z_{jk}^s\right)^H S_{jk}\right)$$

$$\beta_{kj}(x) := \angle\left(\alpha_{kj}^H v_k - \left(z_{jk}^s\right)^H S_{kj}\right)$$

Cycle condition ensures that $(\beta_{jk}(x), \beta_{kj}(x))$ are angle differences across line $(j, k)$, i.e.,

$\exists$ voltage angles $\theta$ s.t. $\beta(x) = C^\mathsf{T}\theta$

**Angle recovery:**

1. Tree topology $\implies \theta = C\left(C^\mathsf{T}C\right)^{-1}\beta(x) + \phi\mathbf{1}$
2. $V_j := \sqrt{v_j}e^{i\theta_j}, \qquad I_{jk} := \sqrt{\ell_{jk}}e^{i\left(\theta_j - \angle S_{jk}\right)}$

# Summary

## BFM for radial network

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk} \left(z^s_{jk}\right)^{\mathrm{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj} \left(z^s_{kj}\right)^{\mathrm{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\alpha^{\mathrm{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathrm{H}} S_{jk} = \left(\alpha^{\mathrm{H}}_{kj} v_k - \left(z^s_{kj}\right)^{\mathrm{H}} S_{kj}\right)^{\mathrm{H}}$$

BFM-radial

$y^s_{jk} = y^s_{kj}$

$y^m_{jk} = y^m_{kj} = 0$

$$\sum_{k:j\to k} S_{jk} = \sum_{i:i\to j} \left(S_{ij} - z_{ij}\ell_{ij}\right) + s_j$$

$$v_j - v_k = 2\,\mathrm{Re}\left(z^{\mathrm{H}}_{jk} S_{jk}\right) - |z_{jk}|^2 \ell_{jk}$$

$$v_j \ell_{jk} = |S_{jk}|^2$$

DistFlow

# Example: power flow solution

## 2-bus network

Two buses 0 and 1 connected by a line with series impedance $z = r + ix$ (graph orientation: up)

$$p_0 - r\ell = -p_1, \quad q_0 - x\ell = -q_1$$

$$v_1 - v_0 = 2\left(rp_1 + xq_1\right) - (r^2 + x^2)\ell$$

$$p_1^2 + q_1^2 = v_1\ell$$

Given: $r = x = 1$ and $v_0 = 1,\ q_1 = 0,$ find $(p_0, q_0, v_1, \ell)$ and show that $(v_1(p_1), p_1)$ forms an ellipse

**Solution**

Eliminate $v_1 \Rightarrow 2\ell^2 - (1 + 2p_1)\ell + p_1^2 = 0.$ Hence $(\Delta := 4p_1(1 - p_1) + 1)$

$$\ell = \frac{1}{4}\left(1 + 2p_1 \pm \sqrt{\Delta}\right), \quad p_0 = \frac{1}{4}\left(1 - 2p_1 \pm \sqrt{\Delta}\right), \quad q_0 = \frac{1}{4}\left(1 + 2p_1 \pm \sqrt{\Delta}\right)$$

$$v_1 = \frac{1}{2}\left(1 + 2p_1 \mp \sqrt{\Delta}\right)$$

# Example: power flow solution

## 2-bus network

### Solution

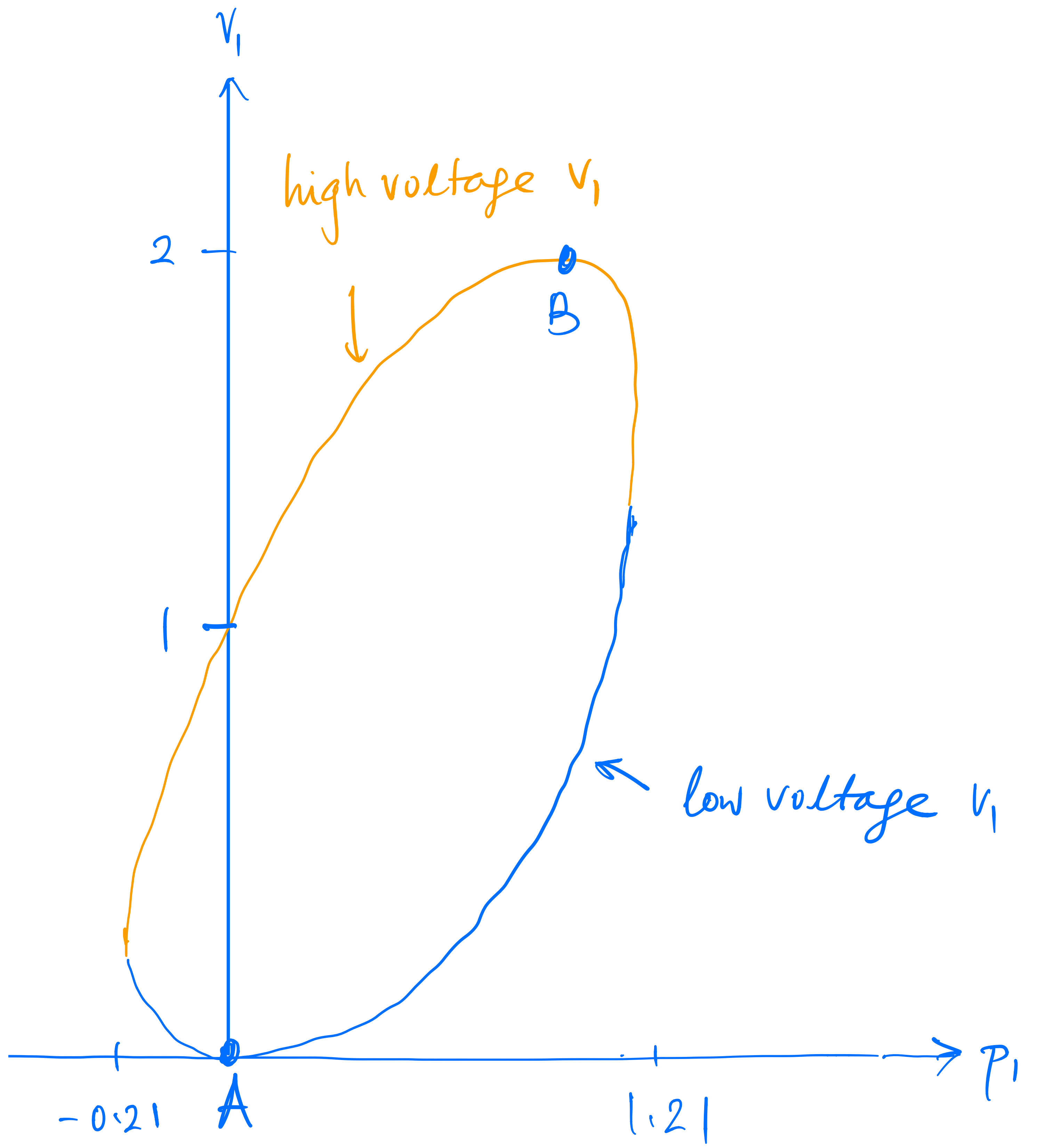
The solution $v_1 = \left(1 + 2p_1 \mp \sqrt{\Delta}\right)/2$ is equivalent to:

$$[p_1 \quad v_1] \underbrace{\begin{bmatrix} 8 & -4 \\ -4 & 4 \end{bmatrix}}_{A} \begin{bmatrix} p_1 \\ v_1 \end{bmatrix} - 2 \underbrace{[0 \quad 2]}_{c^T} \begin{bmatrix} p_1 \\ v_1 \end{bmatrix} + 1 = 1$$

Points $x \in \mathbb{R}^n$ satisfying

$$(x-c)^T A (x-c) = x^T A x - 2c^T x + \|c\|^2 = 1$$

form an ellipse if $A$ is real (symmetric) and positive definite



# Hollow solution set

**Theorem**

Suppose network graph $G$ is connected. Let $\mathbb{T} := \{$power flow solutions of DistFlow model$\}$. If $\hat{x}$ and $\tilde{x}$ are distinct solutions in $\mathbb{T}$ with $\hat{v}_0 = \tilde{v}_0$, then no convex combination of $\hat{x}$ and $\tilde{x}$ can be in $\mathbb{T}$. In particular, $\mathbb{T}$ is nonconvex.

# Outline

1. Radial network
2. Equivalence
   - Extension to general network
   - Equivalence of BFM and BIM
3. Backward forward sweep
4. Linear power flow model

# Power flow models

Bus injection model

$$s_j = \sum_{k:j\sim k} \left(y_{jk}^s\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right) + \left(y_{jj}^m\right)^{\mathsf{H}} |V_j|^2$$

Branch flow models

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}\right) - \left|z_{jk}^s\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right) - \left|z_{kj}^s\right|^2 \ell_{kj}$$

$$\alpha_{jk}^{\mathsf{H}} v_j - \left(z_{jk}^s\right)^{\mathsf{H}} S_{jk} = \left(\alpha_{kj}^{\mathsf{H}} v_k - \left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

- Different vars and equations
- Both describe Kirchhoff's and Ohm's laws
- Are they equivalent? In what sense?

# Power flow models

BIM applies to general networks
BFM applies to radial networks only

To show their equivalence, we first need to extend BFM to general networks with cycles

# General network

## Complex form

Let $\tilde{y}_{jk} := y^s_{jk} + y^m_{jk}$ and $\tilde{y}_{kj} := y^s_{kj} + y^m_{kj}$

BFM for general network:

$$s_j = \sum_{k:j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k, \qquad I_{kj} = \tilde{y}_{kj} V_k - y^s_{kj} V_j$$

$$S_{jk} = V_j I^{\mathsf{H}}_{jk}, \qquad S_{kj} = V_k I^{\mathsf{H}}_{kj}$$

Does **not** assume $y^s_{jk} = y^s_{kj}$ nor $y^m_{jk} = y^m_{kj} = 0$

This model looks similar to BIM complex form!

- It is a bridge between BFM and BIM

# General network

## Real form

$$s_j = \sum_{k:j\sim k} S_{jk}$$ power balance

$$\left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$ branch power magnitude

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$ Ohm's law, KCL (magnitude)

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$\exists \theta \in \mathbb{R}^{N+1}$ s.t. $\beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$ cycle condition

$2(N+1) + 6M$ real equations in $3(N+1) + 6M$ real vars $x := (s, v, \ell, S) \in \mathbb{R}^{3(N+1)+6M}$

# General network

## Real form

Major simplification for radial network: nonlinear cycle condition becomes linear in $x$

All other equations remain the same

$$\beta_{jk}(x) := \angle\left(\alpha_{jk}^{\mathsf{H}} v_j - \left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}\right)$$

$$\beta_{kj}(x) := \angle\left(\alpha_{kj}^{\mathsf{H}} v_k - \left(z_{jk}^s\right)^{\mathsf{H}} S_{kj}\right)$$

$$\beta(x) = \begin{bmatrix} C^T \\ -C^T \end{bmatrix} \theta \text{ for some } \theta \in \mathbb{R}^{N+1}$$

general network

$$\alpha_{jk}^{\mathsf{H}} v_j - \left(z_{jk}^s\right)^{\mathsf{H}} S_{jk} = \left(\alpha_{kj}^{\mathsf{H}} v_k - \left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

radial network

## Equivalence

**BFM-radial**

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk} = \left(\alpha^{\mathsf{H}}_{kj} v_k - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

→ $y^s_{jk} = y^s_{kj}$, $y^m_{jk} = y^m_{kj} = 0$ →

**DistFlow**

$$\sum_{k:j\to k} S_{jk} = \sum_{i:i\to j}\left(S_{ij} - z^s_{ij}\ell_{ij}\right) + s_j$$

$$v_j - v_k = 2\,\mathrm{Re}\left(z^{s\mathsf{H}}_{jk} S_{jk}\right) - |z^s_{jk}|^2 \ell_{jk}$$

$$v_j \ell_{jk} = |S_{jk}|^2$$

↑ radial network ↑

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\exists \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$$

↔

**BFM-complex**

$$s_j = \sum_{k:j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y^s_{kj} V_j$$

$$S_{jk} = V_j I^{\mathsf{H}}_{jk}, \quad S_{kj} = V_k I^{\mathsf{H}}_{kj}$$

↕

$$s_j = \sum_{k:j\sim k}\left(y^s_{jk}\right)^{\mathsf{H}}\left(|V_j|^2 - V_j V^{\mathsf{H}}_k\right) + \left(y^m_{jj}\right)^{\mathsf{H}} |V_j|^2$$

**BIM-complex**

## Equivalence

BFM-radial

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk} = \left(\alpha^{\mathsf{H}}_{kj} v_k - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

$y^s_{jk} = y^s_{kj}$
$y^m_{jk} = y^m_{kj} = 0$

DistFlow

$$\sum_{k:j\to k} S_{jk} = \sum_{i:i\to j}\left(S_{ij} - z^s_{ij}\ell_{ij}\right) + s_j$$

$$v_j - v_k = 2\,\mathrm{Re}\left(z^{s\mathsf{H}}_{jk} S_{jk}\right) - |z^s_{jk}|^2 \ell_{jk}$$

$$v_j \ell_{jk} = |S_{jk}|^2$$

radial network

proof focuses on these two

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\exists \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$$

BFM-complex

$$s_j = \sum_{k:j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y^s_{kj} V_j$$

$$S_{jk} = V_j I^{\mathsf{H}}_{jk}, \quad S_{kj} = V_k I^{\mathsf{H}}_{kj}$$

$$s_j = \sum_{k:j\sim k}\left(y^s_{jk}\right)^{\mathsf{H}}\left(|V_j|^2 - V_j V^{\mathsf{H}}_k\right) + \left(y^m_{jj}\right)^{\mathsf{H}} |V_j|^2$$

BIM-complex

# Equivalence

Branch flow models have been most useful for radial networks

- Different variants have different vars and different equations
- Are they equivalent, in what sense?

All BFM variants are equivalent to each other, and to BIM

- BFM-radial: tree topology (cycle condition: linear)
- DistFlow: tree topology with $y^s_{jk} = y^s_{kj}$ and $y^m_{jk} = y^m_{kj} = 0$ (cycle condition: vacuous)
- BFM-real: BFM for general topology (cycle condition: nonlinear)
- BFM-complex: bridge to BIM-complex

We next state and prove these equivalence relations

# Equivalence

## Solution set

BIM-complex

$$s_j = \sum_{k:j\sim k} \left(y_{jk}^s\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right) + \left(y_{jj}^m\right)^{\mathsf{H}} |V_j|^2$$

Solution set

$$\mathbb{V} := \{(s, V) \in \mathbb{C}^{2(n+1)} \mid V \text{ satisfies BIM}\}$$

# Equivalence

## Solution set

Branch flow models: solution sets

$$\tilde{\mathbb{X}} := \{\tilde{x} : (s, V, I, S) \in \mathbb{C}^{2(N+1)+4M} \mid \tilde{x} \text{ satisfies BFM complex}\}$$

$$\mathbb{X}_{\text{meshed}} := \{x : (s, v, \ell, S) \in \mathbb{R}^{3(N+1)+6M)} \mid x \text{ satisfies BFM real}\}$$

$$\mathbb{X}_{\text{tree}} := \{x : (s, v, \ell, S) \in \mathbb{R}^{9N+3} \mid x \text{ satisfies BFM radial}\}$$

$$\mathbb{X}_{\text{df}} := \left\{x : (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies BFM radial}, y_{jk}^s = y_{kj}^s, y_{jk}^m = y_{kj}^m = 0\right\}$$

<u>Definition</u>: Two sets $A$ and $B$ are equivalent $(A \equiv B)$ if there is a bijection between them

- $x$ is a power flow solution of $A$ iff $g(x)$ is a power flow solution of $B$

# Equivalence

**Theorem**

Suppose $G$ is connected

1. $\mathbb{V} \equiv \tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$
2. If $G$ is a tree, then $\mathbb{X}_{\text{meshed}} \equiv \mathbb{X}_{\text{tree}}$
3. If $G$ is a tree and $y^s_{jk} = y^s_{kj}$, $y^m_{jk} = y^m_{kj} = 0$, then $\mathbb{X}_{\text{tree}} \equiv \mathbb{X}_{\text{df}}$

# Equivalence

Bus injection models and branch flow models are equivalent

- Any result proved in one model holds also in another model

Some results are easier to formulate / prove in one model than the other

- BIM: semidefinite relaxation of OPF (later)
- BFM: some exact relation proofs

Should freely use whichever is more convenient for problem at hand

BFM is particularly suitable for modeling distribution systems

- Tree topology allows efficient computation of power flows (BFS)
- Models and relaxations extend to unbalanced $3\phi$ networks
- Seems to be much more numerically stable than BIM for large networks

## Equivalence

$\mathbb{X}_{\text{tree}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk} = \left(\alpha^{\mathsf{H}}_{kj} v_k - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

$y^s_{jk} = y^s_{kj}$

$y^m_{jk} = y^m_{kj} = 0$

$\mathbb{X}_{\text{df}}$

$$\sum_{k:j\to k} S_{jk} = \sum_{i:i\to j}\left(S_{ij} - z^s_{ij}\ell_{ij}\right) + s_j$$

$$v_j - v_k = 2\,\mathrm{Re}\left(z^{s\mathsf{H}}_{jk} S_{jk}\right) - |z^s_{jk}|^2 \ell_{jk}$$

$$v_j \ell_{jk} = |S_{jk}|^2$$

radial network

proof focuses on these two

$\mathbb{X}_{\text{meshed}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\exists \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$$

$\tilde{\mathbb{X}}$

$$s_j = \sum_{k:j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y^s_{kj} V_j$$

$$S_{jk} = V_j I^{\mathsf{H}}_{jk}, \quad S_{kj} = V_k I^{\mathsf{H}}_{kj}$$

$$s_j = \sum_{k:j\sim k}\left(y^s_{jk}\right)^{\mathsf{H}}\left(|V_j|^2 - V_j V^{\mathsf{H}}_k\right) + \left(y^m_{jj}\right)^{\mathsf{H}} |V_j|^2$$

$\mathbb{V}$

# Equivalence proof

**Proof $\mathbb{V} \equiv \tilde{\mathbb{X}}$ and $\mathbb{X}_{\text{tree}} \equiv \mathbb{X}_{\text{df}}$**

Straightforward.

$\tilde{\mathbb{X}}$

$$s_j = \sum_{k:j\sim k} S_{jk},$$
$$I_{jk} = \tilde{y}_{jk}V_j - y^s_{jk}V_k, \quad I_{kj} = \tilde{y}_{kj}V_k - y^s_{kj}V_j$$
$$S_{jk} = V_j I^{\mathsf{H}}_{jk}, \quad S_{kj} = V_k I^{\mathsf{H}}_{kj}$$

↕

$$s_j = \sum_{k:j\sim k} \left(y^s_{jk}\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right) + \left(y^m_{jj}\right)^{\mathsf{H}} |V_j|^2$$

$\mathbb{V}$

$\mathbb{X}_{\text{tree}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$
$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\text{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$
$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\text{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$
$$\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk} = \left(\alpha^{\mathsf{H}}_{kj} v_k - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

$y^s_{jk} = y^s_{kj}$ →

$\mathbb{X}_{\text{df}}$

$$\sum_{k:j\to k} S_{jk} = \sum_{i:i\to j} \left(S_{ij} - z^s_{ij}\ell_{ij}\right) + s_j$$
$$v_j - v_k = 2\,\text{Re}\left(z^{s\mathsf{H}}_{jk} S_{jk}\right) - |z^s_{jk}|^2 \ell_{jk}$$
$$v_j \ell_{jk} = |S_{jk}|^2$$

# Equivalence proof

## Proof $\tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$

$\mathbb{X}_{\text{meshed}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \quad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \quad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}\right) - \left|z_{jk}^s\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right) - \left|z_{kj}^s\right|^2 \ell_{kj}$$

$$\exists \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$$

$\tilde{\mathbb{X}}$

$$s_j = \sum_{k:j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y_{jk}^s V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y_{kj}^s V_j$$

$$S_{jk} = V_j I_{jk}^{\mathsf{H}}, \quad S_{kj} = V_k I_{kj}^{\mathsf{H}}$$

Fix $\tilde{x} := (s, V, I, S) \in \tilde{\mathbb{X}}$. Define

$$v_j := |V_j|^2, \qquad \ell_{jk} := |I_{jk}|^2, \qquad \ell_{kj} := |I_{kj}|^2$$

Will show $x := (s, v, \ell, S) \in \mathbb{X}_{\text{meshed}}$

It suffices to show

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}\right) - \left|z_{jk}^s\right|^2 \ell_{jk}$$

$$\exists \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$$

For the 1st equation, write $V_k = \alpha_{jk} V_j - z_{jk}^s \left(\dfrac{S_{jk}}{V_j}\right)^{\mathsf{H}}$ and taking square magnitude on both sides.

# Equivalence proof

## Proof $\tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$

$\mathbb{X}_{\text{meshed}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\text{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\text{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\text{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\text{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\exists \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$$

$\tilde{\mathbb{X}}$

$$s_j = \sum_{k:j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y^s_{kj} V_j$$

$$S_{jk} = V_j I^{\text{H}}_{jk}, \quad S_{kj} = V_k I^{\text{H}}_{kj}$$

Fix $\tilde{x} := (s, V, I, S) \in \tilde{\mathbb{X}}$. Define

$$v_j := |V_j|^2, \qquad \ell_{jk} := |I_{jk}|^2, \qquad \ell_{kj} := |I_{kj}|^2$$

Will show $x := (s, v, \ell, S) \in \mathbb{X}_{\text{meshed}}$

For the 2nd equation, we have

$$V_j V_k^{\text{H}} = \alpha^{\text{H}}_{jk} |V_j|^2 - \left(z^s_{jk}\right)^{\text{H}} S_{jk}, \qquad V_k V_j^{\text{H}} = \alpha^{\text{H}}_{kj} |V_k|^2 - \left(z^s_{kj}\right)^{\text{H}} S_{kj}$$

# Equivalence proof

## Proof $\tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$

$\mathbb{X}_{\text{meshed}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \quad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \quad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\exists \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$$

$\tilde{\mathbb{X}}$

$$s_j = \sum_{k:j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y^s_{kj} V_j$$

$$S_{jk} = V_j I^{\mathsf{H}}_{jk}, \quad S_{kj} = V_k I^{\mathsf{H}}_{kj}$$

Fix $\tilde{x} := (s, V, I, S) \in \tilde{\mathbb{X}}$. Define

$$v_j := |V_j|^2, \qquad \ell_{jk} := |I_{jk}|^2, \qquad \ell_{kj} := |I_{kj}|^2$$

Will show $x := (s, v, \ell, S) \in \mathbb{X}_{\text{meshed}}$

For the 2nd equation, we have

$$V_j V_k^{\mathsf{H}} = \alpha^{\mathsf{H}}_{jk} |V_j|^2 - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}, \qquad V_k V_j^{\mathsf{H}} = \alpha^{\mathsf{H}}_{kj} |V_k|^2 - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}$$

Recall the nonlinear functions

$$\beta_{jk}(x) := \angle\left(\alpha^H_{jk} v_j - \left(z^s_{jk}\right)^H S_{jk}\right) = \angle V_j - \angle V_k$$

$$\beta_{kj}(x) := \angle\left(\alpha^H_{kj} v_k - \left(z^s_{jk}\right)^H S_{kj}\right) = \angle V_k - \angle V_j$$

$\therefore\ \theta_j := \angle V_j$

# Equivalence proof

## Proof $\tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$

$\mathbb{X}_{\text{meshed}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\exists \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$$

$\longleftrightarrow$

$\tilde{\mathbb{X}}$

$$s_j = \sum_{k:j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y^s_{kj} V_j$$

$$S_{jk} = V_j I_{jk}^{\mathsf{H}}, \quad S_{kj} = V_k I_{kj}^{\mathsf{H}}$$

Conversely, fix $x := (s, v, \ell, S) \in \mathbb{X}_{\text{meshed}}$. Construct $(V, I)$ from $x$:

$$V_j := \sqrt{v_j}\, e^{i\theta_j}, \qquad I_{jk} := \sqrt{\ell_{jk}}\, e^{i\left(\theta_j - \angle S_{jk}\right)}$$

Will show $\tilde{x} := (s, V, I, S) \in \tilde{\mathbb{X}}$

It suffices to show

$$S_{jk} = V_j I_{jk}^{\mathsf{H}}, \qquad I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k,$$

# Equivalence proof

## Proof $\tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$

$\mathbb{X}_{\text{meshed}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \quad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \quad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\text{Re}\left(\alpha_{jk} \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\text{Re}\left(\alpha_{kj} \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$\exists \theta \in \mathbb{R}^{N+1}$ s.t. $\beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$

$\longleftrightarrow$

$\tilde{\mathbb{X}}$

$$s_j = \sum_{k:j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y^s_{kj} V_j$$

$$S_{jk} = V_j I^{\mathsf{H}}_{jk}, \quad S_{kj} = V_k I^{\mathsf{H}}_{kj}$$

Conversely, fix $x := (s, v, \ell, S) \in \mathbb{X}_{\text{meshed}}$. Construct $(V, I)$ from $x$:

$$V_j := \sqrt{v_j} e^{i\theta_j}, \qquad I_{jk} := \sqrt{\ell_{jk}} e^{i\left(\theta_j - \angle S_{jk}\right)}$$

Will show $\tilde{x} := (s, V, I, S) \in \tilde{\mathbb{X}}$

It suffices to show

$$S_{jk} = V_j I^{\mathsf{H}}_{jk}, \qquad I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k,$$

For the 1st equation, we have from $\left|S_{jk}\right|^2 = v_j \ell_{jk}$ and construction of $(V, I)$:

$$|S_{jk}| = |V_j I^{\mathsf{H}}_{jk}|, \qquad \angle S_{jk} = \angle V_j - \angle I_{jk}$$

i.e., $S_{jk} = V_j I^{\mathsf{H}}_{jk}$

# Equivalence proof

## Proof $\tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$

$\mathbb{X}_{\text{meshed}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad |S_{jk}|^2 = v_j \ell_{jk}, \qquad |S_{kj}|^2 = v_k \ell_{kj}$$

$$|\alpha_{jk}|^2 v_j - v_k = 2\,\text{Re}\left(\alpha_{jk} \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - |z^s_{jk}|^2 \ell_{jk}$$

$$|\alpha_{kj}|^2 v_k - v_j = 2\,\text{Re}\left(\alpha_{kj} \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - |z^s_{kj}|^2 \ell_{kj}$$

$\exists \theta \in \mathbb{R}^{N+1}$ s.t. $\beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$

$\tilde{\mathbb{X}}$

$$s_j = \sum_{k:j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y^s_{kj} V_j$$

$$S_{jk} = V_j I^{\mathsf{H}}_{jk}, \quad S_{kj} = V_k I^{\mathsf{H}}_{kj}$$

Conversely, fix $x := (s, v, \ell, S) \in \mathbb{X}_{\text{meshed}}$. Construct $(V, I)$ from $x$:

$$V_j := \sqrt{v_j} e^{i\theta_j}, \qquad I_{jk} := \sqrt{\ell_{jk}} e^{i\left(\theta_j - \angle S_{jk}\right)}$$

Will show $\tilde{x} := (s, V, I, S) \in \tilde{\mathbb{X}}$

It suffices to show

$$S_{jk} = V_j I^{\mathsf{H}}_{jk}, \qquad I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k,$$

Note that the 2nd equation is equivalent to (recall $\tilde{y}^s_{jk} := y^s_{jk} + y^m_{jk}$):

$$z^s_{jk} \left(S_{jk}/V_j\right)^{\mathsf{H}} = \alpha_{jk} V_j - V_k \iff V_j V^{\mathsf{H}}_k = \alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}$$

We now show that $V_j V^{\mathsf{H}}_k$ and $\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}$ have equal magnitudes and angles.

# Equivalence proof

## Proof $\tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$

$\mathbb{X}_{\text{meshed}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \quad |S_{jk}|^2 = v_j \ell_{jk}, \quad |S_{kj}|^2 = v_k \ell_{kj}$$

$$|\alpha_{jk}|^2 v_j - v_k = 2\,\text{Re}\left(\alpha_{jk} \left(z_{jk}^s\right)^{\text{H}} S_{jk}\right) - |z_{jk}^s|^2 \ell_{jk}$$

$$|\alpha_{kj}|^2 v_k - v_j = 2\,\text{Re}\left(\alpha_{kj} \left(z_{kj}^s\right)^{\text{H}} S_{kj}\right) - |z_{kj}^s|^2 \ell_{kj}$$

$$\exists \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$$

$\tilde{\mathbb{X}}$

$$s_j = \sum_{k:j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y_{jk}^s V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y_{kj}^s V_j$$

$$S_{jk} = V_j I_{jk}^{\text{H}}, \quad S_{kj} = V_k I_{kj}^{\text{H}}$$

Conversely, fix $x := (s, v, \ell, S) \in \mathbb{X}_{\text{meshed}}$. Construct $(V, I)$ from $x$:

$$V_j := \sqrt{v_j} e^{i\theta_j}, \qquad I_{jk} := \sqrt{\ell_{jk}} e^{i\left(\theta_j - \angle S_{jk}\right)}$$

Will show $\tilde{x} := (s, V, I, S) \in \tilde{\mathbb{X}}$

It suffices to show

$$S_{jk} = V_j I_{jk}^{\text{H}}, \qquad I_{jk} = \tilde{y}_{jk} V_j - y_{jk}^s V_k,$$

By definition, $\beta_{jk}(x) := \angle\left(\alpha_{jk}^{\text{H}} v_j - \left(z_{jk}^s\right)^{\text{H}} S_{jk}\right) = \theta_j - \theta_k = \angle\left(V_j V_k^{\text{H}}\right)$

# Equivalence proof

## Proof $\tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$

$\mathbb{X}_{\text{meshed}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \quad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \quad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\text{Re}\left(\alpha_{jk}\left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}\right) - \left|z_{jk}^s\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\text{Re}\left(\alpha_{kj}\left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right) - \left|z_{kj}^s\right|^2 \ell_{kj}$$

$$\exists \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$$

$\longleftrightarrow$ $\tilde{\mathbb{X}}$

$$s_j = \sum_{k:j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y_{jk}^s V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y_{kj}^s V_j$$

$$S_{jk} = V_j I_{jk}^{\mathsf{H}}, \quad S_{kj} = V_k I_{kj}^{\mathsf{H}}$$

Conversely, fix $x := (s, v, \ell, S) \in \mathbb{X}_{\text{meshed}}$. Construct $(V, I)$ from $x$:

$$V_j := \sqrt{v_j} e^{i\theta_j}, \qquad I_{jk} := \sqrt{\ell_{jk} e^{i\left(\theta_j - \angle S_{jk}\right)}}$$

Will show $\tilde{x} := (s, V, I, S) \in \tilde{\mathbb{X}}$

It suffices to show

$$S_{jk} = V_j I_{jk}^{\mathsf{H}}, \qquad I_{jk} = \tilde{y}_{jk} V_j - y_{jk}^s V_k,$$

For magnitude:

$$\left|\alpha_{jk}^{\mathsf{H}} v_j - z_{jk}^{s\mathsf{H}} S_{jk}\right|^2 = |\alpha_{jk}|^2 v_j^2 - 2 v_j \text{Re}\left(\alpha_{jk} z_{jk}^{s\mathsf{H}} S_{jk}\right) + |z_{jk}^s|^2 |S_{jk}|^2$$

$$= v_j \left( |\alpha_{jk}|^2 v_j - 2\text{Re}\left(\alpha_{jk} z_{jk}^{s\mathsf{H}} S_{jk}\right) + |z_{jk}^s|^2 \ell_{jk} \right) = v_j v_k$$

# Equivalence proof

## Proof $\tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$

$\mathbb{X}_{\text{meshed}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \quad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \quad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}\right) - \left|z_{jk}^s\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right) - \left|z_{kj}^s\right|^2 \ell_{kj}$$

$\exists \theta \in \mathbb{R}^{N+1}$ s.t. $\beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$

$\tilde{\mathbb{X}}$

$$s_j = \sum_{k:j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y_{jk}^s V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y_{kj}^s V_j$$

$$S_{jk} = V_j I_{jk}^{\mathsf{H}}, \quad S_{kj} = V_k I_{kj}^{\mathsf{H}}$$

Conversely, fix $x := (s, v, \ell, S) \in \mathbb{X}_{\text{meshed}}$. Construct $(V, I)$ from $x$:

$$V_j := \sqrt{v_j} e^{i\theta_j}, \qquad I_{jk} := \sqrt{\ell_{jk}} e^{i\left(\theta_j - \angle S_{jk}\right)}$$

Will show $\tilde{x} := (s, V, I, S) \in \tilde{\mathbb{X}}$

It suffices to show

$$S_{jk} = V_j I_{jk}^{\mathsf{H}}, \qquad I_{jk} = \tilde{y}_{jk} V_j - y_{jk}^s V_k,$$

This completes the proof of $\tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$

# Equivalence proof

## Proof $\mathbb{X}_{\text{meshed}} \equiv \mathbb{X}_{\text{tree}}$

Suppose $G$ is a tree.

Will show $x := (s, v, \ell, S) \in \mathbb{X}_{\text{meshed}} \iff x \in \mathbb{X}_{\text{tree}}$

It suffices to show nonlinear cycle condition becomes linear:

$$\beta_{jk}(x) \ := \ \theta_j - \theta_k \ = -\beta_{kj}(x)$$

$$\iff \alpha_{jk}^{\mathsf{H}} v_j - \left(z_{jk}^s\right)^{\mathsf{H}} S_{jk} \ = \ \left(\alpha_{kj}^{\mathsf{H}} v_k - \left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

$\mathbb{X}_{\text{tree}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \quad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \quad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}\right) - \left|z_{jk}^s\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right) - \left|z_{kj}^s\right|^2 \ell_{kj}$$

$$\alpha_{jk}^{\mathsf{H}} v_j - \left(z_{jk}^s\right)^{\mathsf{H}} S_{jk} = \left(\alpha_{kj}^{\mathsf{H}} v_k - \left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

↑ radial network

$\mathbb{X}_{\text{meshed}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \quad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \quad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}\right) - \left|z_{jk}^s\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right) - \left|z_{kj}^s\right|^2 \ell_{kj}$$

$$\exists \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$$

*Necessity*: suppose $x$ satisfies LHS. Then angles of RHS satisfy:

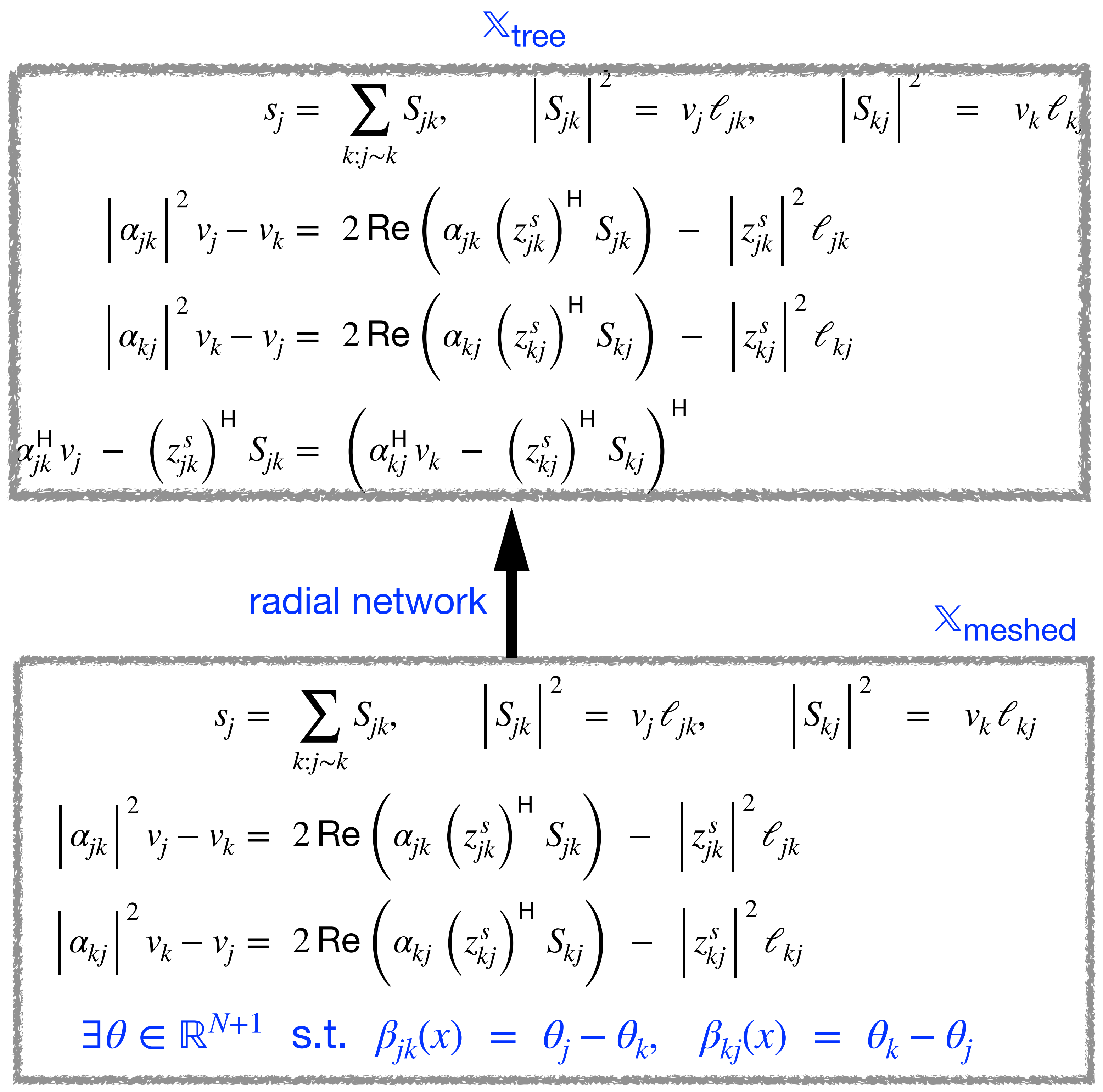
$$\angle\left(\alpha_{jk}^{\mathsf{H}} v_j - z_{jk}^{s\mathsf{H}} S_{jk}\right) \ = \ \beta_{jk}(x) \ = -\beta_{kj}(x) \ = -\angle\left(\alpha_{kj}^{\mathsf{H}} v_k - z_{kj}^{s\mathsf{H}} S_{kj}\right)$$

We now show that they have equal magnitudes as well:

$$\left|\alpha_{jk}^{\mathsf{H}} v_j - \left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}\right|^2 \ = \ |\alpha_{jk}|^2 v_j^2 + |z_{jk}^s|^2 |S_{jk}|^2 - 2 v_j \mathrm{Re}\left(\alpha_{jk} z_{jk}^{s\mathsf{H}} S_{jk}\right) \ = \ v_j v_k \ = \ \left|\alpha_{kj}^{\mathsf{H}} v_k - \left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right|^2$$

# Equivalence proof

## Proof $\mathbb{X}_{\text{meshed}} \equiv \mathbb{X}_{\text{tree}}$




Suppose $G$ is a tree.

Will show $x := (s, v, \ell, S) \in \mathbb{X}_{\text{meshed}} \iff x \in \mathbb{X}_{\text{tree}}$

It suffices to show nonlinear cycle condition becomes linear:

$$\beta_{jk}(x) := \angle\left(\alpha_{jk}^{\mathsf{H}} v_j - z_{jk}^{s\mathsf{H}} S_{jk}\right) = \theta_j - \theta_k$$

$$\iff \alpha_{jk}^{\mathsf{H}} v_j - \left(z_{jk}^s\right)^{\mathsf{H}} S_{jk} = \left(\alpha_{kj}^{\mathsf{H}} v_k - \left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

*Sufficiency*: suppose $x$ satisfies RHS.

Recall the angle recovery procedure where, since $G$ is a tree, there is a unique voltage angle (up to a reference angle) $\theta := C(C^{\mathsf{T}}C)^{-1}\beta(x) + \phi\mathbf{1}$ s.t. $\beta(x) = C^{\mathsf{T}}\theta$

i.e., $x$ satisfies the LHS.

This completes the equivalence proof.

# Outline

1. Radial network
2. Equivalence
3. Backward forward sweep
    - General BFS
    - Example algorithms
4. Linear power flow model

# Backward forward sweep

## General formulation

Efficient solution method for power flow equations

- Special Gauss-Seidel method that is applicable only to radial networks

Partition variables into two groups $x$ and $y$

- Typically, $x$ are branch variables (e.g. line currents) and $y$ are nodal variables (bus voltages)

Design power flow equations as fixed points: $x = f(x, y), \quad y = g(x, y)$

- Choose $(f, g)$ to have a spatially recursive structure enabled by tree topology

Consists of an outer loop where each outer iteration is implemented by two inner loops

- Outer loop: temporal update over $t$ of $(x(t), y(t))$ to converge to a fixed point
- Backward sweep at $t$ : spatial Gauss-Seidel update over nodes $j$ of $x_j(t)$, with $y(t-1)$ held fixed
- Forward sweep at $t$ : spatial Gauss-Seidel update over nodes $j$ of $y_j(t)$, with newly computed $x(t)$ held fixed

Different BFS algorithms differ in choice of variables $(x, y)$ and design of $(f, g)$

- $(f, g)$ that is spatially recursive automatically translates into a BFS algorithm

# Backward forward sweep

## Spatially recursive Gauss-Seidel

At each outer iteration $t$, spatial Gauss-Seidel update over $j$ normally takes the form

$$x_j(t) := f_j(x_1(t), \ldots, x_{j-1}(t), x_j(t-1), \ldots, x_{n_1}(t-1);\; y(t-1))$$

$$y_j(t) := g_j(x(t);\; y_1(t), \ldots, y_{j-1}(t), y_j(t-1), \ldots, y_{n_2}(t-1))$$

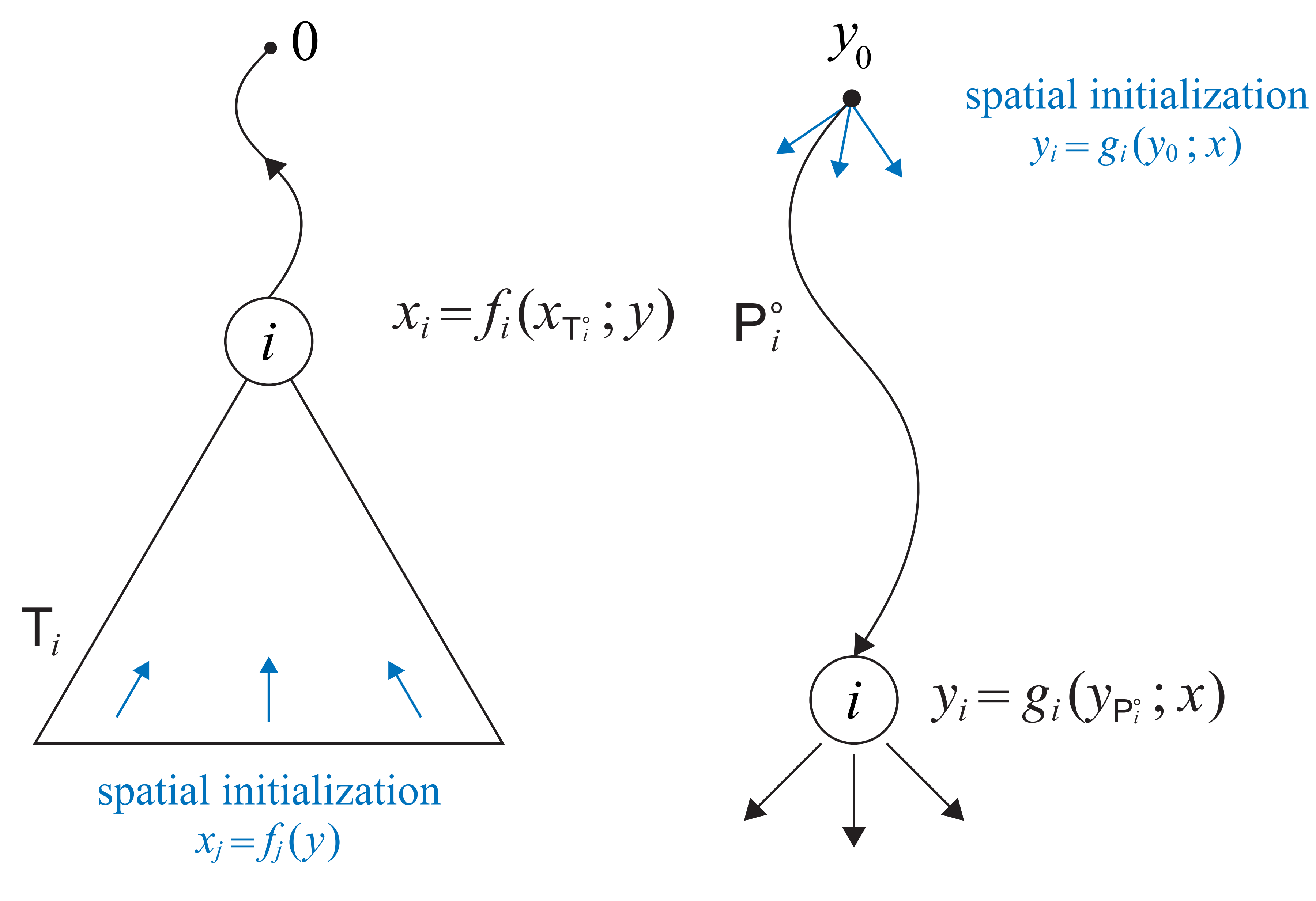
Functions $(f, g)$ are spatially recursive if

- Given $y$, $f_j$ depends on $x$ only through $x_{\mathsf{T}_j^\circ}$ ($\mathsf{T}_j^\circ$ : subtree rooted at $j$)
- Given $x$, $g_j$ depends on $y$ only through $y_{\mathsf{P}_j^\circ}$ ($\mathsf{P}_j^\circ$ : path from $0$ to $j$)

Gauss-Seidel update at $t$ with spatially recursive $(f, g)$

$$\text{BS:} \quad x_j(t) := f_j\left(x_{\mathsf{T}_j^\circ}(t);\; y(t-1)\right)$$

$$\text{FS:} \quad y_j(t) := g_j\left(x(t);\; y_{\mathsf{P}_j^\circ}(t)\right)$$




(a) Backward sweep (b) Forward sweep

# Example: complex form BFM

**Assumptions:** radial network and $z^s_{jk} = z^s_{kj}$

- Can use directed graph (with down orientation) and involve line variables only in direction of the lines
- Can uniquely identify a line variable by its from-node or to-node

Then complex form BFM becomes

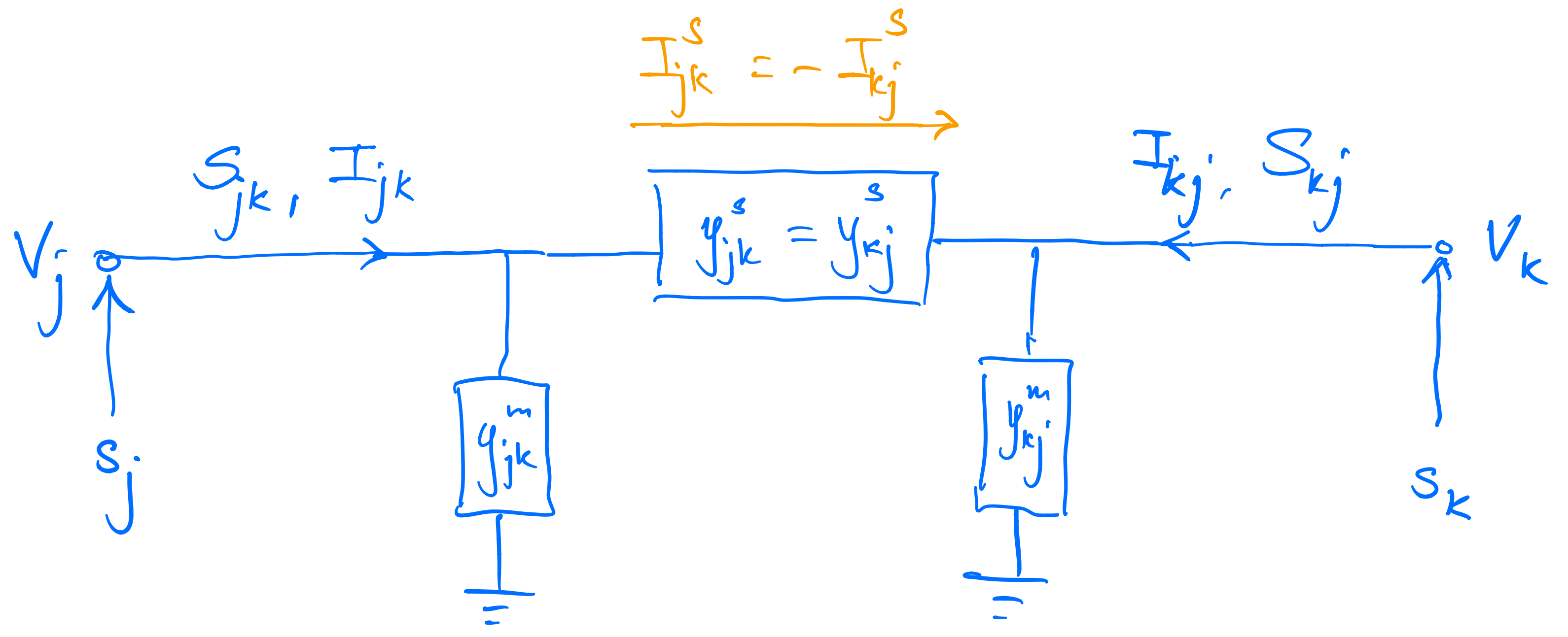
$$s_j \; = \; \sum_{k:j\sim k} V_j \, I^{\mathsf{H}}_{jk}, \qquad I_{jk} \; = \; \tilde{y}_{jk} V_j - y^s_{jk} V_k$$

**Given**: $(V_0, s_j, j \in N)$, **find** $(s_0, V_j, j \in N, \; I_{jk}, S_{jk}, j \to k \in E)$

# Example: complex form BFM

Complex form BFM

$$s_j = \sum_{k:j\sim k} V_j I_{jk}^{\mathsf{H}}, \qquad I_{jk} = \tilde{y}_{jk} V_j - y_{jk}^s V_k$$



**Given**: $(V_0, s_j, j \in N)$, **find** $(s_0, V_j, j \in N,\ I_{jk}, S_{jk}, j \to k \in E)$

Design partitioning $(x, y)$ and corresponding spatially recursive power flow equations $(f, g)$

- $x$ : line currents $I_{jk}^s$ across impedance $z_{jk}^s$
- $y$ : nodal voltages $V_j$
- Given a solution $(V_j, I_{jk}^s)$, all other quantities (e.g. $I_{jk}, S_{jk}$) can be computed
- Can also design BFS that computes sending-end line currents $I_{jk}$ instead of $I_{jk}^s$ (Exercise)

# Example: complex form BFM

## Spatially recursive $(f, g)$

Since $I^s_{jk} := I_{jk} \; - \; y^m_{jk} V_j$, KCL at each bus $j$

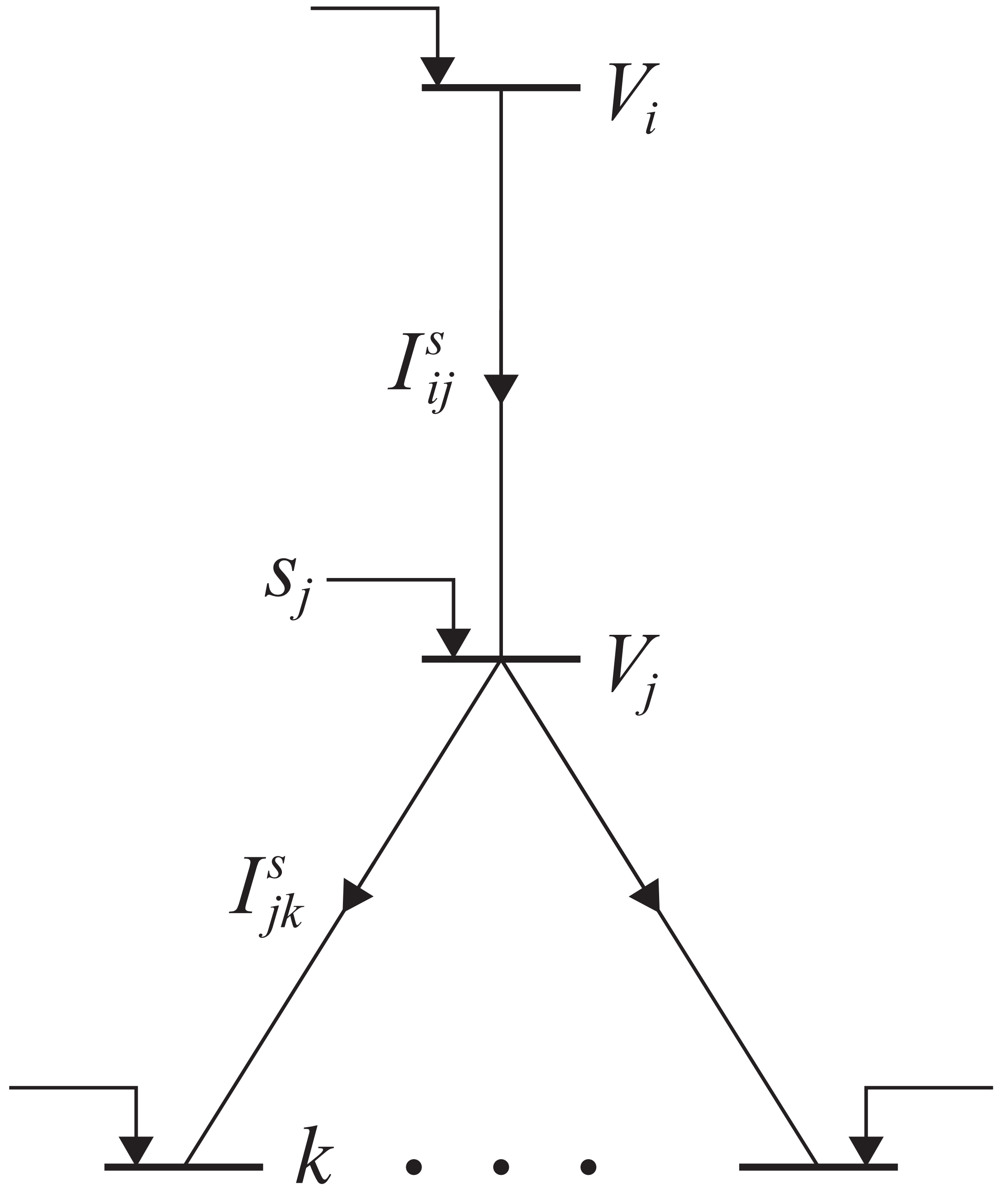
$$\left(\frac{s_j}{V_j}\right)^{\mathsf{H}} + \left(I^s_{ij} - y^m_{ji} V_j\right) \; = \; \sum_{k: j \to k} \left(I^s_{jk} + y^m_{jk} V_j\right)$$

Spatially recursive power flow equations $(f, g)$:

$$I^s_{ij} \; = \; \sum_{k: j \to k} I^s_{jk} \; - \left( \left(\frac{s_j}{V_j}\right)^{\mathsf{H}} - \; y^m_{jj} V_j \right) \quad =: \quad f_j\left(x_{\mathsf{T}^\circ_j}; y\right)$$

$$V_j \; = \; V_i \; - \; z^s_{ij} I^s_{ij} \quad =: \quad g_j\left(x; y_{\mathsf{P}^\circ_j}\right)$$

where $i := i(j)$ is unique parent of $j$ and $y^m_{jj} := y^m_{ji} + \sum_k y^m_{jk}$



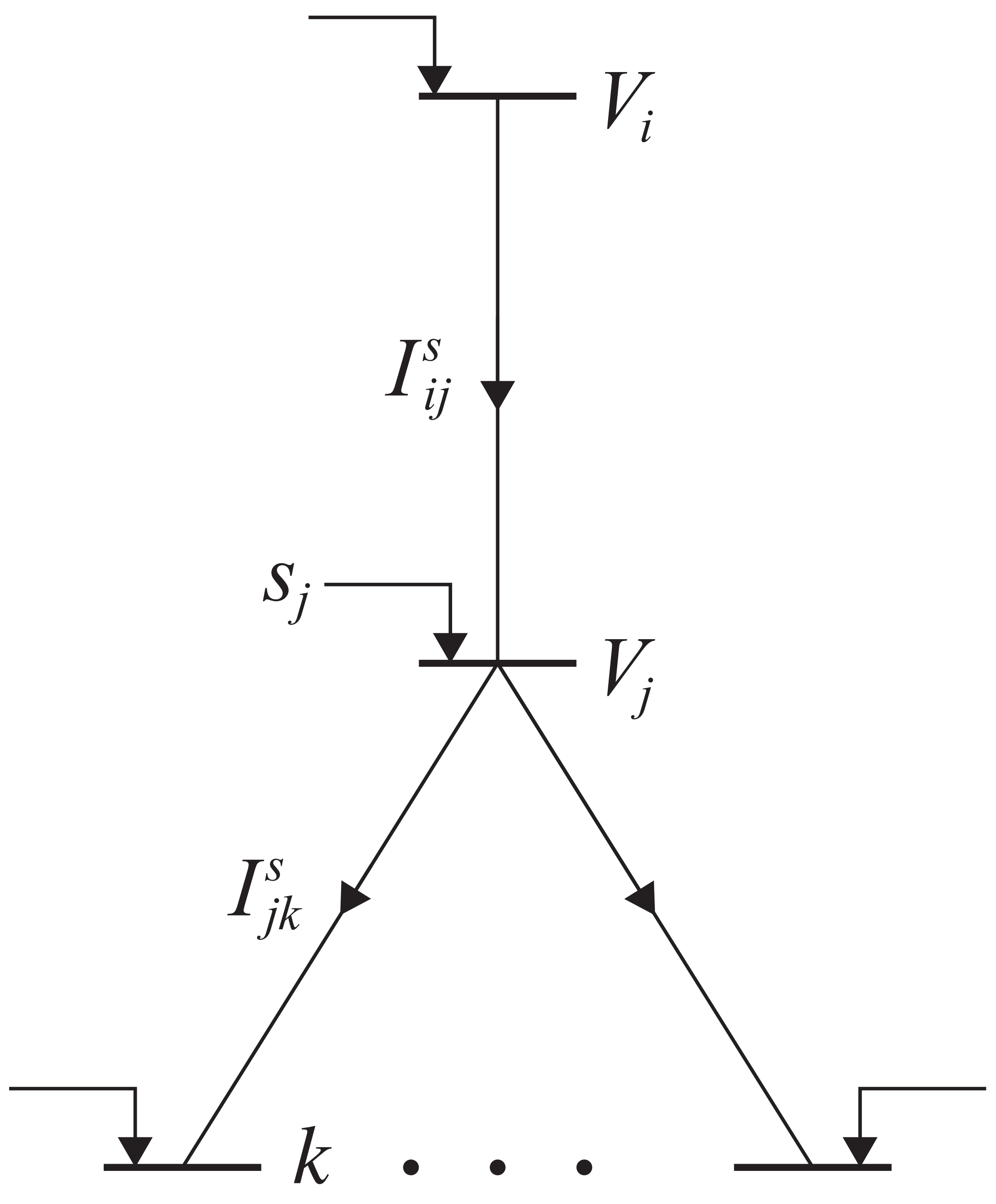
# Example: complex form BFM

## BFS

Spatially recursive power flow equations $(f, g)$:

$$I_{ij}^s \;=\; \sum_{k:j\to k} I_{jk}^s \;-\; \left( \left( \frac{s_j}{V_j} \right)^{\mathsf{H}} - \; y_{jj}^m V_j \right) \;=:\; f_j\left(x_{\mathsf{T}_j^\circ}; y\right)$$

$$V_j \;=\; V_i \;-\; z_{ij}^s I_{ij}^s \;=:\; g_j\left(x; y_{\mathsf{P}_j^\circ}\right)$$




This translates automatically to a BFS algorithm with inner loops:

BS: $x_j(t) \;:=\; f_j\left(x_{\mathsf{T}_j^\circ}(t);\; y(t-1)\right)$

FS: $y_j(t) \;:=\; g_j\left(x(t);\; y_{\mathsf{P}_j^\circ}(t)\right)$

D. Shirmohammadi, H. W. Hong, A. Semlyen, and G. X. Luo. A compensation-based power flow method for weakly meshed distribution and transmission networks. *IEEE Transactions on Power Systems*, 3(2):753–762, May 1988.

# Example: complex form BFM

## Outer loop

**while** *stopping criterion not met* **do**

(a) $t \leftarrow t+1;$

(b) *Backward sweep*: **for** *j starting from leaf nodes and iterating towards bus 0* **do**

$$x_j(t) \quad \leftarrow \quad f_j\left(x_{\mathsf{T}_j^\circ}(t);\ y(t-1)\right), \qquad j \in \overline{N}$$

(c) *Forward sweep*: **for** *j starting from children of bus 0 and iterating towards leaf nodes* **do**

$$y_j(t) \quad \leftarrow \quad g_j\left(x(t);\ y_{\mathsf{P}_j^\circ}(t)\right), \qquad j \in N$$

# Example: complex form BFM

(b) *Backward sweep*: **for** *j starting from leaf nodes and iterating towards bus 0* **do**

$$I_{ij}^{s}(t) \;\leftarrow\; \sum_{k:j\to k} I_{jk}^{s}(t) \;-\; \left( \left( \frac{s_j}{V_j(t-1)} \right)^{\mathsf{H}} - y_{jj}^{m} V_j(t-1) \right), \quad i \to j \in E$$

Given all voltages $V(t-1)$

Given all currents $I_{jk}^{s}(t)$ in previous layer (in $\mathrm{T}_j^{\circ}$)

Update all currents $I_{ij}^{s}(t)$ in present layer (reverse breadth-first search)

# Example: complex form BFM

(c) *Forward sweep*: **for** *j starting from children of bus 0 and iterating towards leaf nodes* **do**

$$V_j(t) \;=\; V_i(t) - z^s_{ij} I^s_{ij}(t), \qquad j \in N$$

Given all currents $I^s(t-1)$

Given voltage $V_i(t)$ at parent of $j$ (in $P^\circ_j$)

Update $V_j(t)$ (breadth-first or depth-first search)

# Example: DistFlow model

**Assumptions:** radial network, $z^s_{jk} = z^s_{kj}$ and $y^m_{jk} = y^m_{kj} = 0$

- Can use directed graph (with up orientation) and involve line variables only in direction of the lines
- Can uniquely identify a line variable by its from-node or to-node

**Given**: $(V_0, s_j, j \in N)$, **find** $(s_0, v_j, j \in N,\ \ell_{jk}, S_{jk}, j \to k \in E)$

Design partitioning $(x, y)$ and corresponding spatially recursive power flow equations $(f, g)$

- Line flows: $x := \left(S_{ji(j)}, \ell_{ji(j)}, j \in N\right) = \left(S_{jk}, \ell_{jk}, j \to k \in E\right)$
- Nodal voltages: $y := (v_j, j \in N)$
- Given a solution $(x, y)$, $s_0$ can be computed

# Example: DistFlow model

## Spatially recursive $(f, g)$

Backward sweep function $f_j\left(x_{\mathsf{T}_j^\circ}; y\right)$:

$$S_{ji} = s_j + \sum_{k:k\to j}\left(S_{kj} - z_{kj}^s \ell_{kj}\right), \qquad \ell_{ji} = \frac{|S_{ji}|^2}{v_j}$$

Forward sweep function $g\left(x, y_{\mathsf{P}_j^\circ}\right)$:

$$v_j = v_i + 2\,\mathrm{Re}\left(z_{ji}^{s\mathsf{H}} S_{ji}\right) - |z_{ji}^s|^2 \ell_{ji}$$

This translates automatically to a BFS algorithm with inner loops:

BS: $x_j(t) := f_j\left(x_{\mathsf{T}_j^\circ}(t);\ y(t-1)\right)$

FS: $y_j(t) := g_j\left(x(t);\ y_{\mathsf{P}_j^\circ}(t)\right)$

# Outline

1. Radial network
2. Equivalence
3. Backward forward sweep
4. Linear power flow model
    - With shunt admittances
    - Without shunt admittances
    - Linear solution and properties

# Linear models

## Advantages

Linear approximations of BFM have two advantages

1. Given nodal injections $s$, voltages $v^{\text{lin}}$ and line flows $S^{\text{lin}}$ can be solved explicitly
2. The linear solution $\left(v^{\text{lin}}, S^{\text{lin}}\right)$ provides bounds on $(v, S)$ from power flow solutions to nonlinear DistFlow models

Linear approximations are reasonable when line losses $z_{jk}\ell_{jk}$ are small compared with line flows $S_{jk}$

# With shunt admittances

$$s_j = \sum_{k:j\sim k} S_{jk}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk} = \left(\alpha^{\mathsf{H}}_{kj} v_k - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

$$\left|S_{jk}\right|^2 = v_j \ell_{jk}, \quad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

BFM-radial

$\ell := 0$

$$s_j = \sum_{k:j\sim k} S_{jk}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right)$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)$$

$$\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk} = \left(\alpha^{\mathsf{H}}_{kj} v_k - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

BFM-linear

# With shunt admittances

$6N + 2$ linear equations in $7N + 3$ real variables $(s, v, S)$

Power flow problem: given $(v_0, s_j, j \in N)$, solve for remaining $5N + 2$ vars $(s_0, v_j, j \in N)$

More equations than unknowns, but they are typically linearly dependent

$$s_j = \sum_{k:j\sim k} S_{jk}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}\right)$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right)$$

$$\alpha_{jk}^{\mathsf{H}} v_j - \left(z_{jk}^s\right)^{\mathsf{H}} S_{jk} = \left(\alpha_{kj}^{\mathsf{H}} v_k - \left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

BFM-linear

# Example

## 2-bus network

Buses $j$ and $k$ connected by a transformer characterized by $(K, \tilde{y}^s, \tilde{y}^m)$ (voltage gain $K$ may be complex). Let $\tilde{\alpha} := (1 + \tilde{z}^s \tilde{y}^m)$.

Linear BFM: 4 linear equations in 6 vars $(s, v)$

$$v_j - v_k \Big/ \left|K\right|^2 = 2\,\text{Re}\left(\left(\tilde{z}^s\right)^{\mathsf{H}} s_j\right)$$

$$\left|\tilde{\alpha}/K\right|^2 v_k - v_j = 2\,\text{Re}\left(\tilde{\alpha}\left(\tilde{z}^s\right)^{\mathsf{H}} s_k\right)$$

$$v_j - \left(\tilde{z}^s\right)^{\mathsf{H}} s_j = \left(\tilde{\alpha}/\left|K\right|^2\right) v_k - \tilde{z}^s \bar{s}_k$$

# Example

## 2-bus network

**Power flow problem:** given $(p_k, q_k, v_j)$, find $(p_j, q_j, v_k)$

Assume: $\tilde{y}^m = 0$ s.t. $\tilde{\alpha} = 1$. Then

$$\underbrace{\begin{bmatrix} 2\tilde{r} & 2\tilde{x} & -1/|K|^2 \\ 0 & 0 & 1/|K|^2 \\ \tilde{r} & \tilde{x} & 1/|K|^2 \\ -\tilde{x} & \tilde{r} & 1/|K|^2 \end{bmatrix}}_{A} \begin{bmatrix} p_j \\ q_j \\ v_k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 2\tilde{r} & 2\tilde{x} & 1 \\ \tilde{r} & \tilde{x} & 1 \\ \tilde{x} & -\tilde{r} & 0 \end{bmatrix} \begin{bmatrix} p_k \\ q_k \\ v_j \end{bmatrix}$$

Elementary row operation reduces $A$ to a rank-3 matrix: $\begin{bmatrix} (\tilde{r}/\tilde{x})(\tilde{r}^2 + \tilde{x}^2) & 0 & 0 \\ 0 & \tilde{r}^2 + \tilde{x}^2 & 0 \\ 0 & 0 & 1/|K|^2 \\ 0 & 0 & 0 \end{bmatrix}$

# Without shunt admittances

BFM-radial

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk} = \left(\alpha^{\mathsf{H}}_{kj} v_k - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

$y^s_{jk} = y^s_{kj}$

$y^m_{jk} = y^m_{kj} = 0$

DistFlow

$$\sum_{k:j\rightarrow k} S_{jk} = \sum_{i:i\rightarrow j}\left(S_{ij} - z^s_{ij}\ell_{ij}\right) + s_j$$

$$v_j - v_k = 2\,\mathrm{Re}\left(z^{s\mathsf{H}}_{jk} S_{jk}\right) - |z^s_{jk}|^2 \ell_{jk}$$

$$v_j \ell_{jk} = |S_{jk}|^2$$

$\ell := 0$

$$\sum_{k:j\rightarrow k} S_{jk} = \sum_{i:i\rightarrow j} S_{ij} + s_j$$

$$v_j - v_k = 2\,\mathrm{Re}\left(z^{s\mathsf{H}}_{jk} S_{jk}\right)$$

LinDistFlow

**BFM-radial**

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk} = \left(\alpha^{\mathsf{H}}_{kj} v_k - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

→ $y^s_{jk} = y^s_{kj}$, $y^m_{jk} = y^m_{kj} = 0$ → **DistFlow**

$$\sum_{k:j\to k} S_{jk} = \sum_{i:i\to j}\left(S_{ij} - z^s_{ij}\ell_{ij}\right) + s_j$$

$$v_j - v_k = 2\,\mathrm{Re}\left(z^{s\mathsf{H}}_{jk} S_{jk}\right) - |z^s_{jk}|^2 \ell_{jk}$$

$$v_j \ell_{jk} = |S_{jk}|^2$$

DistFlow → $\ell := 0$ → LinDistFlow

BFM-radial → $\ell := 0$ → **BFM-linear**

$$s_j = \sum_{k:j\sim k} S_{jk}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right)$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)$$

$$\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk} = \left(\alpha^{\mathsf{H}}_{kj} v_k - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

BFM-linear → $y^s_{jk} = y^s_{kj}$, $y^m_{jk} = y^m_{kj} = 0$ → **LinDistFlow**

$$\sum_{k:j\to k} S_{jk} = \sum_{i:i\to j} S_{ij} + s_j$$

$$v_j - v_k = 2\,\mathrm{Re}\left(z^{s\mathsf{H}}_{jk} S_{jk}\right)$$

# Without shunt admittances

## LinDistFlow in vector form

Let

- $C$ : bus-by-line $(N+1) \times N$ incidence matrix
- $D_r := \text{diag}\left(r_l, l \in E\right) \succ 0$ : diagonal matrix of line resistances
- $D_x := \text{diag}\left(x_l, l \in E\right) \succ 0$ : diagonal matrix of line reactances

Then LinDistFlow is:

$$s = CS, \qquad C^{\mathsf{T}} v = 2\left(D_r P + D_x Q\right)$$

Important features because of tree topology

- $C$ is of rank 1 with null$(C)$ = span($\mathbf{1}$)
- Reduced $N \times N$ incidence matrix $\hat{C}$ is nonsingular
- $\hat{C}^{-1}$ has a simple structure

These features allow explicit linear solutions and structural properties

# Linear solution

Let $C \; =: \; \begin{bmatrix} c_0^\top \\ \hat{C} \end{bmatrix}$

Then LinDistFlow is:

$$\hat{s} = \hat{C}S, \quad s_0 = c_0^\top S$$
$$v_0 c_0 + C^\top \hat{v} = 2\left(D_r P + D_x Q\right)$$

**Given** $(v_0, s_j, j \in N)$, the remaining variables $\left(s_0, v_j, j \in N, S_l, l \in E\right)$ can be obtained explicitly

# Linear solution

**Theorem** [linear solution]

1. Linear solution is:

$$S = \hat{C}^{-1}\hat{s}, \qquad s_0 = c_0^\top \hat{C}^{-1}\hat{s}$$

$$\hat{v} = v_0\mathbf{1} + 2\left(R\hat{p} + X\hat{q}\right)$$

voltages = $v_0$ + correction term $\left(\hat{p}, \hat{q}\right)$

where $R := \hat{C}^{-\top}D_r\hat{C}^{-1}$ and $X := \hat{C}^{-\top}D_x\hat{C}^{-1}$

2. $R \succ 0$ and $X \succ 0$ are positive matrices with

$$R_{jk} = \sum_{l\in P_j\cap P_k} r_l, \qquad X_{jk} = \sum_{l\in P_j\cap P_k} x_l$$

Since entries of $(R, X)$ are nonnegative,
positive injections $(p, q)$ always increase $v$

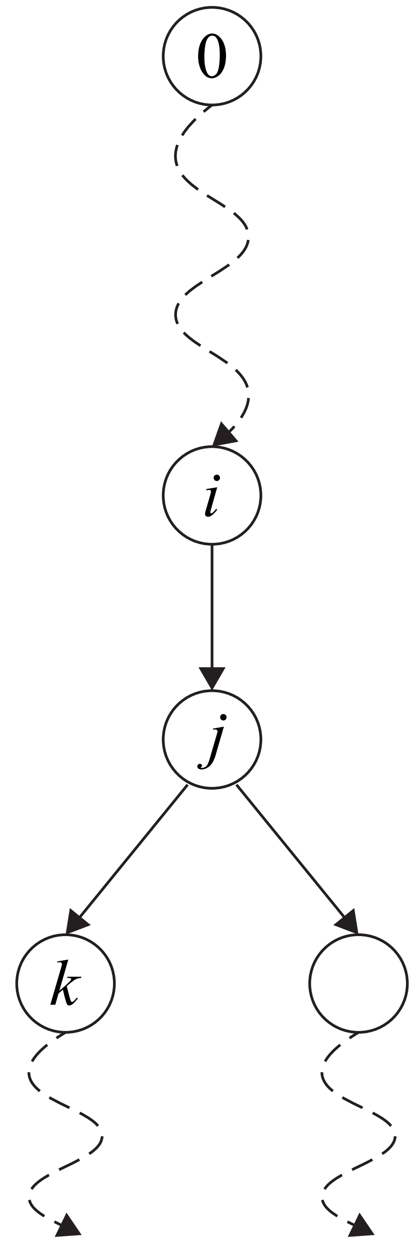
# Analytical properties

## Special graph orientations

**Down orientation:** pointing away from bus 0

$$\sum_{k:j\to k} S_{jk}^{\text{lin}} = S_{ij}^{\text{lin}} + s_j$$

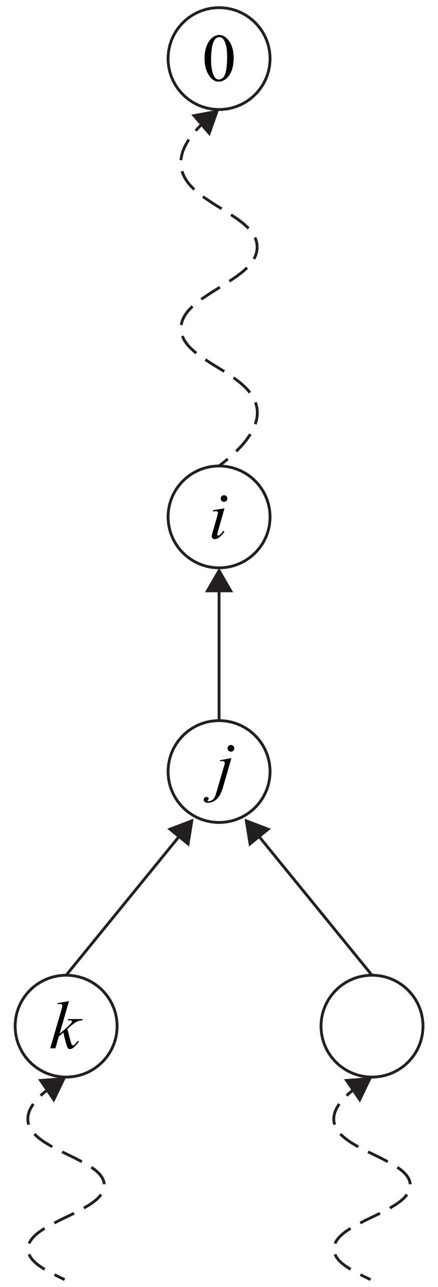
$$v_j^{\text{lin}} - v_k^{\text{lin}} = 2\,\text{Re}\left(z_{jk}^{\text{H}} S_{jk}^{\text{lin}}\right)$$




**Up orientation:** pointing towards bus 0

$$\overline{S}_{ji}^{\text{lin}} = \sum_{k:k\to j} \overline{S}_{kj}^{\text{lin}} + s_j$$

$$\overline{v}_k^{\text{lin}} - \overline{v}_j^{\text{lin}} = 2\,\text{Re}\left(z_{kj}^{\text{H}} \overline{S}_{kj}^{\text{lin}}\right)$$



# Analytical properties

<u>**Corollary**</u>

1. For lines $(i,j) \in E$, $S_{ij}^{\text{lin}} = \overline{S}_{ji}^{\text{lin}}$. Moreover

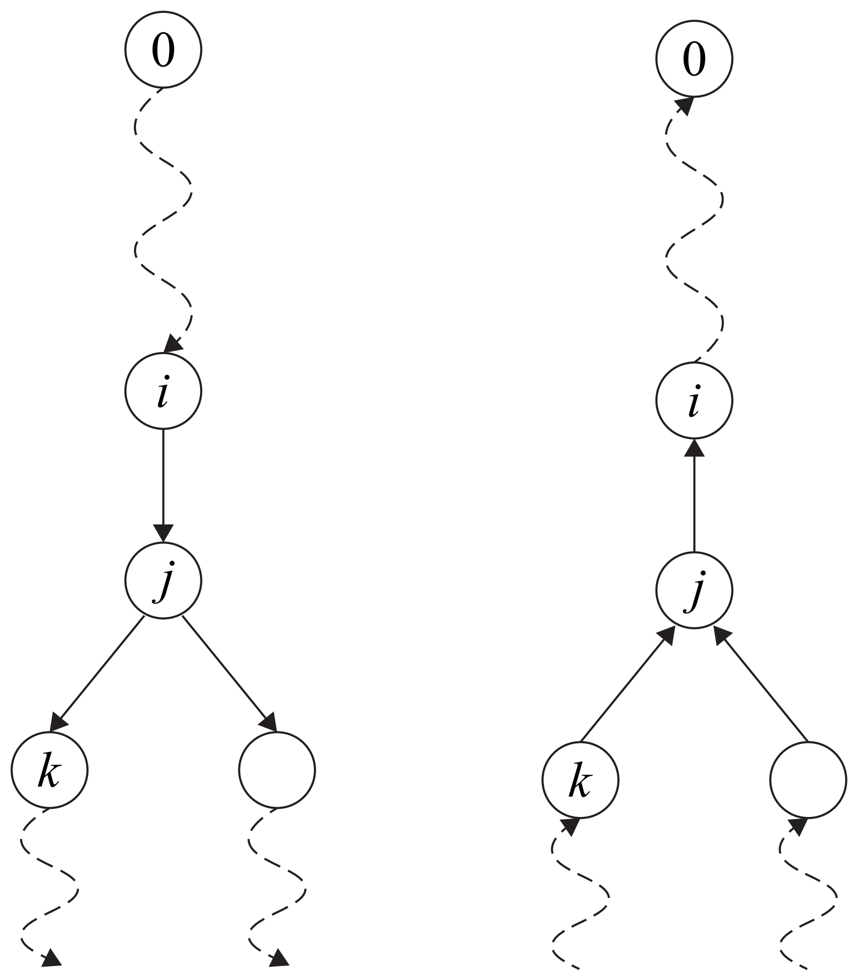
$$S_{ij}^{\text{lin}} = -\sum_{k \in \mathrm{T_j}} s_k, \qquad i \to j$$

line flow $S_{ij}^{\text{lin}}$ to $j$ supplies all loads $-s_k$ in subtree $\mathrm{T_j}$

$$\overline{S}_{ji}^{\text{lin}} = \sum_{k \in \mathrm{T_j}} s_k, \qquad j \to i$$

line flow $\overline{S}_{ij}^{\text{lin}}$ from $j$ come from all injections $s_k$ in subtree $\mathrm{T_j}$

2. For buses $j \in \overline{N}$, $v_j^{\text{lin}} = \overline{v}_j^{\text{lin}} = v_0 + 2\sum_k \left( R_{jk} p_k \ + \ X_{jk} q_k \right)$



# Analytical properties

## Nonlinear DistFlow solution

Linear DistFlow model ignores line losses $\implies$ simple relation between line flows $(S_{ij}, \overline{S}_{ji})$ and injections $s_k$

Given $s$, nonlinear DistFlow solutions $(v, \ell, S)$ satisfy the recursion in up orientation

$$S_{ij} = -\sum_{k \in \mathsf{T}_j} s_k + \left( z_{ij}\ell_{ij} + \sum_{l \in \mathsf{T}_j} z_l \ell_l \right), \qquad v_j = v_0 - \sum_{l \in \mathsf{P}_j} \left( 2\,\mathrm{Re}\left( z_l^{\mathsf{H}} S_l \right) - |z_l|^2 \ell_l \right)$$

and solutions $(\overline{v}, \overline{\ell}, \overline{S})$ satisfy the recursion in down orientation

$$\overline{S}_{ji} = \sum_{k \in \mathsf{T}_j} s_k - \sum_{l \in \mathsf{T}_j} z_l \overline{\ell}_l, \qquad \overline{v}_j = v_0 + \sum_{l \in \mathsf{P}_j} \left( 2\,\mathrm{Re}\left( z_l^{\mathsf{H}} \overline{S}_l \right) - |z_l|^2 \overline{\ell}_l \right)$$

line losses

# Bounds on nonlinear solutions

**Corollary** [bounds on nonlinear solutions]

1. For $i \to j \in E$, $S_{ij} \geq S_{ij}^{\text{lin}}$
2. For $j \to i \in E$, $\overline{S}_{ji} \geq \overline{S}_{ji}^{\text{lin}}$
3. For $j \in \overline{N}$, $v_j = \overline{v}_j \leq \overline{v}_j^{\text{lin}} = v_j^{\text{lin}}$

LinDistFlow ignores losses and underestimates required power to supply loads

proving $\overline{v}_j \leq \overline{v}_j^{\text{lin}}$ is easy;
proving directly $v_j \leq v_j^{\text{lin}}$ is not

# Summary

1. Radial network
   - BFM with and without shunt admittances
   - Nonlinear (quadratic) power flow equations
2. Equivalence
   - BFM variants are all equivalent, and equivalent to BIM
3. Backward forward sweep
   - Gauss-Seidel method that exploits spatially recursive structure enabled by tree topology
4. Linear power flow model
   - Linear BFM with and without shunt admittances
   - Explicit linear solution and bounds on nonlinear solutions
5. Application: volt/var control
   - Local and memoryless control can stabilize voltages and implicitly minimizes cost determined by control design