Power System Analysis

Chapter 5 Branch flow models: radial networks

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Outline

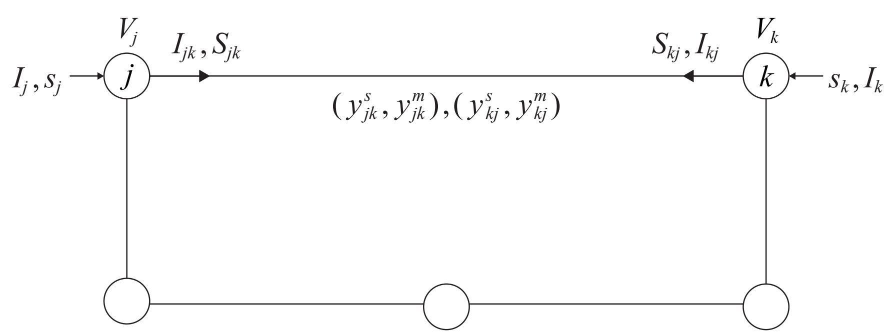
1. Radial networks
2. Equivalence
3. Backward forward sweep
4. Linear power flow model

Outline

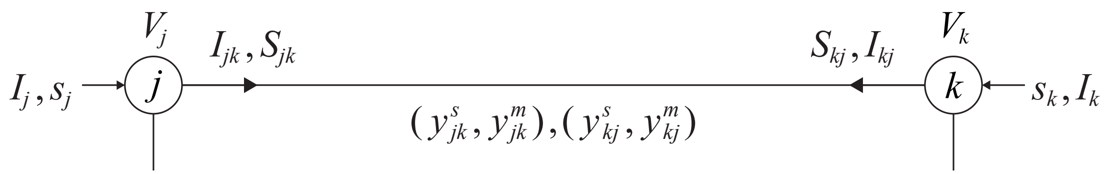
1. Radial network
   - Line model
   - With shunt admittances
   - Without shunt admittances
   - Power flow solution
2. Equivalence
3. Backward forward sweep
4. Linear power flow model

Line model

1. Network $G := (\overline{N}, E)$
   - $\overline{N} := \{0\} \cup N := \{0\} \cup \{1,\ldots,N\}$: buses/nodes/terminals
   - $E \subseteq \overline{N} \times \overline{N}$: lines/branches/links/edges
2. Each line (j,k) is parameterized by $\left(y^s_{jk}, y^m_{jk}\right)$ and $\left(y^s_{kj}, y^m_{kj}\right)$
   - (y^s_{jk}, y^m_{jk}) : series and shunt admittances from j to k
   - (y^s_{kj}, y^m_{kj}) : series and shunt admittances from k to j
   - Models transmission or distribution lines, single-phase transformers



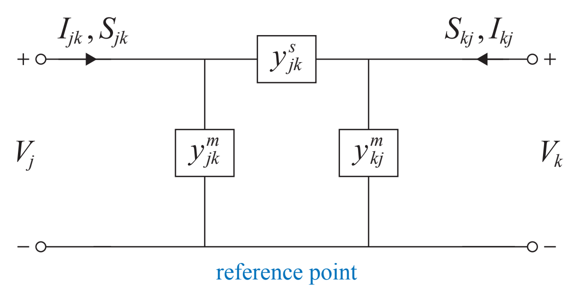
Line model



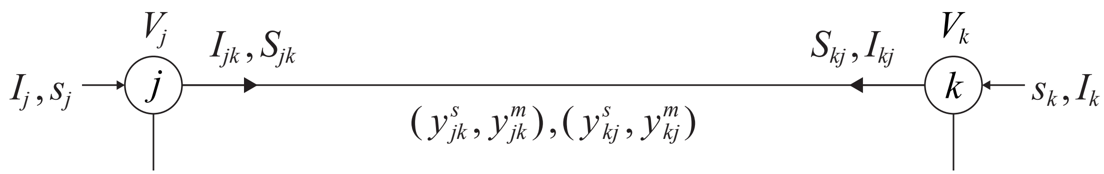
Sending-end currents

$$I_{jk} = y^s_{jk}(V_j - V_k) + y^m_{jk} V_j, \qquad I_{kj} = y^s_{kj}(V_k - V_j) + y^m_{kj} V_k,$$

If $y^s_{jk} = y^s_{kj}$: same relation but equivalent to Π circuit:



Line model



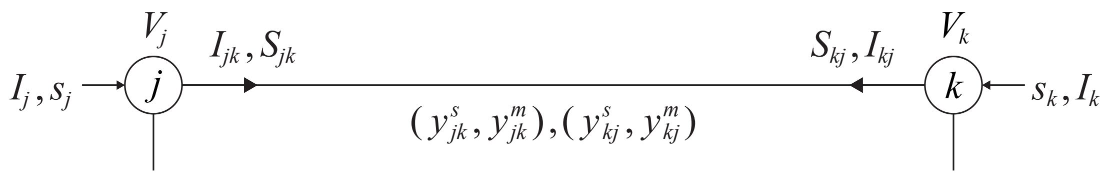
Sending-end currents

$$I_{jk} \;=\; y^s_{jk}(V_j - V_k) \;+\; y^m_{jk}\, V_j, \qquad I_{kj} \;=\; y^s_{kj}(V_k - V_j) \;+\; y^m_{kj}\, V_k,$$

Recall: bus injection models relate nodal variables (s, V) and are suitable for general networks

$$s_j = \sum_{k:j\sim k} \left(y^s_{jk}\right)^H \left(|V_j|^2 - V_j V_k^H \right) \;+\; \left(y^m_{jj}\right)^H |V_j|^2$$

Line model



Sending-end currents

$$I_{jk} = y^s_{jk}(V_j - V_k) + y^m_{jk} V_j, \qquad I_{kj} = y^s_{kj}(V_k - V_j) + y^m_{kj} V_k,$$

Branch flow models: key features

- Involve branch variables as well
- Particularly suitable for distribution systems which are mostly radial networks
- Variables contain no voltage/current phase angles (only magnitudes)
- Can recover voltage/current angles due to tree topology
- Equivalent to bus injection model

Radial network

With shunt admittances: variables

For each bus j

- $s_j := (p_j, q_j)$ or $s_j := p_j + iq_j$: power injection
- v_j : squared voltage magnitude

For each branch (j, k)

- $\left(\ell_{jk}, \ell_{kj}\right)$: squared magnitude of sending-end current $j \to k$, and $k \to j$
- $S_{jk} := \left(P_{jk}, Q_{jk}\right)$ or $S_{jk} := P_{jk} + iQ_{jk}$: sending-end power $j \to k$; also S_{kj} from $k \to j$

The variables v_j and $\left(\ell_{jk}, \ell_{kj}\right)$ contain no angle information

Angles must be recovered from a power flow solution $x := (s, v, \ell, S) \in \mathbb{R}^{3(N+1)+6M}$

- This is easy for radial networks; trickier for meshed networks

Radial network

With shunt admittances

For each line (j, k) let:

$$\alpha_{jk} := 1 + z^s_{jk}\, y^m_{jk}, \qquad \alpha_{kj} := 1 + z^s_{kj}\, y^m_{kj}$$

$$z^s_{jk} := \left(y^s_{jk}\right)^{-1}, \quad z^s_{kj} := \left(y^s_{kj}\right)^{-1}$$

$\alpha_{jk} = \alpha_{kj}$ if and only if $y^m_{jk} = y^m_{kj}$

$\alpha_{jk} = \alpha_{kj} = 1$ if and only if $y^m_{jk} = y^m_{kj} = 0$

Radial network

With shunt admittances

$$s_j = \sum_{k:j\sim k} S_{jk}$$

power balance

Radial network

With shunt admittances

$$s_j = \sum_{k: j \sim k} S_{jk}$$ power balance

$$\left| S_{jk} \right|^2 = v_j \ell_{jk}, \qquad \left| S_{kj} \right|^2 = v_k \ell_{kj}$$ branch power magnitude

The complex notation is only shorthand for real equations

$$p_j = \sum_k P_{jk}, \qquad q_j = \sum_k Q_{jk}$$

$$v_j \ell_{jk} = P_{jk}^2 + Q_{jk}^2, \qquad v_k \ell_{kj} = P_{kj}^2 + Q_{kj}^2$$

Radial network

With shunt admittances

$$s_j = \sum_{k:j\sim k} S_{jk}$$ power balance

$$\left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$ branch power magnitude

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk} \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$ Ohm's law, KCL (magnitude)

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj} \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

Radial network

With shunt admittances

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$$\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk} = \left(\alpha^{\mathsf{H}}_{kj} v_k - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

cycle condition:
$V_j \bar{V}_k = (V_k \bar{V}_j)^{\mathsf{H}}$

$2(N+1)+6M$ real equations in $3(N+1)+6M$ real vars $x := (s, v, \ell, S) \in \mathbb{R}^{3(N+1)+6M}$

Radial network

With shunt admittances

$$s_j = \sum_{k:j\sim k} S_{jk}$$ power balance

$$\left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$ branch power magnitude

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$ Ohm's law, KCL (magnitude)

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk} = \left(\alpha^{\mathsf{H}}_{kj} v_k - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$ cycle condition: $V_j \bar{V}_k = (V_k \bar{V}_j)^{\mathsf{H}}$

Any $x := (s, v, \ell, S) \in \mathbb{R}^{3(N+1)+6M}$ that satisfies these equations with $(v, \ell) \geq 0$ is a **power flow solution**

Radial network

With shunt admittances

All equations are linear in x, except the quadratic equalities

$$\left|S_{jk}\right|^2 = v_j \ell_{jk}, \quad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

There may be 0, 1, or >1 power flow solutions

This can be relaxed to second-order cone constraint in OPF (later)

Example

2-bus network

Buses j and k connected by a transformer characterized by $(K, \tilde{y}^s, \tilde{y}^m)$ (voltage gain K may be complex)

Line parameters are:

$$y^s_{jk} := \frac{\tilde{y}^s}{K}, \quad y^m_{jk} := \left(1 - \frac{1}{K}\right)\tilde{y}^s, \quad y^s_{kj} := \frac{\tilde{y}^s}{\bar{K}} \quad y^m_{kj} := \frac{1}{|K|^2}\left((1-K)\tilde{y}^s + \tilde{y}^m\right)$$

BFM:

$$v_j - v_k \Big/ \left|K\right|^2 = 2\,\mathrm{Re}\left(\left(\tilde{z}^s\right)^{\mathsf{H}} s_j\right) - \left|\tilde{z}^s\right|^2 \ell_{jk}$$

$$\left|\tilde{\alpha}/K\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\tilde{\alpha}\left(\tilde{z}^s\right)^{\mathsf{H}} s_k\right) - \left|K\tilde{z}^s\right|^2 \ell_{kj}$$

$$\left|s_j\right|^2 = v_j \ell_{jk}, \qquad \left|s_k\right|^2 = v_k \ell_{kj}$$

$$v_j - \left(\tilde{z}^s\right)^{\mathsf{H}} s_j = \left(\tilde{\alpha}/|K|^2\right) v_k - \tilde{z}^s \bar{s}_k$$

$$\tilde{z}^s := (\tilde{y}^s)^{-1}$$

$$\tilde{\alpha} := (1 + \tilde{z}^s \tilde{y}^m)$$

Radial network

Without shunt admittances

Assume: $y^s_{jk} = y^s_{kj}$ and $y^m_{jk} = y^m_{kj} = 0$

Then

1. $\alpha_{jk} = \alpha_{kj} = 1$
2. $\ell_{kj} = \ell_{jk}$ and $S_{kj} + S_{jk} = z^s_{jk}\ell_{jk}$

Can use directed graph with vars $\left(\ell_{jk}, S_{jk}\right)$ defined only in direction of lines $j \to k \in E$

Substitute $\left(\ell_{kj}, S_{kj}\right)$ in terms of $\left(\ell_{jk}, S_{jk}\right)$ into previous power flow equations yields original DistFlow equations of [Baran-Wu 1989]

Radial network

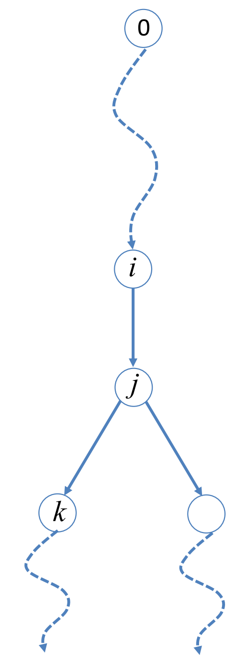
Without shunt admittances

DistFlow equations [Baran-Wu 1989]:

$$\sum_{k:j\to k} S_{jk} = \sum_{i:i\to j} \left(S_{ij} - z^s_{ij}\ell_{ij} \right) + s_j$$ power balance

$$v_j - v_k = 2\,\mathrm{Re}\left(z^{sH}_{jk} S_{jk} \right) - |z^s_{jk}|^2 \ell_{jk}$$ Ohm's law (magnitude)

$$v_j \ell_{jk} = |S_{jk}|^2$$ branch power magnitude



- Cycle condition becomes vacuous (because $S_{kj} := z^s_{jk}\ell_{jk} - S_{jk}$)
- $2(N+1) + 2M$ real equations in $3(N+1) + 3M$ real vars
- e.g. given $(v_0, s_j, j \in N)$, there are $4N+2$ equations in $4N+2$ vars $(s_0, v_j, j \in N, \ell, S)$

Radial network

Without shunt admittances

All equations are linear in x, except the quadratic equalities

$$v_j \ell_j = \left| S_{jk} \right|^2$$

There may be 0, 1, or >1 power flow solutions

This can be relaxed to second-order cone constraint in OPF (later)

Angle recovery

Given power flow solution $x := (s, v, \ell, S)$, define nonlinear functions

$$\beta_{jk}(x) := \angle\left(\alpha_{jk}^H v_j - \left(z_{jk}^s\right)^H S_{jk}\right)$$

$$\beta_{kj}(x) := \angle\left(\alpha_{kj}^H v_k - \left(z_{jk}^s\right)^H S_{kj}\right)$$

Cycle condition ensures that $(\beta_{jk}(x), \beta_{kj}(x))$ are angle differences across line (j, k), i.e.,

$\exists$ voltage angles θ s.t. $\beta(x) = C^\mathsf{T}\theta$

Angle recovery:

1. Tree topology $\implies$ $\theta = C\left(C^\mathsf{T}C\right)^{-1}\beta(x) + \phi\mathbf{1}$
2. $V_j := \sqrt{v_j}e^{i\theta_j}, \qquad I_{jk} := \sqrt{\ell_{jk}}e^{i\left(\theta_j - \angle S_{jk}\right)}$

Summary

BFM for radial network

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk} = \left(\alpha^{\mathsf{H}}_{kj} v_k - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

BFM-radial

$y^s_{jk} = y^s_{kj}$

$y^m_{jk} = y^m_{kj} = 0$

$$\sum_{k:j\to k} S_{jk} = \sum_{i:i\to j}\left(S_{ij} - z_{ij}\ell_{ij}\right) + s_j$$

$$v_j - v_k = 2\,\mathrm{Re}\left(z^{\mathsf{H}}_{jk} S_{jk}\right) - |z_{jk}|^2 \ell_{jk}$$

$$v_j \ell_{jk} = |S_{jk}|^2$$

DistFlow

Example: power flow solution

2-bus network

Two buses 0 and 1 connected by a line with series impedance $z = r + ix$ (graph orientation: up)

$$p_0 - r\ell = -p_1, \quad q_0 - x\ell = -q_1$$

$$v_1 - v_0 = 2\left(rp_1 + xq_1\right) - (r^2 + x^2)\ell$$

$$p_1^2 + q_1^2 = v_1\ell$$

Given: $r = x = 1$ and $v_0 = 1,\ q_1 = 0,$ find (p_0, q_0, v_1, ℓ) and show that $(v_1(p_1), p_1)$ forms an ellipse

Solution

Eliminate $v_1 \ \Rightarrow\ 2\ell^2 - (1 + 2p_1)\ell + p_1^2 = 0.$ Hence $(\Delta := 4p_1(1 - p_1) + 1)$

$$\ell = \frac{1}{4}\left(1 + 2p_1 \pm \sqrt{\Delta}\right), \quad p_0 = \frac{1}{4}\left(1 - 2p_1 \pm \sqrt{\Delta}\right), \quad q_0 = \frac{1}{4}\left(1 + 2p_1 \pm \sqrt{\Delta}\right)$$

$$v_1 = \frac{1}{2}\left(1 + 2p_1 \mp \sqrt{\Delta}\right)$$

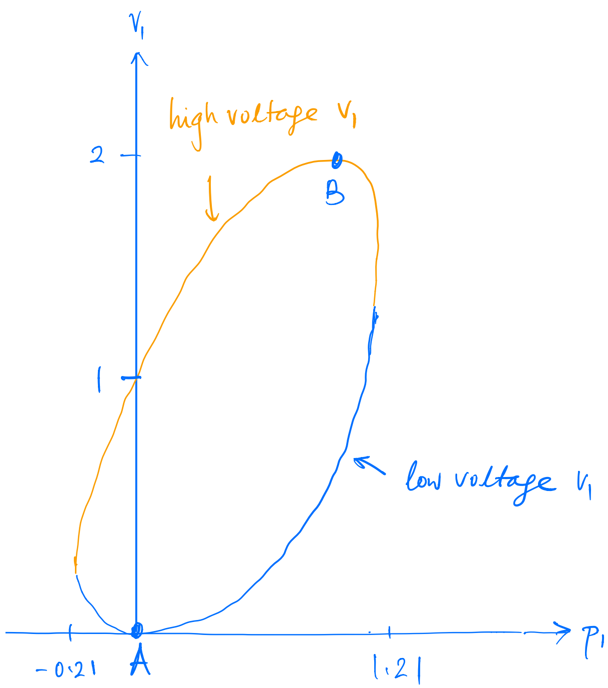
Example: power flow solution

2-bus network

Solution

The solution $v_1 = \left(1 + 2p_1 \mp \sqrt{\Delta}\right)/2$ is equivalent to:

$$[p_1 \quad v_1] \underbrace{\begin{bmatrix} 8 & -4 \\ -4 & 4 \end{bmatrix}}_{A} \begin{bmatrix} p_1 \\ v_1 \end{bmatrix} - 2 \underbrace{[0 \quad 2]}_{c^T} \begin{bmatrix} p_1 \\ v_1 \end{bmatrix} + 1 = 1$$



Points $x \in \mathbb{R}^n$ satisfying

$$(x - c)^T A (x - c) = x^T A x - 2c^T x + \|c\|^2 = 1$$

form an ellipse if A is real (symmetric) and positive definite

Hollow solution set

Let

$$\mathbb{X}_{\text{df}} \ := \ \{x := (s, v, \ell, S) \in \mathbb{R}^{6N+3} : x \text{ satisfies DistFlow equations } \}$$

Theorem

Suppose network graph G is connected. If $\hat{x}$ and $\tilde{x}$ are distinct solutions in $\mathbb{X}_{\text{df}}$ with $\hat{v}_0 = \tilde{v}_0$, then no convex combination of $\hat{x}$ and $\tilde{x}$ can be in $\mathbb{X}_{\text{df}}$. In particular, $\mathbb{X}_{\text{df}}$ is nonconvex.

Outline

1. Radial network
2. Equivalence
    - Extension to general network
    - Equivalence of BFM and BIM
3. Backward forward sweep
4. Linear power flow model

Power flow models

Bus injection model

$$s_j = \sum_{k:j\sim k} \left(y_{jk}^s\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V_k^{\mathsf{H}} \right) + \left(y_{jj}^m\right)^{\mathsf{H}} |V_j|^2$$

Branch flow models

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk} \left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}\right) - \left|z_{jk}^s\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj} \left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right) - \left|z_{kj}^s\right|^2 \ell_{kj}$$

$$\alpha_{jk}^{\mathsf{H}} v_j - \left(z_{jk}^s\right)^{\mathsf{H}} S_{jk} = \left(\alpha_{kj}^{\mathsf{H}} v_k - \left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

- Different vars and equations
- Both describe Kirchhoff's and Ohm's laws
- Are they equivalent? In what sense?

Power flow models

BIM applies to general networks
BFM applies to radial networks only

To show their equivalence, we first need to extend BFM to general networks with cycles

General network

Complex form

Let $\tilde{y}_{jk} := y_{jk}^s + y_{jk}^m$ and $\tilde{y}_{kj} := y_{kj}^s + y_{kj}^m$

BFM for general network:

$$s_j = \sum_{k:j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y_{jk}^s V_k, \qquad I_{kj} = \tilde{y}_{kj} V_k - y_{kj}^s V_j$$

$$S_{jk} = V_j I_{jk}^{\mathsf{H}}, \qquad S_{kj} = V_k I_{kj}^{\mathsf{H}}$$

Does **not** assume $y_{jk}^s = y_{kj}^s$ nor $y_{jk}^m = y_{kj}^m = 0$

This model looks similar to BIM complex form!

- It is a bridge between BFM and BIM

General network

Real form

$$s_j = \sum_{k:j\sim k} S_{jk}$$ power balance

$$\left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$ branch power magnitude

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\text{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$ Ohm's law, KCL (magnitude)

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\text{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\exists \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$$ cycle condition

$2(N+1) + 6M$ real equations in $3(N+1) + 6M$ real vars $x := (s, v, \ell, S) \in \mathbb{R}^{3(N+1)+6M}$

General network

Real form

Major simplification for radial network: nonlinear cycle condition becomes linear in x

All other equations remain the same

$$\beta_{jk}(x) := \angle\left(\alpha_{jk}^{\mathsf{H}} v_j - \left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}\right)$$

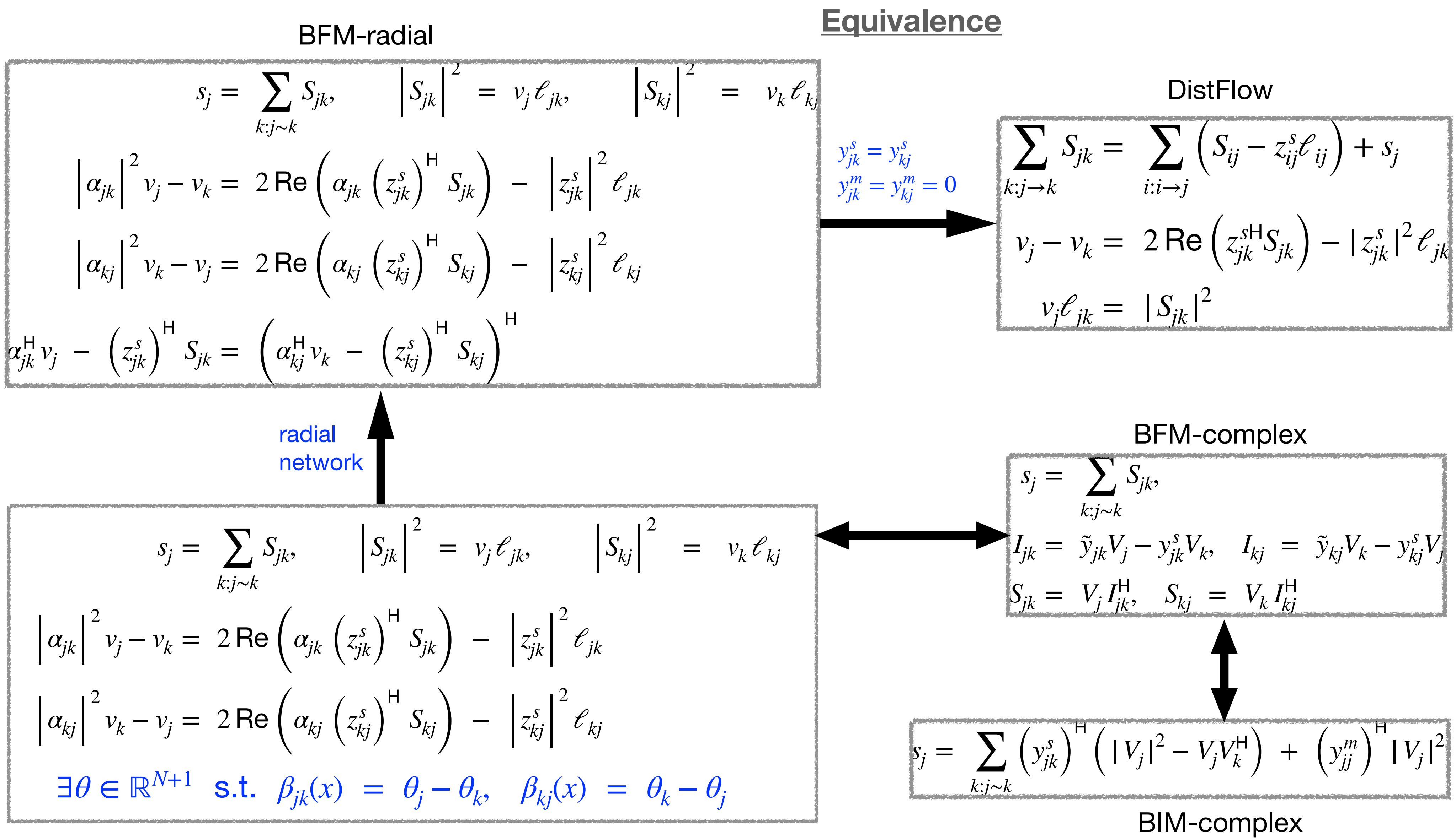
$$\beta_{kj}(x) := \angle\left(\alpha_{kj}^{\mathsf{H}} v_k - \left(z_{jk}^s\right)^{\mathsf{H}} S_{kj}\right)$$

$$\beta(x) = \begin{bmatrix} C^T \\ -C^T \end{bmatrix} \theta \text{ for some } \theta \in \mathbb{R}^{N+1}$$

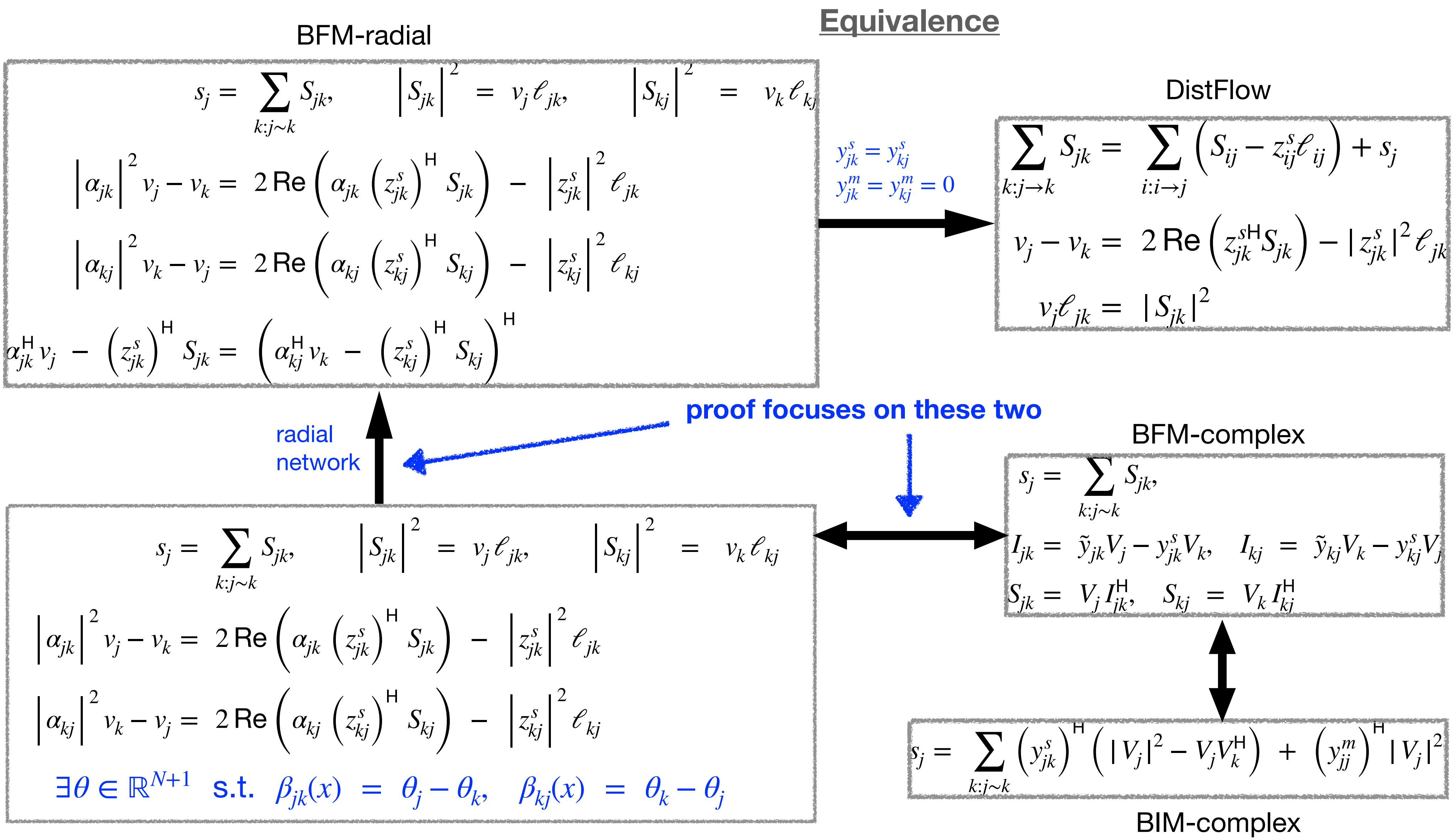
general network

$$\alpha_{jk}^{\mathsf{H}} v_j - \left(z_{jk}^s\right)^{\mathsf{H}} S_{jk} = \left(\alpha_{kj}^{\mathsf{H}} v_k - \left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

radial network



Equivalence
BFM-radial
$s_j = \sum_{k:j\sim k} S_{jk}, \quad |S_{jk}|^2 = v_j \ell_{jk}, \quad |S_{kj}|^2 = v_k \ell_{kj}$
$|\alpha_{jk}|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk} (z^s_{jk})^{\mathsf{H}} S_{jk}\right) - |z^s_{jk}|^2 \ell_{jk}$
$|\alpha_{kj}|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj} (z^s_{kj})^{\mathsf{H}} S_{kj}\right) - |z^s_{kj}|^2 \ell_{kj}$
$\alpha^{\mathsf{H}}_{jk} v_j - (z^s_{jk})^{\mathsf{H}} S_{jk} = \left(\alpha^{\mathsf{H}}_{kj} v_k - (z^s_{kj})^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$
$y^s_{jk} = y^s_{kj}$
$y^m_{jk} = y^m_{kj} = 0$
DistFlow
$\sum_{k:j\to k} S_{jk} = \sum_{i:i\to j} \left(S_{ij} - z^s_{ij}\ell_{ij}\right) + s_j$
$v_j - v_k = 2\,\mathrm{Re}\left(z^{s\mathsf{H}}_{jk} S_{jk}\right) - |z^s_{jk}|^2 \ell_{jk}$
$v_j \ell_{jk} = |S_{jk}|^2$
radial network
$s_j = \sum_{k:j\sim k} S_{jk}, \quad |S_{jk}|^2 = v_j \ell_{jk}, \quad |S_{kj}|^2 = v_k \ell_{kj}$
$|\alpha_{jk}|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk} (z^s_{jk})^{\mathsf{H}} S_{jk}\right) - |z^s_{jk}|^2 \ell_{jk}$
$|\alpha_{kj}|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj} (z^s_{kj})^{\mathsf{H}} S_{kj}\right) - |z^s_{kj}|^2 \ell_{kj}$
$\exists \theta \in \mathbb{R}^{N+1}$ s.t. $\beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$
BFM-complex
$s_j = \sum_{k:j\sim k} S_{jk},$
$I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y^s_{kj} V_j$
$S_{jk} = V_j I^{\mathsf{H}}_{jk}, \quad S_{kj} = V_k I^{\mathsf{H}}_{kj}$
$s_j = \sum_{k:j\sim k} (y^s_{jk})^{\mathsf{H}} \left(|V_j|^2 - V_j V^{\mathsf{H}}_k\right) + (y^m_{jj})^{\mathsf{H}} |V_j|^2$
BIM-complex



Equivalence
BFM-radial
$s_j = \sum_{k:j\sim k} S_{jk}, \quad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \quad \left|S_{kj}\right|^2 = v_k \ell_{kj}$
$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}\right) - \left|z_{jk}^s\right|^2 \ell_{jk}$
$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right) - \left|z_{kj}^s\right|^2 \ell_{kj}$
$\alpha_{jk}^{\mathsf{H}} v_j - \left(z_{jk}^s\right)^{\mathsf{H}} S_{jk} = \left(\alpha_{kj}^{\mathsf{H}} v_k - \left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$
$y_{jk}^s = y_{kj}^s$
$y_{jk}^m = y_{kj}^m = 0$
DistFlow
$\sum_{k:j\to k} S_{jk} = \sum_{i:i\to j}\left(S_{ij} - z_{ij}^s \ell_{ij}\right) + s_j$
$v_j - v_k = 2\,\mathrm{Re}\left(z_{jk}^{s\mathsf{H}} S_{jk}\right) - |z_{jk}^s|^2 \ell_{jk}$
$v_j \ell_{jk} = |S_{jk}|^2$
proof focuses on these two
radial
network
BFM-complex
$s_j = \sum_{k:j\sim k} S_{jk},$
$I_{jk} = \tilde{y}_{jk} V_j - y_{jk}^s V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y_{kj}^s V_j$
$S_{jk} = V_j I_{jk}^{\mathsf{H}}, \quad S_{kj} = V_k I_{kj}^{\mathsf{H}}$
$s_j = \sum_{k:j\sim k} S_{jk}, \quad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \quad \left|S_{kj}\right|^2 = v_k \ell_{kj}$
$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}\right) - \left|z_{jk}^s\right|^2 \ell_{jk}$
$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right) - \left|z_{kj}^s\right|^2 \ell_{kj}$
$\exists \theta \in \mathbb{R}^{N+1}$ s.t. $\beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$
$s_j = \sum_{k:j\sim k} \left(y_{jk}^s\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right) + \left(y_{jj}^m\right)^{\mathsf{H}} |V_j|^2$
BIM-complex

Equivalence

Branch flow models have been most useful for radial networks

- Different variants have different vars and different equations
- Are they equivalent, in what sense?

All BFM variants are equivalent to each other, and to BIM

- BFM-radial: tree topology (cycle condition: linear)
- DistFlow: tree topology with $y^s_{jk} = y^s_{kj}$ and $y^m_{jk} = y^m_{kj} = 0$ (cycle condition: vacuous)
- BFM-real: BFM for general topology (cycle condition: nonlinear)
- BFM-complex: bridge to BIM-complex

We next state and prove these equivalence relations

Equivalence

Solution set

BIM-complex

$$s_j = \sum_{k:j\sim k} \left(y_{jk}^s\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right) + \left(y_{jj}^m\right)^{\mathsf{H}} |V_j|^2$$

Solution set

$$\mathbb{V} := \{(s, V) \in \mathbb{C}^{2(n+1)} \mid V \text{ satisfies BIM}\}$$

Equivalence

Solution set

Branch flow models: solution sets

$$\tilde{\mathbb{X}} := \{\tilde{x} : (s, V, I, S) \in \mathbb{C}^{2(N+1)+4M} \mid \tilde{x} \text{ satisfies BFM complex}\}$$

$$\mathbb{X}_{\text{meshed}} := \{x : (s, v, \ell, S) \in \mathbb{R}^{3(N+1)+6M)} \mid x \text{ satisfies BFM real}\}$$

$$\mathbb{X}_{\text{tree}} := \{x : (s, v, \ell, S) \in \mathbb{R}^{9N+3} \mid x \text{ satisfies BFM radial}\}$$

$$\mathbb{X}_{\text{df}} := \left\{x : (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies BFM radial}, y^s_{jk} = y^s_{kj}, y^m_{jk} = y^m_{kj} = 0\right\}$$

Definition: Two sets A and B are equivalent ($A \equiv B$) if there is a bijection between them

- x is a power flow solution of A iff $g(x)$ is a power flow solution of B

Equivalence

Theorem

Suppose G is connected

1. $\mathbb{V} \equiv \tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$
2. If G is a tree, then $\mathbb{X}_{\text{meshed}} \equiv \mathbb{X}_{\text{tree}}$
3. If G is a tree and $y^s_{jk} = y^s_{kj}$, $y^m_{jk} = y^m_{kj} = 0$, then $\mathbb{X}_{\text{tree}} \equiv \mathbb{X}_{\text{df}}$

Equivalence

Bus injection models and branch flow models are equivalent

- Any result proved in one model holds also in another model

Some results are easier to formulate / prove in one model than the other

- BIM: semidefinite relaxation of OPF (later)
- BFM: some exact relation proofs

Should freely use whichever is more convenient for problem at hand

BFM is particularly suitable for modeling distribution systems

- Tree topology allows efficient computation of power flows (BFS)
- Models and relaxations extend to unbalanced 3ϕ networks
- Seems to be much more numerically stable than BIM for large networks

Equivalence

$\mathbb{X}_{\text{tree}}$

$$s_j = \sum_{k: j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\text{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\text{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\text{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\text{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\alpha^{\text{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\text{H}} S_{jk} = \left(\alpha^{\text{H}}_{kj} v_k - \left(z^s_{kj}\right)^{\text{H}} S_{kj}\right)^{\text{H}}$$

$y^s_{jk} = y^s_{kj}$

$y^m_{jk} = y^m_{kj} = 0$

$\mathbb{X}_{\text{df}}$

$$\sum_{k: j\to k} S_{jk} = \sum_{i: i\to j}\left(S_{ij} - z^s_{ij}\ell_{ij}\right) + s_j$$

$$v_j - v_k = 2\,\text{Re}\left(z^{s\text{H}}_{jk} S_{jk}\right) - |z^s_{jk}|^2 \ell_{jk}$$

$$v_j \ell_{jk} = |S_{jk}|^2$$

radial network

proof focuses on these two

$\mathbb{X}_{\text{meshed}}$

$$s_j = \sum_{k: j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\text{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\text{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\text{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\text{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\exists \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$$

$\tilde{\mathbb{X}}$

$$s_j = \sum_{k: j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y^s_{kj} V_j$$

$$S_{jk} = V_j I^{\text{H}}_{jk}, \quad S_{kj} = V_k I^{\text{H}}_{kj}$$

$$s_j = \sum_{k: j\sim k}\left(y^s_{jk}\right)^{\text{H}}\left(|V_j|^2 - V_j V^{\text{H}}_k\right) + \left(y^m_{jj}\right)^{\text{H}} |V_j|^2$$

$\mathbb{V}$

Equivalence proof

Proof $\mathbb{V} \equiv \tilde{\mathbb{X}}$ and $\mathbb{X}_{\text{tree}} \equiv \mathbb{X}_{\text{df}}$

Straightforward.

$\tilde{\mathbb{X}}$

$$s_j = \sum_{k:j\sim k} S_{jk},$$
$$I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y^s_{kj} V_j$$
$$S_{jk} = V_j I_{jk}^{\mathsf{H}}, \quad S_{kj} = V_k I_{kj}^{\mathsf{H}}$$

$\mathbb{V}$

$$s_j = \sum_{k:j\sim k} \left(y^s_{jk}\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V_k^{\mathsf{H}} \right) + \left(y^m_{jj}\right)^{\mathsf{H}} |V_j|^2$$

$\mathbb{X}_{\text{tree}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \quad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \quad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$
$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk} \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$
$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj} \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$
$$\alpha_{jk}^{\mathsf{H}} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk} = \left(\alpha_{kj}^{\mathsf{H}} v_k - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

$y^s_{jk} = y^s_{kj}$

$\mathbb{X}_{\text{df}}$

$$\sum_{k:j\to k} S_{jk} = \sum_{i:i\to j} \left(S_{ij} - z^s_{ij} \ell_{ij}\right) + s_j$$
$$v_j - v_k = 2\,\mathrm{Re}\left(z^{s\mathsf{H}}_{jk} S_{jk}\right) - |z^s_{jk}|^2 \ell_{jk}$$
$$v_j \ell_{jk} = |S_{jk}|^2$$

Equivalence proof

Proof $\tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$

$\mathbb{X}_{\text{meshed}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\text{Re}\left(\alpha_{jk}\left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}\right) - \left|z_{jk}^s\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\text{Re}\left(\alpha_{kj}\left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right) - \left|z_{kj}^s\right|^2 \ell_{kj}$$

$$\exists \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$$

$\longleftrightarrow$

$\tilde{\mathbb{X}}$

$$s_j = \sum_{k:j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y_{jk}^s V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y_{kj}^s V_j$$

$$S_{jk} = V_j I_{jk}^{\mathsf{H}}, \quad S_{kj} = V_k I_{kj}^{\mathsf{H}}$$

Fix $\tilde{x} := (s, V, I, S) \in \tilde{\mathbb{X}}$. Define

$$v_j := |V_j|^2, \qquad \ell_{jk} := |I_{jk}|^2, \qquad \ell_{kj} := |I_{kj}|^2$$

Will show $x := (s, v, \ell, S) \in \mathbb{X}_{\text{meshed}}$

It suffices to show

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\text{Re}\left(\alpha_{jk}\left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}\right) - \left|z_{jk}^s\right|^2 \ell_{jk}$$

$$\exists \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$$

For the 1st equation, write $V_k = \alpha_{jk} V_j - z_{jk}^s \left(\frac{S_{jk}}{V_j}\right)^{\mathsf{H}}$ and taking square magnitude on both sides.

Equivalence proof

Proof $\tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$

$\mathbb{X}_{\text{meshed}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}\right) - \left|z_{jk}^s\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right) - \left|z_{kj}^s\right|^2 \ell_{kj}$$

$$\exists \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$$

↔

$\tilde{\mathbb{X}}$

$$s_j = \sum_{k:j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y_{jk}^s V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y_{kj}^s V_j$$

$$S_{jk} = V_j I_{jk}^{\mathsf{H}}, \quad S_{kj} = V_k I_{kj}^{\mathsf{H}}$$

Fix $\tilde{x} := (s, V, I, S) \in \tilde{\mathbb{X}}$. Define

$$v_j := |V_j|^2, \qquad \ell_{jk} := |I_{jk}|^2, \qquad \ell_{kj} := |I_{kj}|^2$$

Will show $x := (s, v, \ell, S) \in \mathbb{X}_{\text{meshed}}$

For the 2nd equation, we have

$$V_j V_k^{\mathsf{H}} = \alpha_{jk}^{\mathsf{H}} |V_j|^2 - \left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}, \qquad V_k V_j^{\mathsf{H}} = \alpha_{kj}^{\mathsf{H}} |V_k|^2 - \left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}$$

Equivalence proof

Proof $\tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$

$\mathbb{X}_{\text{meshed}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\exists \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$$

$\longleftrightarrow$

$\tilde{\mathbb{X}}$

$$s_j = \sum_{k:j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y^s_{kj} V_j$$

$$S_{jk} = V_j I^{\mathsf{H}}_{jk}, \quad S_{kj} = V_k I^{\mathsf{H}}_{kj}$$

Fix $\tilde{x} := (s, V, I, S) \in \tilde{\mathbb{X}}$. Define

$$v_j := |V_j|^2, \qquad \ell_{jk} := |I_{jk}|^2, \qquad \ell_{kj} := |I_{kj}|^2$$

Will show $x := (s, v, \ell, S) \in \mathbb{X}_{\text{meshed}}$

For the 2nd equation, we have

$$V_j V_k^{\mathsf{H}} = \alpha^{\mathsf{H}}_{jk} |V_j|^2 - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}, \qquad V_k V_j^{\mathsf{H}} = \alpha^{\mathsf{H}}_{kj} |V_k|^2 - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}$$

Recall the nonlinear functions

$$\beta_{jk}(x) := \angle\left(\alpha^H_{jk} v_j - \left(z^s_{jk}\right)^H S_{jk}\right) = \angle V_j - \angle V_k$$

$$\beta_{kj}(x) := \angle\left(\alpha^H_{kj} v_k - \left(z^s_{jk}\right)^H S_{kj}\right) = \angle V_k - \angle V_j$$

$$\therefore\ \theta_j := \angle V_j$$

Equivalence proof

Proof $\tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$

$\mathbb{X}_{\text{meshed}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\text{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\text{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\text{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\text{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\exists \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$$

$\longleftrightarrow$

$\tilde{\mathbb{X}}$

$$s_j = \sum_{k:j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y^s_{kj} V_j$$

$$S_{jk} = V_j I^{\text{H}}_{jk}, \quad S_{kj} = V_k I^{\text{H}}_{kj}$$

Conversely, fix $x := (s, v, \ell, S) \in \mathbb{X}_{\text{meshed}}$. Construct (V, I) from x:

$$V_j := \sqrt{v_j} e^{i\theta_j}, \qquad I_{jk} := \sqrt{\ell_{jk}} e^{i\left(\theta_j - \angle S_{jk}\right)}$$

Will show $\tilde{x} := (s, V, I, S) \in \tilde{\mathbb{X}}$

It suffices to show

$$S_{jk} = V_j I^{\text{H}}_{jk}, \qquad I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k,$$

Equivalence proof

Proof $\tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$

$\mathbb{X}_{\text{meshed}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\text{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\text{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\exists \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$$

$\longleftrightarrow$

$\tilde{\mathbb{X}}$

$$s_j = \sum_{k:j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y^s_{kj} V_j$$

$$S_{jk} = V_j I^{\mathsf{H}}_{jk}, \quad S_{kj} = V_k I^{\mathsf{H}}_{kj}$$

Conversely, fix $x := (s, v, \ell, S) \in \mathbb{X}_{\text{meshed}}$. Construct (V, I) from x:

$$V_j := \sqrt{v_j}\, e^{i\theta_j}, \qquad I_{jk} := \sqrt{\ell_{jk}}\, e^{i\left(\theta_j - \angle S_{jk}\right)}$$

Will show $\tilde{x} := (s, V, I, S) \in \tilde{\mathbb{X}}$

It suffices to show

$$S_{jk} = V_j I^{\mathsf{H}}_{jk}, \qquad I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k,$$

For the 1st equation, we have from $\left|S_{jk}\right|^2 = v_j \ell_{jk}$ and construction of (V, I):

$$|S_{jk}| = |V_j I^{\mathsf{H}}_{jk}|, \qquad \angle S_{jk} = \angle V_j - \angle I_{jk}$$

i.e., $S_{jk} = V_j I^{\mathsf{H}}_{jk}$

Equivalence proof

Proof $\tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$

$\mathbb{X}_{\text{meshed}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \quad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \quad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\exists \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$$

$\longleftrightarrow$

$\tilde{\mathbb{X}}$

$$s_j = \sum_{k:j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y^s_{kj} V_j$$

$$S_{jk} = V_j I^{\mathsf{H}}_{jk}, \quad S_{kj} = V_k I^{\mathsf{H}}_{kj}$$

Conversely, fix $x := (s, v, \ell, S) \in \mathbb{X}_{\text{meshed}}$. Construct (V, I) from x:

$$V_j := \sqrt{v_j} e^{i\theta_j}, \qquad I_{jk} := \sqrt{\ell_{jk}} e^{i\left(\theta_j - \angle S_{jk}\right)}$$

Will show $\tilde{x} := (s, V, I, S) \in \tilde{\mathbb{X}}$

It suffices to show

$$S_{jk} = V_j I^{\mathsf{H}}_{jk}, \qquad I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k,$$

Note that the 2nd equation is equivalent to (recall $\tilde{y}^s_{jk} := y^s_{jk} + y^m_{jk}$):

$$z^s_{jk}\left(S_{jk}/V_j\right)^{\mathsf{H}} = \alpha_{jk} V_j - V_k \iff V_j V^{\mathsf{H}}_k = \alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}$$

We now show that $V_j V^{\mathsf{H}}_k$ and $\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}$ have equal magnitudes and angles.

Equivalence proof

Proof $\tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$

$\mathbb{X}_{\text{meshed}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \quad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \quad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\exists \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$$

$\longleftrightarrow$

$\tilde{\mathbb{X}}$

$$s_j = \sum_{k:j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y^s_{kj} V_j$$

$$S_{jk} = V_j I^{\mathsf{H}}_{jk}, \quad S_{kj} = V_k I^{\mathsf{H}}_{kj}$$

Conversely, fix $x := (s, v, \ell, S) \in \mathbb{X}_{\text{meshed}}$. Construct (V, I) from x:

$$V_j := \sqrt{v_j} e^{i\theta_j}, \qquad I_{jk} := \sqrt{\ell_{jk}} e^{i\left(\theta_j - \angle S_{jk}\right)}$$

Will show $\tilde{x} := (s, V, I, S) \in \tilde{\mathbb{X}}$

It suffices to show

$$S_{jk} = V_j I^{\mathsf{H}}_{jk}, \qquad I_{jk} = \tilde{y}_{jk} V_j - y^s_{jk} V_k,$$

By definition, $\beta_{jk}(x) := \angle\left(\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) = \theta_j - \theta_k = \angle\left(V_j V^{\mathsf{H}}_k\right)$

Equivalence proof

Proof $\tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$

$\mathbb{X}_{\text{meshed}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\text{Re}\left(\alpha_{jk}\left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}\right) - \left|z_{jk}^s\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\text{Re}\left(\alpha_{kj}\left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right) - \left|z_{kj}^s\right|^2 \ell_{kj}$$

$$\exists \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$$

$\longleftrightarrow$

$\tilde{\mathbb{X}}$

$$s_j = \sum_{k:j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y_{jk}^s V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y_{kj}^s V_j$$

$$S_{jk} = V_j I_{jk}^{\mathsf{H}}, \quad S_{kj} = V_k I_{kj}^{\mathsf{H}}$$

Conversely, fix $x := (s, v, \ell, S) \in \mathbb{X}_{\text{meshed}}$. Construct (V, I) from x:

$$V_j := \sqrt{v_j}\, e^{i\theta_j}, \qquad I_{jk} := \sqrt{\ell_{jk}}\, e^{i\left(\theta_j - \angle S_{jk}\right)}$$

Will show $\tilde{x} := (s, V, I, S) \in \tilde{\mathbb{X}}$

It suffices to show

$$S_{jk} = V_j I_{jk}^{\mathsf{H}}, \qquad I_{jk} = \tilde{y}_{jk} V_j - y_{jk}^s V_k,$$

For magnitude:

$$\left|\alpha_{jk}^{\mathsf{H}} v_j - z_{jk}^{s\mathsf{H}} S_{jk}\right|^2 = |\alpha_{jk}|^2 v_j^2 - 2 v_j \text{Re}\left(\alpha_{jk} z_{jk}^{s\mathsf{H}} S_{jk}\right) + |z_{jk}^s|^2 |S_{jk}|^2$$

$$= v_j \left(|\alpha_{jk}|^2 v_j - 2\text{Re}\left(\alpha_{jk} z_{jk}^{s\mathsf{H}} S_{jk}\right) + |z_{jk}^s|^2 \ell_{jk} \right) = v_j v_k$$

Equivalence proof

Proof $\tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$

$\mathbb{X}_{\text{meshed}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\text{Re}\left(\alpha_{jk}\left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}\right) - \left|z_{jk}^s\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\text{Re}\left(\alpha_{kj}\left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right) - \left|z_{kj}^s\right|^2 \ell_{kj}$$

$$\exists \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$$

$\longleftrightarrow$

$\tilde{\mathbb{X}}$

$$s_j = \sum_{k:j\sim k} S_{jk},$$

$$I_{jk} = \tilde{y}_{jk} V_j - y_{jk}^s V_k, \quad I_{kj} = \tilde{y}_{kj} V_k - y_{kj}^s V_j$$

$$S_{jk} = V_j I_{jk}^{\mathsf{H}}, \quad S_{kj} = V_k I_{kj}^{\mathsf{H}}$$

Conversely, fix $x := (s, v, \ell, S) \in \mathbb{X}_{\text{meshed}}$. Construct (V, I) from x:

$$V_j := \sqrt{v_j} e^{i\theta_j}, \qquad I_{jk} := \sqrt{\ell_{jk}} e^{i\left(\theta_j - \angle S_{jk}\right)}$$

Will show $\tilde{x} := (s, V, I, S) \in \tilde{\mathbb{X}}$

It suffices to show

$$S_{jk} = V_j I_{jk}^{\mathsf{H}}, \qquad I_{jk} = \tilde{y}_{jk} V_j - y_{jk}^s V_k,$$

This completes the proof of $\tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$

Equivalence proof

Proof $\mathbb{X}_{\text{meshed}} \equiv \mathbb{X}_{\text{tree}}$

Suppose G is a tree.

Will show $x := (s, v, \ell, S) \in \mathbb{X}_{\text{meshed}} \iff x \in \mathbb{X}_{\text{tree}}$

It suffices to show nonlinear cycle condition becomes linear:

$$\beta_{jk}(x) \;:=\; \theta_j - \theta_k \;=\; -\beta_{kj}(x)$$

$$\iff \alpha_{jk}^{\mathsf{H}} v_j - \left(z_{jk}^s\right)^{\mathsf{H}} S_{jk} \;=\; \left(\alpha_{kj}^{\mathsf{H}} v_k - \left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

$\mathbb{X}_{\text{tree}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\text{Re}\left(\alpha_{jk}\left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}\right) - \left|z_{jk}^s\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\text{Re}\left(\alpha_{kj}\left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right) - \left|z_{kj}^s\right|^2 \ell_{kj}$$

$$\alpha_{jk}^{\mathsf{H}} v_j - \left(z_{jk}^s\right)^{\mathsf{H}} S_{jk} = \left(\alpha_{kj}^{\mathsf{H}} v_k - \left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

↑ radial network

$\mathbb{X}_{\text{meshed}}$

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\text{Re}\left(\alpha_{jk}\left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}\right) - \left|z_{jk}^s\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\text{Re}\left(\alpha_{kj}\left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right) - \left|z_{kj}^s\right|^2 \ell_{kj}$$

$\exists \theta \in \mathbb{R}^{N+1}$ s.t. $\beta_{jk}(x) = \theta_j - \theta_k, \quad \beta_{kj}(x) = \theta_k - \theta_j$

Necessity: suppose x satisfies LHS. Then angles of RHS satisfy:

$$\angle\left(\alpha_{jk}^{\mathsf{H}} v_j - z_{jk}^{s\mathsf{H}} S_{jk}\right) \;=\; \beta_{jk}(x) \;=\; -\beta_{kj}(x) \;=\; -\angle\left(\alpha_{kj}^{\mathsf{H}} v_k - z_{kj}^{s\mathsf{H}} S_{kj}\right)$$

We now show that they have equal magnitudes as well:

$$\left|\alpha_{jk}^{\mathsf{H}} v_j - \left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}\right|^2 \;=\; |\alpha_{jk}|^2 v_j^2 + |z_{jk}^s|^2 |S_{jk}|^2 - 2 v_j \text{Re}\left(\alpha_{jk} z_{jk}^{s\mathsf{H}} S_{jk}\right) \;=\; v_j v_k \;=\; \left|\alpha_{kj}^{\mathsf{H}} v_k - \left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right|^2$$

Equivalence proof

Proof $\mathbb{X}_{\text{meshed}} \equiv \mathbb{X}_{\text{tree}}$

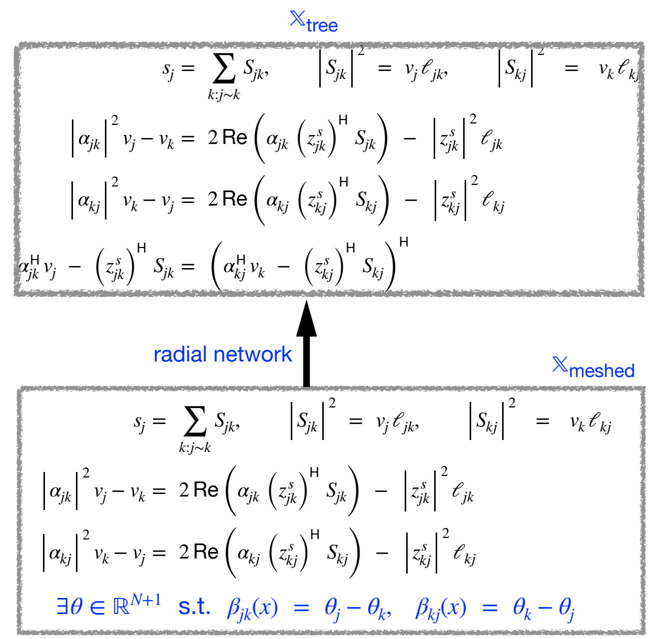
Suppose G is a tree.

Will show $x := (s, v, \ell, S) \in \mathbb{X}_{\text{meshed}} \iff x \in \mathbb{X}_{\text{tree}}$

It suffices to show nonlinear cycle condition becomes linear:

$$\beta_{jk}(x) \;:=\; \angle\left(\alpha_{jk}^{\mathsf{H}} v_j - z_{jk}^{s\mathsf{H}} S_{jk}\right) \;=\; \theta_j - \theta_k$$

$$\iff \alpha_{jk}^{\mathsf{H}} v_j - \left(z_{jk}^{s}\right)^{\mathsf{H}} S_{jk} \;=\; \left(\alpha_{kj}^{\mathsf{H}} v_k - \left(z_{kj}^{s}\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$



Sufficiency: suppose x satisfies RHS.

Recall the angle recovery procedure where, since G is a tree, there is a unique voltage angle (up to a reference angle) $\theta := C(C^{\mathsf{T}}C)^{-1}\beta(x) + \phi\mathbf{1}$ s.t. $\beta(x) = C^{\mathsf{T}}\theta$

i.e., x satisfies the LHS.

This completes the equivalence proof.

Outline

1. Radial network
2. Equivalence
3. Backward forward sweep
    - General BFS
    - Example algorithms
4. Linear power flow model

Backward forward sweep

General formulation

Efficient solution method for power flow equations

- Special Gauss-Seidel method that is applicable only to radial networks

Partition variables into two groups x and y

- Typically, x are branch variables (e.g. line currents) and y are nodal variables (bus voltages)

Design power flow equations as fixed points: $x = f(x, y), \quad y = g(x, y)$

- Choose (f, g) to have a spatially recursive structure enabled by tree topology

Consists of an outer loop where each outer iteration is implemented by two inner loops

- Outer loop: temporal update over t of $(x(t), y(t))$ to converge to a fixed point
- Backward sweep at t : spatial Gauss-Seidel update over nodes j of $x_j(t)$, with $y(t-1)$ held fixed
- Forward sweep at t : spatial Gauss-Seidel update over nodes j of $y_j(t)$, with newly computed $x(t)$ held fixed

Different BFS algorithms differ in choice of variables (x, y) and design of (f, g)

- (f, g) that is spatially recursive automatically translates into a BFS algorithm

Backward forward sweep

Spatially recursive Gauss-Seidel

At each outer iteration t, spatial Gauss-Seidel update over j normally takes the form

$$x_j(t) := f_j(x_1(t), \ldots, x_{j-1}(t), x_j(t-1), \ldots, x_{n_1}(t-1);\ y(t-1))$$

$$y_j(t) := g_j(x(t);\ y_1(t), \ldots, y_{j-1}(t), y_j(t-1), \ldots, y_{n_2}(t-1))$$

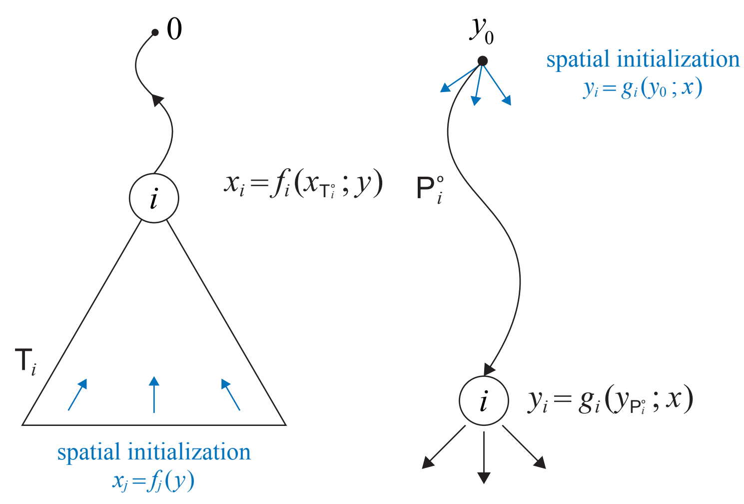
Functions (f, g) are spatially recursive if

- Given y, f_j depends on x only through $x_{\mathsf{T}_j^\circ}$ (T_j° : subtree rooted at j)
- Given x, g_j depends on y only through $y_{\mathsf{P}_j^\circ}$ (P_j° : path from 0 to j)

Gauss-Seidel update at t with spatially recursive (f, g)

$$\text{BS:} \quad x_j(t) \ := \ f_j\left(x_{\mathsf{T}_j^\circ}(t);\ y(t-1)\right)$$

$$\text{FS:} \quad y_j(t) \ := \ g_j\left(x(t);\ y_{\mathsf{P}_j^\circ}(t)\right)$$



(a) Backward sweep (b) Forward sweep

Example: complex form BFM

Assumptions: radial network and $z_{jk}^s = z_{kj}^s$

- Can use directed graph (with down orientation) and involve line variables only in direction of the lines
- Can uniquely identify a line variable by its from-node or to-node

Then complex form BFM becomes

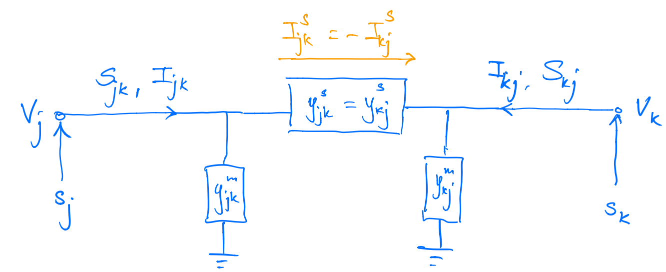
$$s_j = \sum_{k:j\sim k} V_j I_{jk}^{\mathsf{H}}, \qquad I_{jk} = \tilde{y}_{jk} V_j - y_{jk}^s V_k$$

Given: $(V_0, s_j, j \in N)$, **find** $(s_0, V_j, j \in N,\ I_{jk}, S_{jk}, j \to k \in E)$

Example: complex form BFM

Complex form BFM

$$s_j = \sum_{k:j\sim k} V_j I_{jk}^{\mathsf{H}}, \qquad I_{jk} = \tilde{y}_{jk} V_j - y_{jk}^s V_k$$



Given: $(V_0, s_j, j \in N)$, **find** $(s_0, V_j, j \in N,\ I_{jk}, S_{jk}, j \to k \in E)$

Design partitioning (x, y) and corresponding spatially recursive power flow equations (f, g)

- x : line currents I_{jk}^s across impedance z_{jk}^s
- y : nodal voltages V_j
- Given a solution (V_j, I_{jk}^s), all other quantities (e.g. I_{jk}, S_{jk}) can be computed
- Can also design BFS that computes sending-end line currents I_{jk} instead of I_{jk}^s (Exercise)

Example: complex form BFM

Spatially recursive (f, g)

Since $I^s_{jk} := I_{jk} - y^m_{jk} V_j$, KCL at each bus j

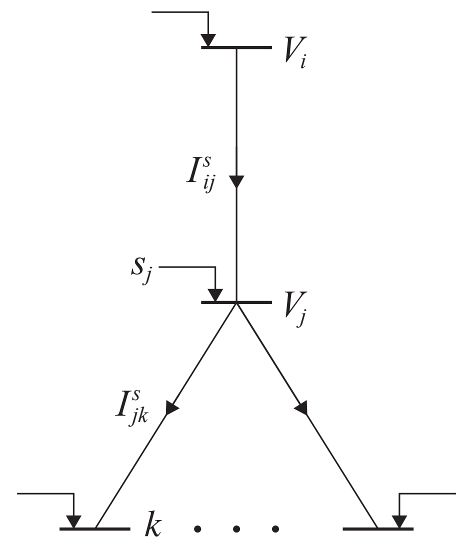
$$\left(\frac{s_j}{V_j}\right)^{\mathsf{H}} + \left(I^s_{ij} - y^m_{ji} V_j\right) = \sum_{k: j\to k} \left(I^s_{jk} + y^m_{jk} V_j\right)$$

Spatially recursive power flow equations (f, g):

$$I^s_{ij} = \sum_{k: j\to k} I^s_{jk} - \left(\left(\frac{s_j}{V_j}\right)^{\mathsf{H}} - y^m_{jj} V_j \right) =: f_j\left(x_{\mathsf{T}^\circ_j}; y\right)$$

$$V_j = V_i - z^s_{ij} I^s_{ij} =: g_j\left(x; y_{\mathsf{P}^\circ_j}\right)$$

where $i := i(j)$ is unique parent of j and $y^m_{jj} := y^m_{ji} + \sum_k y^m_{jk}$



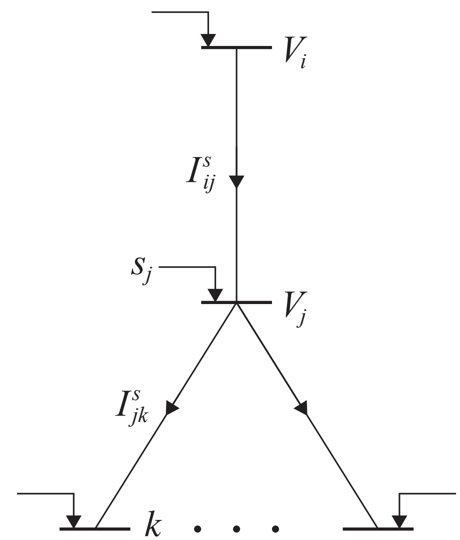
Example: complex form BFM

BFS

Spatially recursive power flow equations (f, g):

$$I_{ij}^s \;=\; \sum_{k:j\to k} I_{jk}^s \;-\; \left(\left(\frac{s_j}{V_j} \right)^{\mathsf{H}} \;-\; y_{jj}^m V_j \right) \;=:\; f_j\left(x_{\mathsf{T}_j^\circ}; y \right)$$

$$V_j \;=\; V_i \;-\; z_{ij}^s I_{ij}^s \;=:\; g_j\left(x; y_{\mathsf{P}_j^\circ} \right)$$



This translates automatically to a BFS algorithm with inner loops:

BS: $x_j(t) \;:=\; f_j\left(x_{\mathsf{T}_j^\circ}(t);\; y(t-1) \right)$

FS: $y_j(t) \;:=\; g_j\left(x(t);\; y_{\mathsf{P}_j^\circ}(t) \right)$

D. Shirmohammadi, H. W. Hong, A. Semlyen, and G. X. Luo. A compensation-based power flow method for weakly meshed distribution and transmission networks. *IEEE Transactions on Power Systems*, 3(2):753–762, May 1988.

Example: complex form BFM

Outer loop

while *stopping criterion not met* **do**

(a) $t \leftarrow t+1$;

(b) *Backward sweep*: **for** *j starting from leaf nodes and iterating towards bus 0* **do**

$$x_j(t) \quad \leftarrow \quad f_j\left(x_{\mathsf{T}_j^\circ}(t);\, y(t-1)\right), \qquad j \in \overline{N}$$

(c) *Forward sweep*: **for** *j starting from children of bus 0 and iterating towards leaf nodes* **do**

$$y_j(t) \quad \leftarrow \quad g_j\left(x(t);\, y_{\mathsf{P}_j^\circ}(t)\right), \qquad j \in N$$

Example: complex form BFM

(b) *Backward sweep*: **for** j *starting from leaf nodes and iterating towards bus 0* **do**

$$I_{ij}^{s}(t) \;\leftarrow\; \sum_{k:j\to k} I_{jk}^{s}(t) \;-\; \left(\left(\frac{s_j}{V_j(t-1)}\right)^{\mathsf{H}} - y_{jj}^{m} V_j(t-1)\right), \quad i\to j\in E$$

Given all voltages $V(t-1)$

Given all currents $I_{jk}^{s}(t)$ in previous layer (in T_j°)

Update all currents $I_{ij}^{s}(t)$ in present layer (reverse breadth-first search)

Example: complex form BFM

(c) *Forward sweep*: **for** *j starting from children of bus 0 and iterating towards leaf nodes* **do**

$$V_j(t) \;=\; V_i(t) \,-\, z_{ij}^s I_{ij}^s(t), \qquad j \in N$$

Given all currents $I^s(t-1)$

Given voltage $V_i(t)$ at parent of j (in P_j°)

Update $V_j(t)$ (breadth-first or depth-first search)

Example: DistFlow model

Assumptions: radial network, $z^s_{jk} = z^s_{kj}$ and $y^m_{jk} = y^m_{kj} = 0$

- Can use directed graph (with up orientation) and involve line variables only in direction of the lines
- Can uniquely identify a line variable by its from-node or to-node

Given: $(V_0, s_j, j \in N)$, **find** $(s_0, v_j, j \in N, \ \ell_{jk}, S_{jk}, j \to k \in E)$

Design partitioning (x, y) and corresponding spatially recursive power flow equations (f, g)

- Line flows: $x := \left(S_{ji(j)}, \ell_{ji(j)}, j \in N\right) = \left(S_{jk}, \ell_{jk}, j \to k \in E\right)$
- Nodal voltages: $y := (v_j, j \in N)$
- Given a solution (x, y), s_0 can be computed

Example: DistFlow model

Spatially recursive (f, g)

Backward sweep function $f_j\left(x_{\mathsf{T}_j^\circ}; y\right)$:

$$S_{ji} = s_j + \sum_{k:k\to j}\left(S_{kj} - z_{kj}^s \ell_{kj}\right), \qquad \ell_{ji} = \frac{|S_{ji}|^2}{v_j}$$

Forward sweep function $g\left(x, y_{\mathsf{P}_j^\circ}\right)$:

$$v_j = v_i + 2\,\mathrm{Re}\left(z_{ji}^{s\mathsf{H}} S_{ji}\right) - |z_{ji}^s|^2 \ell_{ji}$$

This translates automatically to a BFS algorithm with inner loops:

$$\text{BS:} \quad x_j(t) := f_j\left(x_{\mathsf{T}_j^\circ}(t);\ y(t-1)\right)$$

$$\text{FS:} \quad y_j(t) := g_j\left(x(t);\ y_{\mathsf{P}_j^\circ}(t)\right)$$

Outline

1. Radial network
2. Equivalence
3. Backward forward sweep
4. Linear power flow model
   - With shunt admittances
   - Without shunt admittances
   - Linear solution and properties

Linear models
Advantages

Linear approximations of BFM have two advantages

1. Given nodal injections s, voltages v^{lin} and line flows S^{lin} can be solved explicitly
2. The linear solution $\left(v^{\text{lin}}, S^{\text{lin}}\right)$ provides bounds on (v, S) from power flow solutions to nonlinear DistFlow models

Linear approximations are reasonable when line losses $z_{jk}\ell_{jk}$ are small compared with line flows S_{jk}

With shunt admittances

$$s_j = \sum_{k:j\sim k} S_{jk}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \cancel{\left|z^s_{jk}\right|^2} \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \cancel{\left|z^s_{kj}\right|^2} \ell_{kj}$$

$$\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk} = \left(\alpha^{\mathsf{H}}_{kj} v_k - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

$$\cancel{\left|S_{jk}\right|^2 = v_j \ell_{jk}, \quad \left|S_{kj}\right|^2 = v_k \ell_{kj}}$$

BFM-radial

$\ell := 0$ →

$$s_j = \sum_{k:j\sim k} S_{jk}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right)$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)$$

$$\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk} = \left(\alpha^{\mathsf{H}}_{kj} v_k - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

BFM-linear

With shunt admittances

$6N + 2$ linear equations in $7N + 3$ real variables (s, v, S)

Power flow problem: given $(v_0, s_j, j \in N)$, solve for remaining $5N + 2$ vars $(s_0, v_j, j \in N)$

More equations than unknowns, but they are typically linearly dependent

$$s_j = \sum_{k:j\sim k} S_{jk}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\text{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right)$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\text{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)$$

$$\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk} = \left(\alpha^{\mathsf{H}}_{kj} v_k - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

BFM-linear

Example

2-bus network

Buses j and k connected by a transformer characterized by $(K, \tilde{y}^s, \tilde{y}^m)$ (voltage gain K may be complex). Let $\tilde{\alpha} := (1 + \tilde{z}^s \tilde{y}^m)$.

Linear BFM: 4 linear equations in 6 vars (s, v)

$$v_j - v_k \Big/ \left|K\right|^2 = 2\,\mathrm{Re}\left(\left(\tilde{z}^s\right)^{\mathsf{H}} s_j\right)$$

$$\left|\tilde{\alpha}/K\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\tilde{\alpha}\left(\tilde{z}^s\right)^{\mathsf{H}} s_k\right)$$

$$v_j - \left(\tilde{z}^s\right)^{\mathsf{H}} s_j = \left(\tilde{\alpha}/|K|^2\right) v_k - \tilde{z}^s \bar{s}_k$$

Example

2-bus network

Power flow problem: given (p_k, q_k, v_j), find (p_j, q_j, v_k)

Assume: $\tilde{y}^m = 0$ s.t. $\tilde{\alpha} = 1$. Then

$$\underbrace{\begin{bmatrix} 2\tilde{r} & 2\tilde{x} & -1/|K|^2 \\ 0 & 0 & 1/|K|^2 \\ \tilde{r} & \tilde{x} & 1/|K|^2 \\ -\tilde{x} & \tilde{r} & 1/|K|^2 \end{bmatrix}}_{A} \begin{bmatrix} p_j \\ q_j \\ v_k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 2\tilde{r} & 2\tilde{x} & 1 \\ \tilde{r} & \tilde{x} & 1 \\ \tilde{x} & -\tilde{r} & 0 \end{bmatrix} \begin{bmatrix} p_k \\ q_k \\ v_j \end{bmatrix}$$

Elementary row operation reduces A to a rank-3 matrix: $\begin{bmatrix} (\tilde{r}/\tilde{x})(\tilde{r}^2 + \tilde{x}^2) & 0 & 0 \\ 0 & \tilde{r}^2 + \tilde{x}^2 & 0 \\ 0 & 0 & 1/|K|^2 \\ 0 & 0 & 0 \end{bmatrix}$

Without shunt admittances

BFM-radial

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk} \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj} \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk} = \left(\alpha^{\mathsf{H}}_{kj} v_k - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$$

→ $y^s_{jk} = y^s_{kj}$, $y^m_{jk} = y^m_{kj} = 0$ →

DistFlow

$$\sum_{k:j\to k} S_{jk} = \sum_{i:i\to j} \left(S_{ij} - z^s_{ij}\ell_{ij}\right) + s_j$$

$$v_j - v_k = 2\,\mathrm{Re}\left(z^{s\mathsf{H}}_{jk} S_{jk}\right) - |z^s_{jk}|^2 \ell_{jk}$$

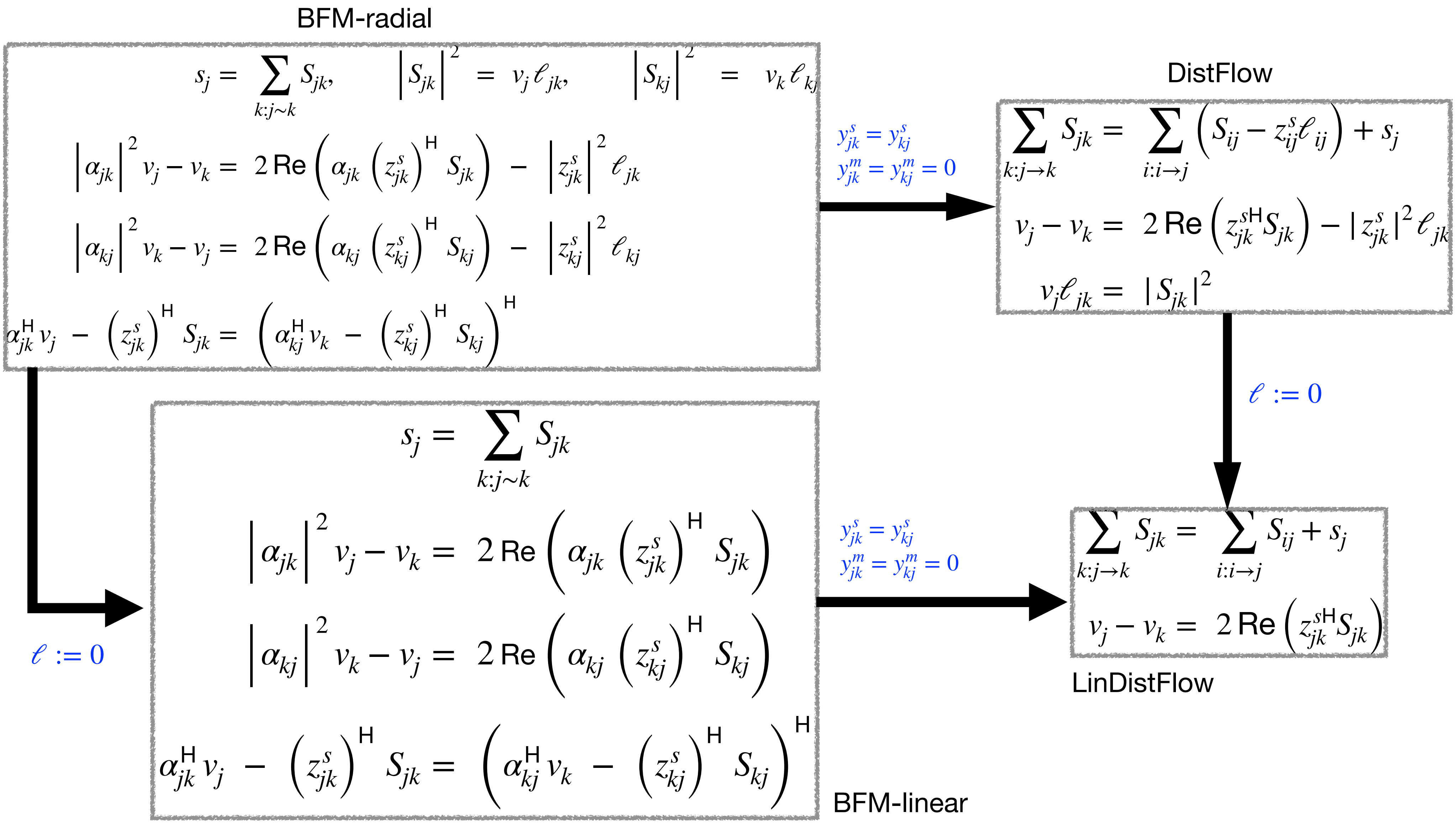
$$v_j \ell_{jk} = |S_{jk}|^2$$

↓ $\ell := 0$ ↓

$$\sum_{k:j\to k} S_{jk} = \sum_{i:i\to j} S_{ij} + s_j$$

$$v_j - v_k = 2\,\mathrm{Re}\left(z^{s\mathsf{H}}_{jk} S_{jk}\right)$$

LinDistFlow



BFM-radial
$s_j = \sum_{k:j\sim k} S_{jk}, \quad |S_{jk}|^2 = v_j \ell_{jk}, \quad |S_{kj}|^2 = v_k \ell_{kj}$
$|\alpha_{jk}|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk} \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - |z^s_{jk}|^2 \ell_{jk}$
$|\alpha_{kj}|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj} \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - |z^s_{kj}|^2 \ell_{kj}$
$\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk} = \left(\alpha^{\mathsf{H}}_{kj} v_k - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$
$y^s_{jk} = y^s_{kj}$
$y^m_{jk} = y^m_{kj} = 0$
DistFlow
$\sum_{k:j\to k} S_{jk} = \sum_{i:i\to j} \left(S_{ij} - z^s_{ij} \ell_{ij}\right) + s_j$
$v_j - v_k = 2\,\mathrm{Re}\left(z^{s\mathsf{H}}_{jk} S_{jk}\right) - |z^s_{jk}|^2 \ell_{jk}$
$v_j \ell_{jk} = |S_{jk}|^2$
$\ell := 0$
$\ell := 0$
$s_j = \sum_{k:j\sim k} S_{jk}$
$|\alpha_{jk}|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk} \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right)$
$|\alpha_{kj}|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj} \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)$
$\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk} = \left(\alpha^{\mathsf{H}}_{kj} v_k - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$
BFM-linear
$y^s_{jk} = y^s_{kj}$
$y^m_{jk} = y^m_{kj} = 0$
$\sum_{k:j\to k} S_{jk} = \sum_{i:i\to j} S_{ij} + s_j$
$v_j - v_k = 2\,\mathrm{Re}\left(z^{s\mathsf{H}}_{jk} S_{jk}\right)$
LinDistFlow

Without shunt admittances

LinDistFlow in vector form

Let

- C : bus-by-line $(N+1) \times N$ incidence matrix
- $D_r := \text{diag}\left(r_l, l \in E\right) \succ 0$: diagonal matrix of line resistances
- $D_x := \text{diag}\left(x_l, l \in E\right) \succ 0$: diagonal matrix of line reactances

Then LinDistFlow is:

$$s = CS, \qquad C^\top v = 2\left(D_r P + D_x Q\right)$$

Important features because of tree topology

- C is of rank 1 with null(C) = span(**1**)
- Reduced $N \times N$ incidence matrix $\hat{C}$ is nonsingular
- $\hat{C}^{-1}$ has a simple structure

These features allow explicit linear solutions and structural properties

Linear solution

Let $C \;=:\; \begin{bmatrix} c_0^{\mathsf{T}} \\ \hat{C} \end{bmatrix}$

Then LinDistFlow is:

$$\hat{s} = \hat{C}S, \quad s_0 = c_0^{\mathsf{T}} S$$
$$v_0 c_0 + C^{\mathsf{T}} \hat{v} = 2\left(D_r P + D_x Q\right)$$

Given $(v_0, s_j, j \in N)$, the remaining variables $\left(s_0, v_j, j \in N, S_l, l \in E\right)$ can be obtained explicitly

Linear solution

Theorem [linear solution]

1. Linear solution is:

$$S = \hat{C}^{-1}\hat{s}, \qquad s_0 = c_0^{\mathsf{T}}\hat{C}^{-1}\hat{s}$$

$$\hat{v} = v_0\mathbf{1} + 2\left(R\hat{p} + X\hat{q}\right)$$

voltages = v_0 + correction term $\left(\hat{p}, \hat{q}\right)$

where $R := \hat{C}^{-\mathsf{T}}D_r\hat{C}^{-1}$ and $X := \hat{C}^{-\mathsf{T}}D_x\hat{C}^{-1}$

2. $R \succ 0$ and $X \succ 0$ are positive matrices with

$$R_{jk} = \sum_{l \in P_j \cap P_k} r_l, \qquad X_{jk} = \sum_{l \in P_j \cap P_k} x_l$$

Since entries of (R, X) are nonnegative, positive injections (p, q) always increase v

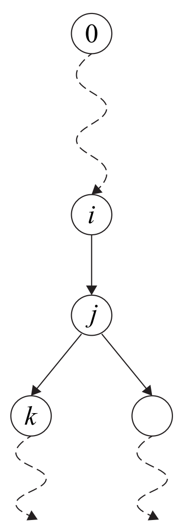
Analytical properties

Special graph orientations

Down orientation: pointing away from bus 0

$$\sum_{k:j\to k} S_{jk}^{\text{lin}} = S_{ij}^{\text{lin}} + s_j$$

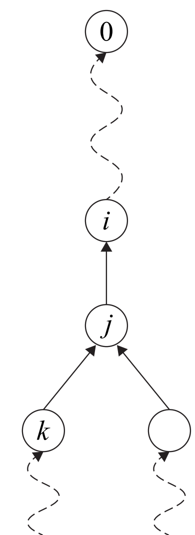
$$v_j^{\text{lin}} - v_k^{\text{lin}} = 2\,\text{Re}\left(z_{jk}^{\text{H}} S_{jk}^{\text{lin}}\right)$$



Up orientation: pointing towards bus 0

$$\overline{S}_{ji}^{\text{lin}} = \sum_{k:k\to j} \overline{S}_{kj}^{\text{lin}} + s_j$$

$$\overline{v}_k^{\text{lin}} - \overline{v}_j^{\text{lin}} = 2\,\text{Re}\left(z_{kj}^{\text{H}} \overline{S}_{kj}^{\text{lin}}\right)$$



Analytical properties

Corollary

1. For lines $(i,j) \in E$, $S_{ij}^{\text{lin}} = \overline{S}_{ji}^{\text{lin}}$. Moreover

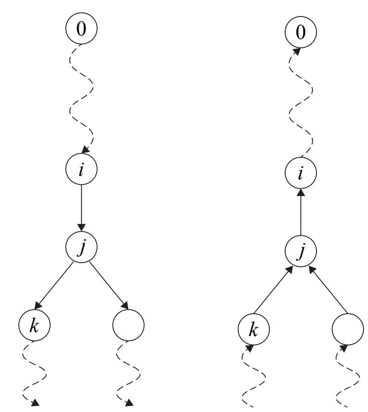
$$S_{ij}^{\text{lin}} = -\sum_{k \in \text{T}_\text{j}} s_k, \qquad i \to j$$

line flow S_{ij}^{lin} to j supplies all loads $-s_k$ in subtree T_j

$$\overline{S}_{ji}^{\text{lin}} = \sum_{k \in \text{T}_\text{j}} s_k, \qquad j \to i$$

line flow $\overline{S}_{ij}^{\text{lin}}$ from j come from all injections s_k in subtree T_j

2. For buses $j \in \overline{N}$, $v_j^{\text{lin}} = \overline{v}_j^{\text{lin}} = v_0 + 2\sum_k \left(R_{jk}p_k \ + \ X_{jk}q_k\right)$



Analytical properties

Nonlinear DistFlow solution

Linear DistFlow model ignores line losses $\implies$ simple relation between line flows $(S_{ij}, \overline{S}_{ji})$ and injections s_k

Given s, nonlinear DistFlow solutions (v, ℓ, S) satisfy the recursion in up orientation

$$S_{ij} = -\sum_{k \in \mathsf{T}_j} s_k + \left(z_{ij}\ell_{ij} + \sum_{l \in \mathsf{T}_j} z_l \ell_l \right), \qquad v_j = v_0 - \sum_{l \in \mathsf{P}_j} \left(2\,\mathrm{Re}\left(z_l^{\mathsf{H}} S_l \right) - |z_l|^2 \ell_l \right)$$

and solutions $(\overline{v}, \overline{\ell}, \overline{S})$ satisfy the recursion in down orientation

$$\overline{S}_{ji} = \sum_{k \in \mathsf{T}_j} s_k - \sum_{l \in \mathsf{T}_j} z_l \overline{\ell}_l, \qquad \overline{v}_j = v_0 + \sum_{l \in \mathsf{P}_j} \left(2\,\mathrm{Re}\left(z_l^{\mathsf{H}} \overline{S}_l \right) - |z_l|^2 \overline{\ell}_l \right)$$

line losses

Bounds on nonlinear solutions

Corollary [bounds on nonlinear solutions]

1. For $i \to j \in E$, $S_{ij} \geq S_{ij}^{\text{lin}}$
2. For $j \to i \in E$, $\overline{S}_{ji} \geq \overline{S}_{ji}^{\text{lin}}$
3. For $j \in \overline{N}$, $v_j = \overline{v}_j \leq \overline{v}_j^{\text{lin}} = v_j^{\text{lin}}$

LinDistFlow ignores losses and underestimates required power to supply loads

proving $\overline{v}_j \leq \overline{v}_j^{\text{lin}}$ is easy;
proving directly $v_j \leq v_j^{\text{lin}}$ is not

Summary

1. Radial network
   - BFM with and without shunt admittances
   - Nonlinear (quadratic) power flow equations
2. Equivalence
   - BFM variants are all equivalent, and equivalent to BIM
3. Backward forward sweep
   - Gauss-Seidel method that exploits spatially recursive structure enabled by tree topology
4. Linear power flow model
   - Linear BFM with and without shunt admittances
   - Explicit linear solution and bounds on nonlinear solutions
5. Application: volt/var control
   - Local and memoryless control can stabilize voltages and implicitly minimizes cost determined by control design