Power System Analysis

Chapter 6 Example applications

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Outline

1. Economic dispatch
2. Voltage control
3. Radial network identification

Outline

1. Economic dispatch
    - Economic dispatch
    - LMP and properties
2. Voltage control
3. Radial network identification

Electricity market

Goal: to illustrate the use of DC power flow model that is widely used for market applications

Setup

1. A connected network $G := (\overline{N}, E)$ with $N + 1$ buses and M lines modeled by DC power flow
2. p : real power injections p_j at buses j
   - generator: $p_j > 0$, incurs cost $f_j(p_j)$
   - load: $p_j < 0$, enjoys utility $-f_j(p_j)$
   - capacity limits: $p^{\min} \leq p \leq p^{\max}$
   - supply = demand: $\mathbf{1}^\mathsf{T} p = 0$
3. $P := BC^\mathsf{T} L^\dagger p =: S^\mathsf{T} p$: line power flows
   - $B := \text{diag}\left(b_l, l \in E\right) \succ 0$, C : incidence matrix, $L^\dagger$: pseudo-inverse of Laplacian $L := CBC^\mathsf{T}$
   - line limits: $P^{\min} \leq P = BC^\mathsf{T} L^\dagger p \leq P^{\max}$
   - $S = \left(\partial P / \partial p\right)^\mathsf{T}$ shift factor that maps line vars (e.g. line congestion prices) to nodal var (e.g. nodal congestion prices)

Economic dispatch

LMP

$$\min_{p^{\min} \le p \le p^{\max}} \quad \sum_{j \in \overline{N}} f_j(p_j)$$

$$\text{subject to} \quad \mathbf{1}^\top p = 0 \qquad [\gamma]$$

$$P^{\min} \le S^\top p \le P^{\max} \qquad [\kappa^-, \kappa^+]$$

- p : primal variable
- Associated with each constraint is a Lagrange multiplier: $\gamma \in \mathbb{R},\ \kappa^- \in \mathbb{R}^M_+,\ \kappa^+ \in \mathbb{R}^M_+$
- Given an optimal dispatch p^* and optimal Lagrange multiplier $\left(\gamma^*, \kappa^{-*}, \kappa^{+*}\right)$, define locational marginal price (LMP):

$$\lambda^* := \gamma^* \mathbf{1} - L^\dagger C B \kappa^* = \gamma^* \mathbf{1} + S \kappa^*$$

where $\kappa^* := \kappa^{-*} - \kappa^{+*}$

Economic dispatch

Settlement rule

Locational marginal price (LMP):

$$\lambda^* := \gamma^* \mathbf{1} + S\kappa^*$$

Settlement rule

- System operator (SO) solves economic dispatch to obtain optimal dispatch p^* and $\left(\gamma^*, \kappa^{-*}, \kappa^{+*}\right)$, and compute LMP λ^*
- Generator that generates $p_j > 0$: is paid $\lambda_j^* p_j$
- Load that consumes $-p_j > 0$: pays $-\lambda_j^* p_j$
- Some markets allow participants to choose their own p_j, some markets dispatch binding p_j^* (e.g. many US markets)

Optimality condition

Assume: cost functions f_j are convex and optimal value of ED is finite

- Optimal Lagrange multiplier $\left(\gamma^*, \kappa^{-*}, \kappa^{+*}\right)$ and hence LMP λ^* exist; moreover strong duality holds
- p^* is an optimal dispatch if and only if p^* and $\left(\gamma^*, \kappa^{-*}, \kappa^{+*}\right)$ satisfy the KKT condition:
    - Primal feasibility: $p^{\min} \leq p^* \leq p^{\max}, \quad \mathbf{1}^\top p^* = 0, \quad P^{\min} \leq BC^\top L^\dagger p^* \leq P^{\max}$
    - Dual feasibility: $\kappa^{-*} \geq 0, \quad \kappa^{+*} \geq 0$
    - Stationarity: $$f_j'(p_j^*) \begin{cases} = \lambda_j^* \text{ if } & p_j^{\min} < p_j^* < p_j^{\max} \\ > \lambda_j^* \text{ only if } & p_j^* = p_j^{\min} \\ < \lambda_j^* \text{ only if } & p_j^* = p_j^{\max} \end{cases}$$ marginal unit
    - Complementary slackness:

$$\left(\kappa^{-*}\right)^\top \left(P^{\min} - BC^\top L^\dagger p^*\right) = 0, \quad \left(\kappa^{+*}\right)^\top \left(BC^\top L^\dagger p^* - P^{\max}\right) = 0$$

LMP properties

We study properties of optimal dispatch p^* and LMP λ^*

- Competitive equilibrium
- Nodal and line congestion price κ^*
- Revenue adequacy
- Price reference bus

These properties are consequences of DC power flow equation and KKT condition

Competitive equilibrium

An important justification for pricing according to LMP is that optimal dispatch and LMP (p^*, λ^*) is a competitive equilibrium:

- Market clearing: supply = demand, $\mathbf{1}^\mathsf{T} p^* = 0$
- Power flows satisfy line limits: $P^{\min} \leq S^\mathsf{T} p \leq P^{\max}$
- Welfare optimization: p^* solves economic dispatch
- Incentive compatibility: individually optimal p_j^* that solve

$$\max_{p_j^{\min} \leq p_j \leq p_j^{\max}} \quad \lambda_j^* p_j - f_j(p_j)$$

  turn out to be socially optimal

LMP consists of Lagrange multipliers associated with non-local constraints (only) that couple individual decisions p_j

- It prices externalities of unit j's decisions and aligns individual optimality with social optimality

Nodal and line congestion prices

LMP: $\lambda^* := \gamma^* \mathbf{1} + S\kappa^*$

Energy price γ^*

- Same prices $\lambda_j^* = \gamma^*$ at all buses j if no congestion ($P^{\min} < BC^\top L^\dagger p < P^{\max} \;\Rightarrow\; \kappa^* = 0$)
- In general, energy price $\gamma^* = \dfrac{1}{N+1}\mathbf{1}\lambda^*$, the average LMP (system λ)

Nodal and line congestion prices

LMP: $\lambda^* := \gamma^* \mathbf{1} + S\kappa^*$

Line congestion price κ^*

- Interpret $\kappa^* := \kappa^{-*} - \kappa^{+*}$ as line congestion prices, for two reasons
- κ_l^* : shadow price of line capacities $(P_l^{\min}, P_l^{\max})$ at l because (Envelop Theorem)

$$\frac{\partial f^*}{\partial P_l^{\min}}(P^{\min}, P^{\max}) = \kappa_l^{-*}$$

$$\frac{\partial f^*}{\partial P_l^{\max}}(P^{\min}, P^{\max}) = -\kappa_l^{+*}$$

i.e., each unit of additional capacities reduces optimal cost f^* by $(\kappa^{-*}, \kappa^{+*}) \geq 0$

- $-\kappa_l^* P_l \geq 0$: cost of carrying P_l on line l (due to complementary slackness)

Nodal and line congestion prices

LMP: $\lambda^* := \gamma^* \mathbf{1} + S\kappa^*$

Nodal congestion price $c^* := S\kappa^*$

- c_j^* : marginal cost of serving 1 additional load at node j
- Main observation : $S = \left(\dfrac{\partial P}{\partial p}\right)^{\mathsf{T}}$ because $P = S^{\mathsf{T}} p$
  - $\dfrac{\partial P_l}{\partial p_j}\Delta p_j = S_{jl}\Delta p_j$ = increase in power flow at line l due to additional injection Δp_j at node j
  - $\therefore\ -\kappa_l^* \left(S_{jl}\Delta p_j\right)$ = increase in congestion cost at line l due to additional injection Δp_j at node j
  - $\therefore -\sum_l S_{jl}\kappa_l^* \Delta p_j$ = increase in congestion cost over network due to additional injection Δp_j at node j
  - $\therefore\ c_j^* := \sum_l S_j \kappa_l^*$ = increase in congestion cost over network due to 1 additional unit of load at node j

Nodal and line congestion prices

Negative price

LMP: $\lambda_j^* := \gamma^* + c_j^*$

- Since the nodal congestion price c_j^* can be positive or negative, λ_j^* can be negative
- Negative λ_j^* are not uncommon in practice, e.g., in CAISO market during daytime when there is a lot of solar generation
- Negative λ_j^* can be due to congestion or nonzero generation limit $p_j^{\min} > 0$

Revenue adequacy

System operator collects payment $\lambda_j^*(-p_j^*)$ from load j and pays $\lambda_j^* p_j^*$ to generator j

The residue is merchandizing surplus

$$\text{MS} := -\sum_j \lambda_j^* p_j^* \;=\; -\left(\lambda^*\right)^{\mathsf{T}} p^*$$

- Substituting $\lambda^* := \gamma \mathbf{1} + S\kappa^*$ and complementary slackness yield

$$\text{MS} \;=\; \left(\kappa^{+*}\right)^{\mathsf{T}} P^{\max} \;+\; \left(\kappa^{-*}\right)^{\mathsf{T}} (-P^{\min}) \;\geq\; 0$$

  i.e., SO will not run cash negative. This is called revenue adequate

- MS > 0 if and only if there is congestion ($\kappa^{-*} > 0$ or $\kappa^{+*} > 0$)

Price reference bus
Summary

Price reference (slack) bus r : injections p_{-r} at non-price-reference buses can be arbitrarily chosen and always by p_r at bus r, so that $p_r = -\mathbf{1}^\mathsf{T} p_{-r}$

May be different from angle reference bus 0 where $\theta_0 := 0$

Can write everything in terms of injections p_{-r} and shift factor S_r at non-price-ref buses only

- DC power flow equations, economic dispatch (DC OPF), LMP λ^*

Optimal dispatch p^*, LMP λ^* , and line flows P do not depend on choice of r

- Lagrange multiplier γ^* does

Disadvantages of designating a price reference bus r

- Somewhat arbitrary (typically a bus where there is large generator that is rarely bottlenecked)
- Reduced Laplacian matrix $L_r := C_{-r} B C_{-0}^{sfT}$ is not principal submatrix of L, hence may not be symmetric nor nonsingular (unless $r = 0$)
- Reduced shift factor $S_r := L_r^{-1} C_{-0} B$ depends on r (when L_r is nonsingular)
- Seems unnecessary (can express DC power flow, economic dispatch, and LMP in terms of $L^\dagger$

Price reference bus

In terms of p_{-r} and S_r

Partition node-by-line incidence matrix

$$C =: \begin{bmatrix} c_0^\mathsf{T} \\ C_{-0} \end{bmatrix}, \qquad C =: \begin{bmatrix} C_{-r} \\ c_r^\mathsf{T} \end{bmatrix}$$

DC power flow equations become

$$\begin{bmatrix} p_{-r} \\ p_r \end{bmatrix} = \begin{bmatrix} C_{-r} \\ c_r^\mathsf{T} \end{bmatrix} P, \qquad P = B \begin{bmatrix} c_0 C_{-0}^\mathsf{T} \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_{-0} \end{bmatrix}$$

leading to

$$P = \left(B C_{-0}^\mathsf{T} L_r^{-1} \right) p_{-r} =: S_r^\mathsf{T} p_{-r}$$

Economic dispatch becomes:

$$\min_{p^{\min} \le p \le p^{\max}} \sum_{j \in \overline{N}} f_j(p_j) \qquad \text{s.t.} \quad \mathbf{1}^\mathsf{T} p = 0 \qquad [\gamma]$$

$$P^{\min} \le S_r^\mathsf{T} p_{-r} \le P^{\max} \qquad [\kappa^-, \kappa^+]$$

Price reference bus

In terms of p_{-r} and S_r

LMP: $\lambda^* \;=\; \gamma^* \;+\; \begin{bmatrix} S_r \kappa^* \\ 0 \end{bmatrix}$

where $S_r := L_r^{-\mathsf{T}} C_{-0} B$ and $\kappa^* := \kappa^{-*} - \kappa^{+*}$

Theorem

Suppose cost functions f_j are convex (and hence differentiable), so that KKT is N&S optimality condition.

Fix p^* and let

$$\tilde{\gamma}^* \;=\; \gamma^* - s_r^{\mathsf{T}} \kappa^*, \qquad \tilde{\kappa}^{-*} \;=\; \kappa^{-*}, \qquad \tilde{\kappa}^{+*} \;=\; \kappa^{+*}$$

1. $\tilde{\lambda} := \tilde{\gamma}^* \mathbf{1} + S\kappa^* = \lambda^*$
2. $(p^*, \tilde{\lambda}^*)$ is primal-dual optimal for original ED iff (p^*, λ^*) is primal-dual optimal for reduced ED
3. $P^* = S^{\mathsf{T}} p = S_r^{\mathsf{T}} p_{-r}$

Outline

1. Economic dispatch
2. Voltage control
3. Radial network identification
   - Linear DistFlow model
   - Decentralized control: convergence and optimality

volt/var control

Stabilize voltages on distribution grid by adapting reactive power injections

- e.g., at inverters, capacitor banks

Questions we will study

- How to design simple control schemes?
- What is the dynamic behavior of closed-loop system?
- What is the optimality of closed-loop system?

Design and analysis method

- Use LinDistFlow model due to its analytical properties

volt/var control

Network model

At each bus j, there are

- Fixed and given active and reactive load $\left(p_j^0, q_j^0\right)$
- Possibly a DER (e.g. inverter) with fixed p_j (e.g. PV generation) and controllable q_j

Notation: write $s = (p, q) \in \mathbb{R}^{2N}$ and $v \in \mathbb{R}^N$ at non-reference buses, instead of $(\hat{s}, \hat{v})$

From linear solution theorem:

$$v \ = \ v_0 \mathbf{1} + 2\left(R(p - p^0) + X(q - q^0)\right)$$

Or

$$v(q) = 2Xq + \tilde{v}$$

where $\tilde{v} := v_0 \mathbf{1} + 2R(p - p^0) - 2Xq^0$ independent of the control q

volt/var control

Inverter model

At each bus j, the reactive power q_j is constrained to stay in the intersection of

- Capacity limt $\{q_j : p_j^2 + q_j^2 \leq \sigma^2\}$ which depends on p_j (e.g. PV generation), and
- Power factor limit $-\phi_j \leq \tan^{-1}(q_j/p_j) \leq \phi_j$

Hence q_j must lie in

$$U_j \ := \ U_j(p_j) \left\{ \ q_j \ : \ \underline{q}_j \leq q_j \leq \overline{q}_j \ \right\}$$

where $\overline{q}_j := \min \left\{ p_j \tan \phi_j, \sqrt{\sigma^2 - p_j^2} \right\}$ and $\underline{q}_j := \max \left\{ -p_j \tan \phi_j, \ -\sqrt{\sigma^2 - p_j^2} \right\}$

volt/var control

Local memoryless control

Let v^{ref} = given vector of reference voltages at buses $j > 0$

Control goal: design $q \in U$ to drive voltages towards v^{ref}

- Local control: $q_j(t+1)$ depends only on $v_j(t)$, not voltages $v_k(t)$ at buses $k \neq j$
- Memoryless control: $q_j(t+1)$ depends only on $v_j(t)$, not on $\left(v_j(s), s < t\right)$

Restrict control law $u_j : \mathbb{R} \rightarrow \mathbb{R}$ to depend on voltage error $v_j(t) - v_j^{\text{ref}}$

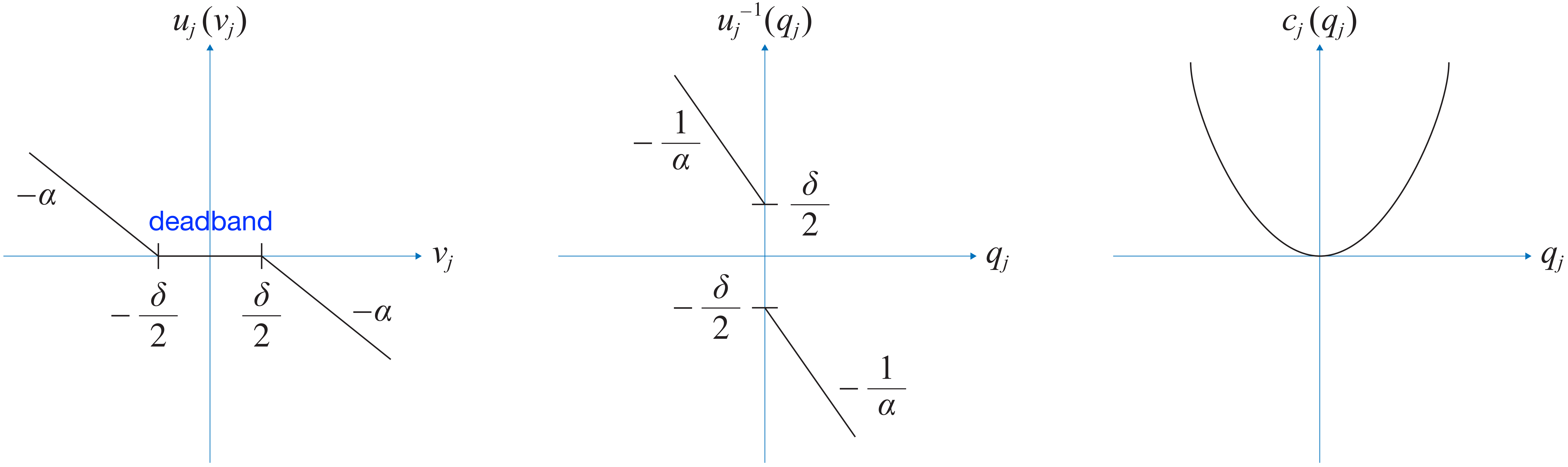
$$q_j(t+1) \;=\; \left[u_j\left(v_j(t) - v_j^{\text{ref}}\right)\right]_{U_j}, \qquad j = 1, \ldots, N$$

i.e. we are to design u_j that map voltage errors $v_j(t) - v_j^{\text{ref}}$ to reactive power settings $q_j(t+1)$

volt/var control

Local memoryless control

Example:

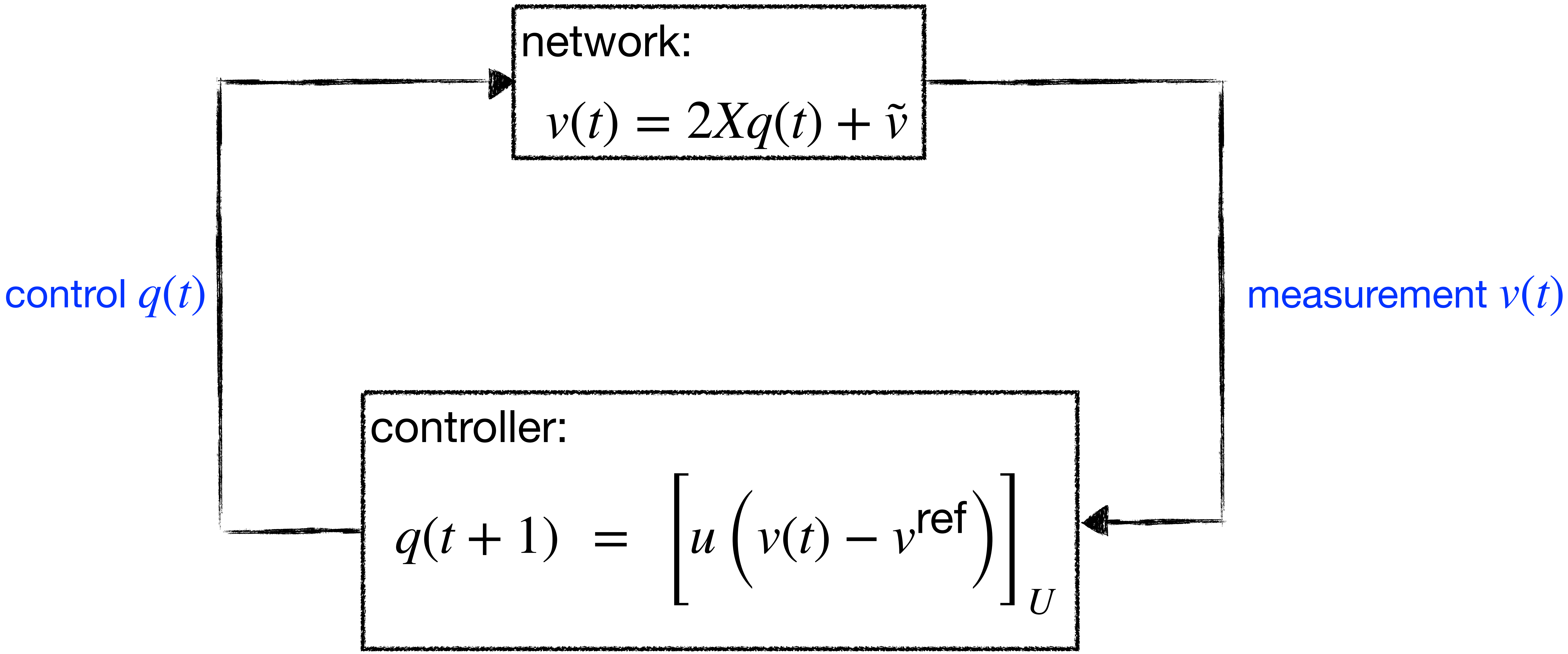


(a) Piecewise linear control $u_j(v_j)$

(b) Inverse $u_j^{-1}(q_j)$

(c) Implied cost $c_j(q_j)$

Closed-loop system



Closed-loop system

Closed-loop system is discrete-time dynamical system:

$$q(t+1) \; = \; \left[u\left(v(q(t)) - v_j^{\text{ref}} \right) \right]_U$$

- $v(q) := 2Xq + \tilde{v}$: maps linearly reactive power control q to network voltage
- $u\left(v - v^{\text{ref}}\right)$: maps voltage error to potential control action
- $[u]_U$: projects potential control action to its feasibility region U

Questions:

- Stability: will $(q(t), v(t))$ converge to an equilibrium point (q^*, v^*) ?
- Optimality: is the equilibrium point (q^*, v^*) optimal, in what sense?

Closed-loop system

Closed-loop system is discrete-time dynamical system:

$$q(t+1) \;=\; \left[u\left(v(q(t)) - v_j^{\mathsf{ref}}\right)\right]_U$$

where $v(q) := 2Xq + \tilde{v}$

Definition:

q^* is an **equilibrium point** if it is a fixed point, i.e., $q^* \;=\; \left[u\left(v(q^*) - v_j^{\mathsf{ref}}\right)\right]_U$

Assumptions:

1. u_j are differentiable; $\exists \alpha_j$ s.t. $\left|u_j'(v_j)\right| \leq \alpha_j$ $\qquad A :=$ diag $\left(\alpha_j, j \in N\right)$
2. u_j are strictly decreasing

Convergence

Theorem [Convergence]

Suppose Assumption 1 holds. If largest singular value $\sigma_{\max}(AX) < 1/2$ then

$A := \text{diag}\left(\alpha_j, j \in N\right)$

1. $\exists$ unique equilibrium point $q^* \in U$
2. $q(t)$ convergest to q^* geometrically, i.e.,

$$\|q(t) - q^*\| \leq \beta^t \|q(0) - q^*\| \rightarrow 0$$

for some $\beta \in [0,1)$

Optimality

Theorem [Optimality]

Suppose Assumptions 1 and 2 hold. The unique equilibrium point q^* of the dynamical system is the unique minimizer of

$$\min_{q \in U} \quad \sum_j c_j(q_j) \; + \; q^{\mathsf{T}} X q \; + \; q^{\mathsf{T}} (\tilde{v} - v^{\mathsf{ref}})$$

where $c_j(q_j) \; := -\int_0^{q_j} u_j^{-1}(\hat{q}_j)\, d\hat{q}_j$

Closed-loop behavior

Questions:

- Stability: will $(q(t), v(t))$ converge to an equilibrium point (q^*, v^*) ?
- Optimality: is the equilibrium point (q^*, v^*) optimal, in what sense?

Answer: under assumptions 1 and 2

- $(q(t), v(t))$ converges geometrically to a unique equilibrium point (q^*, v^*)
- The unique equilibrium point (q^*, v^*) minimizes a cost function determined by control law u_j

Reverse engineering: by choosing a control function u_j, we implicitly choose a cost function $c_j\left(q_j\right)$ that the closed-loop equilibrium optimizes

Closed-loop behavior

Questions:

- Stability: will $(q(t), v(t))$ converge to an equilibrium point (q^*, v^*) ?
- Optimality: is the equilibrium point (q^*, v^*) optimal, in what sense?

Answer: under assumptions 1 and 2

- $(q(t), v(t))$ converges geometrically to a unique equilibrium point (q^*, v^*)
- The unique equilibrium point (q^*, v^*) minimizes a cost function determined by control law u_j

<u>Forward engineering</u>: Choose a cost function $c_j\left(q_j\right)$ and derive control functions u_j as distributed algorithm to solve the optimization problem

Convergence proof

Sketch

Mean value theorem $\implies$ $u_j(v_j) - u(\hat{v}_j) = u_j'(w)(u - \hat{u})$ where $w := \lambda u + (1-\lambda)\hat{u}$ for some $\lambda \in [0,1]$

Assumption 1 and MVT

Hence

$$\|u(v) - u(\hat{v})\|_2^2 = \sum_j \left| u_j(v_j) - u_j(\hat{v}_j) \right|^2 \leq \sum_j \left| \alpha_j(v_j - \hat{v}_j) \right|^2 = \left\| A(v - \hat{v}) \right\|_2^2$$

Therefore

$$\left\| u\left(v(q) - v^{\text{ref}}\right) - u\left(v(\hat{q}) - v^{\text{ref}}\right) \right\|_2 \leq \left\| Av(q) - Av(\hat{q}) \right\|_2$$

Convergence proof

Sketch

Vector-function mean value theorem: if $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable then

$$\|f(y) - f(x)\| \;\leq\; \left\| \frac{\partial f}{\partial x}(z) \right\| \; \|y - x\|$$

for any induced matrix norm $\|\cdot\|$ where $z := \mu x + (1-\mu)y$ for some $\mu \in [0,1]$

Hence

$$\left\| Av(q) - Av(\hat{q}) \right\|_2 \;\leq\; \left\| \frac{\partial Av}{\partial q} \right\|_2 \|q - \hat{q}\|_2 \;\leq\; \|2AX\|_2 \, \|q - \hat{q}\|_2$$

because $\dfrac{\partial Av}{\partial q}(q) = A\dfrac{\partial v}{\partial q}(q) = 2AX$

Convergence proof

Sketch

Therefore

$$\left\| u\left(v(q) - v^{\text{ref}}\right) - u\left(v(\hat{q}) - v^{\text{ref}}\right) \right\|_2 \leq \|2AX\|_2 \, \|q - \hat{q}\|_2$$

Since induced matrix norm $\|AX\|_2 = \sigma_{\max}(AX)$, if $\beta = 2\sigma_{\max}(AX) < 1$ then

$$\left\| u\left(v(q) - v^{\text{ref}}\right) - u\left(v(\hat{q}) - v^{\text{ref}}\right) \right\|_2 \leq \beta \|q - \hat{q}\|_2$$

i.e. $u(q)$ is a contraction mapping.

Since projection $[u]_U$ is non-expansive, i.e., $\|[u]_U - [\hat{u}]_U\|_2 \leq \|u - \hat{u}\|_2$, the mapping $\left[u\left(v(q) - v^{\text{ref}}\right)\right]_U$ is a contraction mapping in q

Convergence proof

Sketch

Contraction theorem implies, for the dynamical system

$$q(t+1) \;=\; \left[u\left(v(q(t)) - v_j^{\text{ref}}\right)\right]_U$$

that

- $\exists$ unique fixed point q^*
- $q(t)$ converges to q^* geometrically

Optimality

Theorem [Optimality]

Suppose Assumptions 1 and 2 hold. The unique equilibrium point q^* of the dynamical system is the unique minimizer of

$$\min_{q \in U} \quad \sum_j c_j(q_j) \ + \ q^{\mathsf{T}} X q \ + \ q^{\mathsf{T}}(\tilde{v} - v^{\mathsf{ref}})$$

where $c_j(q_j) \ := -\int_0^{q_j} u_j^{-1}(\hat{q}_j)\, d\hat{q}_j$

Optimality proof

Sketch

Assumption 1 implies that there is a unique equilibrium pt q^*

Let $C(q) \ := \ \sum_j c_j(q_j) \ + \ q^\top X q \ + \ q^\top \Delta\tilde{v}$ where $\Delta\tilde{v} := \tilde{v} - v^{\text{ref}}$

Assumption 2 and $X \succ 0$ imply that $C(q)$ is strictly convex and hence, if an optimal q^* exists, it is unique

It thus suffices to show that q^* is the unique equilibrium pt if and only if q^* is the unique minimizer

We will show this in 3 steps:

1. Obtain optimality condition (necessary and sufficient because of convexity)
2. Relate $[\nabla C(q^*)]_j$ to $u_j\left(v_j(q_j^*) - v_j^{\text{ref}}\right)$ and q^*
3. Conclude optimality condition is equivalent to $q^* = \left[u\left(v(q^*) - v^{\text{ref}}\right)\right]_U$

Optimality proof

Sketch

Step 1: By convexity, $q^* \in U$ is optimal iff

$$\left(\nabla C(q^*) \right)^{\mathsf{T}} \left(q - q^* \right) \;\geq\; 0 \qquad \forall q \in U$$

This is equivalent to

$$q_j^* \in (\underline{q}_j, \overline{q}_j) \qquad \Longrightarrow \qquad \left[\nabla C(q^*) \right]_j = 0$$

$$q_j^* = \underline{q}_j \qquad \Longleftarrow \qquad \left[\nabla C(q^*) \right]_j > 0$$

$$q_j^* = \overline{q}_j \qquad \Longleftarrow \qquad \left[\nabla C(q^*) \right]_j < 0$$

Optimality proof

Sketch

Step 2: Evaluate

$$\nabla C(q^*) \;=\; \nabla c(q^*) + 2Xq^* + \Delta\tilde{v} \;=\; \nabla c(q^*) + \left(v(q^*) - v^{\text{ref}}\right)$$

where $\nabla c(q^*) = (c_j'(q_j^*) = -u_j^{-1}(q_j^*), i \in N)$

Hence $[\nabla C(q^*)]_j \;= -u_j^{-1}(q^*) + \left(v_j(q^*) - v_j^{\text{ref}}\right)$

Since u_j is strictly decreasing (Assumption 2), we have

$$\left[\nabla C(q^*)\right]_j = 0 \quad\Longleftrightarrow\quad u_j\left(v_j(q_j^*) \;-\; v_j^{\text{ref}}\right) \;=\; q_j^*$$

$$\left[\nabla C(q^*)\right]_j > 0 \quad\Longleftrightarrow\quad u_j\left(v_j(q_j^*) \;-\; v_j^{\text{ref}}\right) \;<\; q_j^*$$

$$\left[\nabla C(q^*)\right]_j < 0 \quad\Longleftrightarrow\quad u_j\left(v_j(q_j^*) \;-\; v_j^{\text{ref}}\right) \;>\; q_j^*$$

Optimality proof

Sketch

Step 3: Use $[\nabla C(q^*)]_j$ to combine the conditions in Steps 1 and 2 into:

$$q_j^* \in (\underline{q}_j, \overline{q}_j) \implies \left[\nabla C(q^*)\right]_j = 0 \iff u_j\left(v_j(q_j^*) - v_j^{\text{ref}}\right) = q_j^*$$

$$q_j^* = \underline{q}_j \impliedby \left[\nabla C(q^*)\right]_j > 0 \iff u_j\left(v_j(q_j^*) - v_j^{\text{ref}}\right) < \underline{q}_j$$

$$q_j^* = \overline{q}_j \impliedby \left[\nabla C(q^*)\right]_j < 0 \iff u_j\left(v_j(q_j^*) - v_j^{\text{ref}}\right) > \overline{q}_j$$

But this is equivalent to:

$$q^* = \left[u\left(v(q^*) - v^{\text{ref}}\right)\right]_U$$

i.e. q^* is the unique equilibrium point

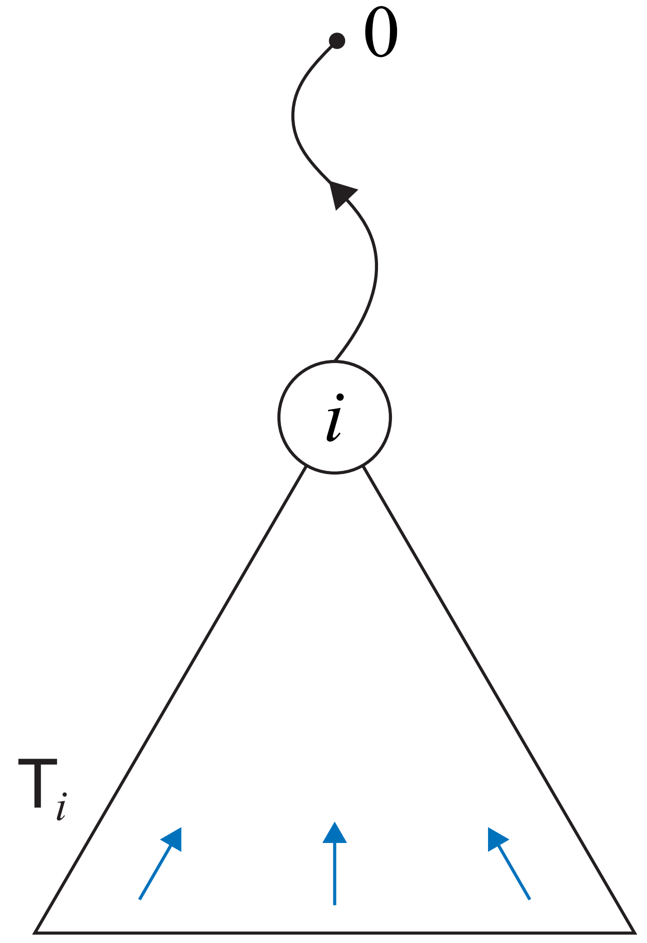
Therefore q^* is the unique equilibrium pt if and only if q^* is the unique minimizer

Outline

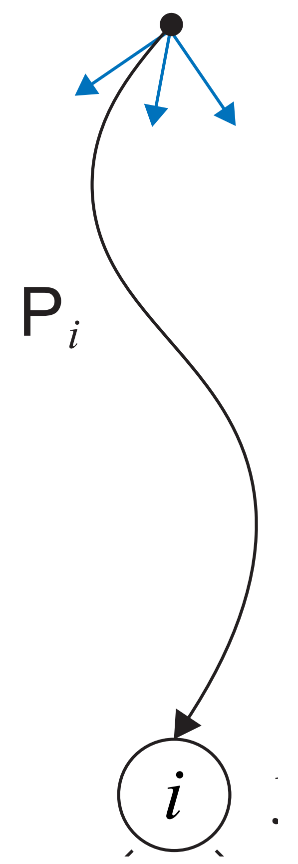
1. Economic dispatch
2. Voltage control
3. Radial network identification
    - Linearized polar-form AC model
    - Covariances of voltage magnitudes

Recall: radial networks

When $y^s_{jk} = y^s_{kj}$ and $y^m_{jk} = y^m_{kj} = 0$



T_i : subtree rooted at bus i



T_i : unique path from 0 to i

<u>**Theorem 10**</u>

Suppose G is connected, Y is complex symmetric ($y^s_{jk} = y^s_{kj}$) and $y^m_{jk} = y^m_{kj} = 0$.

1. Reduced incidence matrix $\hat{C}$ is nonsingular

$$\left[\hat{C}^{-1}\right]_{lj} = \begin{cases} -1 & l \in \mathsf{P}_j \\ 1 & -l \in \mathsf{P}_j \\ 0 & \text{otherwise} \end{cases}$$

2. Reduced admittance matrix $\hat{Y}$ is nonsingular, and

$$\hat{Z} := \hat{Y}^{-1} = \hat{C}^{-\mathsf{T}} D^s_z \, \hat{C}^{-1}$$

$$\hat{Z}_{jk} = \sum_{l \in \mathsf{P}_j \cap \mathsf{P}_k} z^s_l$$

Apply this result to topology identification problem

Topology identification

1. Distribution grid typically consists of a meshed network with sectionalizing and tie switches on some lines
2. At any time switch are configured s.t. operational network is a spanning tree (substation at its root)
3. System operator knows the meshed network, but may not always know accurately switch status and hence operational network

Goal: Identify operational radial network from measurements of voltage magnitudes

Linearized power flow model

Linearization of polar form

Assumptions: For all $(j, k) \in E$

1. $y^s_{jk} = y^s_{kj} = g^s_{jk} + ib^s_{jk}$; $y^m_{jk} = y^m_{kj} = 0$
2. $g^s_{jk} > 0$ and $b^s_{jk} < 0$

Consider flat voltage profile: $V_j^{\text{flat}} = \mu e^{i\theta} \implies \left(p^{\text{flat}}, q^{\text{flat}}\right) = (0,0)$

- All voltages have same magnitude (e.g. $\mu = 1$ pu) and angle

Let

- $(|\hat{V}|, \hat{\theta})$: perturbation variable around V^{flat} at non-reference buses
- $(\hat{p}, \hat{q})$: perturbation variable around $\left(p^{\text{flat}}, q^{\text{flat}}\right) = (0,0)$ at non-reference buses

Linearized power flow model

Linearization of polar form

Polar form power flow model

$$p_j = \sum_{k:k\sim j} \left(g_{jk}^s + g_{jk}^m\right) |V_j|^2 - \sum_{k:k\sim j} |V_j||V_k| \left(g_{jk}^s \cos\theta_{jk} + b_{jk}^s \sin\theta_{jk}\right)$$

$$q_j = -\sum_{k:k\sim j} \left(b_{jk}^s + b_{jk}^m\right) |V_j|^2 - \sum_{k:k\sim j} |V_j||V_k| \left(g_{jk}^s \sin\theta_{jk} - b_{jk}^s \cos\theta_{jk}\right)$$

Linearized power flow model

Linearization of polar form

Polar form power flow model

$$p_j = \sum_{k:k\sim j} \left(g^s_{jk} + g^m_{jk}\right) |V_j|^2 \; - \; \sum_{k:k\sim j} |V_j|\,|V_k| \left(g^s_{jk} \cos\theta_{jk} + b^s_{jk} \sin\theta_{jk}\right)$$

$$q_j = -\sum_{k:k\sim j} \left(b^s_{jk} + b^m_{jk}\right) |V_j|^2 \; - \; \sum_{k:k\sim j} |V_j|\,|V_k| \left(g^s_{jk} \sin\theta_{jk} - b^s_{jk} \cos\theta_{jk}\right)$$

Linearize around $\left(V^{\text{flat}}, p^{\text{flat}}, q^{\text{flat}}\right)$ yields a linear model from $|\hat{V}|$ to $(\hat{p}, \hat{q})$ at non-reference buses:

$$|\hat{V}| \; = \; \hat{R}\hat{p} \; + \; \hat{X}\hat{q} \; + \; \hat{v}_0$$

where

$$\hat{R} := \hat{C}^{-T} D_1 \hat{C}^{-1} \succ 0, \quad \hat{X} := -\hat{C}^{-T} D_2 \hat{C}^{-1} \succ 0$$

$\hat{C}$ is reduced incidence matrix and

$$D_g := \text{diag}\left(g^s_l, l \in E\right) \succ 0, \qquad D_b := \text{diag}\left(b^s_l, l \in E\right) \prec 0$$

$$D_1 := \left(D_g + D_b D_g^{-1} D_b\right)^{-1} \succ 0, \qquad D_2 := \left(D_b + D_g D_b^{-1} D_g\right)^{-1} \prec 0$$

Covariance of voltages and powers

Suppose injections (p, q) vary randomly and induce random fluctuations in $|\hat{V}|$

Define covariance and cross-covariance matrices

$$\Sigma_v := E[|\hat{V}| - E(|\hat{V}|)][(|V| - E(|V|)]^\mathsf{T}$$

$$\Sigma_p := E[\hat{p} - E\hat{p}][\hat{p} - E\hat{p}]^\mathsf{T}, \qquad \Sigma_q := E[\hat{q} - E\hat{q}][\hat{q} - E\hat{q}]^\mathsf{T}$$

$$\Sigma_{pq} := E[\hat{p} - E\hat{p}][\hat{q} - E\hat{q}]^\mathsf{T}, \qquad \Sigma_{qp} := E[\hat{q} - E\hat{q}][\hat{p} - E\hat{p}]^\mathsf{T}$$

Then

$$\Sigma_v \;=\; \hat{R}\Sigma_p\hat{R}^\mathsf{T} + \hat{X}\Sigma_q\hat{X}^\mathsf{T} + \hat{R}\Sigma_{pq}\hat{X}^\mathsf{T} + \hat{X}\Sigma_{qp}\hat{R}^\mathsf{T}$$

Covariance of voltages and powers

Assumptions: power injections at same bus are positively correlated, those at different buses are uncorrelated

3. For all $j \in N$: $\Sigma_p[j,j] > 0, \;\; \Sigma_q[j,j] > 0, \;\; \Sigma_{pq}[j,j] = \Sigma_{qp}[j,j] > 0 \,; \;\; y^m_{jk} = y^m_{kj} = 0$

4. For all $j \neq k$: $\Sigma_p[j,k] = \Sigma_q[j,k] = \Sigma_{pq}[j,k] = \Sigma_{qp}[j,k] = 0$

Covariance of voltages and powers

Assumptions: power injections at same bus are positively correlated, those at different buses are uncorrelated

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4. For all $j \neq k$: $\Sigma_p[j,k] = \Sigma_q[j,k] = \Sigma_{pq}[j,k] = \Sigma_{qp}[j,k] = 0$

<u>**Theorem**</u>

Under assumptions 1-4:

1. If a non-reference bus $j \in N$ is a descendant of bus i, then $\text{var}(|V_j|) > \text{var}(|V_i|)$
2. If bus i is a parent of bus j then the variance of $|V_i| - |V_j|$ is given by:

$$E\left((|V_i| - |V_j|) - E(|V_i| - |V_j|)\right)^2 = \sum_{k \in \mathrm{T}_j} \left(r_{ij}^2 \text{var}(p_k) + x_{ij}^2 \text{var}(q_k) + 2 r_{ij} x_{ij} \text{cov}(p_k, q_k) \right)$$

Covariance of voltages and powers

Theorem

Under assumptions 1-4:

1. If a non-reference bus $j \in N$ is a descendant of bus i, then var$(|V_j|) > $ var$(|V_i|)$
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Implications

Property 1 identifies a leaf node j as one with max var$(|V_j|)$

Property 2 identifies j's parent i as one that most closely satisfies the formula

Algorithm

1. Identify a leaf node j among unidentified nodes.
2. Identify j's parent.
3. Remove j from set of unidentified nodes and goto 1

Covariance of voltages and powers

<u>Proof:</u> part 1

Theorem 10 implies

$$\hat{R}_{jk} = \sum_{l \in \mathsf{P}_j \cap \mathsf{P}_k} r_l > 0, \qquad \hat{X}_{jk} = \sum_{l \in \mathsf{P}_j \cap \mathsf{P}_k} x_l > 0$$

Hence

$$\hat{R}_{jk} = \hat{R}_{ik} + r_{ij}, \qquad \hat{R}_{ik} = \sum_{l \in P_i} r_l, \qquad \text{if } k \in \mathsf{T}_j$$

$$\hat{R}_{ik} = \hat{R}_{jk}, \qquad \text{if } k \notin \mathsf{T}_j$$

Use these to evaluate the diagonal entries of $\text{var}(|V_j|) - \text{var}(|V_i|) = \Sigma_v[j, j] - \Sigma_v[i, i]$, for each of the four terms in

$$\Sigma_v = \hat{R}\Sigma_p\hat{R}^\mathsf{T} + \hat{X}\Sigma_q\hat{X}^\mathsf{T} + \hat{R}\Sigma_{pq}\hat{X}^\mathsf{T} + \hat{X}\Sigma_{qp}\hat{R}^\mathsf{T}$$

Covariance of voltages and powers

Due to covariances Σ_p, Σ_q :

$$\left(\hat{R}\Sigma_p\hat{R}^\top\right)[j,j] - \left(\hat{R}\Sigma_p\hat{R}^\top\right)[i,i] = \sum_{k\in\mathsf{T}_j}\Sigma_p[k,k]\left(2\sum_{l\in\mathsf{P}_i} r_l + r_{ij}\right) r_{ij} > 0$$

similarly: $\left(\hat{X}\Sigma_q\hat{X}^\top\right)[j,j] > \left(\hat{X}\Sigma_q\hat{X}^\top\right)[i,i]$

Due to cross-covariances Σ_{pq}, Σ_{qp} :

$$\left(\hat{R}\Sigma_{pq}\hat{X}^\top\right)[j,j] - \left(\hat{R}\Sigma_{pq}\hat{X}^\top\right)[i,i] = \sum_{k}\Sigma_{pq}[k,k]\left(\hat{R}_{jk}\hat{X}_{jk} - \hat{R}_{ik}\hat{X}_{ik}\right) > 0$$

similarly: $\left(\hat{X}\Sigma_{qp}\hat{R}^\top\right)[j,j] > \left(\hat{X}\Sigma_{qp}\hat{R}^\top\right)[i,i]$

yielding: $\Sigma_v[j,j] > \Sigma_v[i,i]$

Covariance of voltages and powers

<u>Proof</u>: part 2

If bus i is a parent of bus j, then variance of $|V_i| - |V_j|$ is:

$$E\left((|V_i| - E|V_j|) - (|V_i|) - |V_j|)\right)^2 \;=\; \Sigma_v[i,i] + \Sigma_v[j,j] - 2\Sigma_v[i,j]$$

Again use

$$\hat{R}_{jk} \;=\; \hat{R}_{ik} + r_{ij}, \qquad \hat{R}_{ik} \;=\; \sum_{l \in P_i} r_l, \qquad \text{if } k \in \mathsf{T}_j$$

$$\hat{R}_{ik} \;=\; \hat{R}_{jk}, \qquad \text{if } k \notin \mathsf{T}_j$$

to show that the first term of

$$\Sigma_v \;=\; \hat{R}\Sigma_p\hat{R}^\mathsf{T} \;+\; \hat{X}\Sigma_q\hat{X}^\mathsf{T} \;+\; \hat{R}\Sigma_{pq}\hat{X}^\mathsf{T} \;+\; \hat{X}\Sigma_{qp}\hat{R}^\mathsf{T}$$

yields a simple expression:

$$\sigma_1 \;:=\; \left(\hat{R}\Sigma_p\hat{R}^\mathsf{T}\right)[i,i] \;+\; \left(\hat{R}\Sigma_p\hat{R}^\mathsf{T}\right)[j,j] \;-\; 2\left(\hat{R}\Sigma_p\hat{R}^\mathsf{T}\right)[i,j] \;=\; r_{ij}^2 \sum_{k \in \mathsf{T}_j} \Sigma_p[k,k]$$

Covariance of voltages and powers

Similarly, the other terms of

$$\Sigma_v \;=\; \hat{R}\Sigma_p\hat{R}^\mathsf{T} + \hat{X}\Sigma_q\hat{X}^\mathsf{T} + \hat{R}\Sigma_{pq}\hat{X}^\mathsf{T} + \hat{X}\Sigma_{qp}\hat{R}^\mathsf{T}$$

yield

$$\sigma_1 := \left(\hat{R}\Sigma_p\hat{R}^\mathsf{T}\right)[i,i] + \left(\hat{R}\Sigma_p\hat{R}^\mathsf{T}\right)[j,j] - 2\left(\hat{R}\Sigma_p\hat{R}^\mathsf{T}\right)[i,j] \;=\; r_{ij}^2 \sum_{k \in \mathsf{T}_j} \Sigma_p[k,k]$$

$$\sigma_2 := \left(\hat{X}\Sigma_q\hat{X}^\mathsf{T}\right)[i,i] + \left(\hat{X}\Sigma_q\hat{X}^\mathsf{T}\right)[j,j] - 2\left(\hat{X}\Sigma_q\hat{X}^\mathsf{T}\right)[i,j] \;=\; x_{ij}^2 \sum_{k \in \mathsf{T}_j} \Sigma_q[k,k]$$

$$\sigma_3 := \left(\hat{R}\Sigma_{pq}\hat{X}^\mathsf{T}\right)[i,i] + \left(\hat{R}\Sigma_{pq}\hat{X}^\mathsf{T}\right)[j,j] - 2\left(\hat{R}\Sigma_{pq}\hat{X}^\mathsf{T}\right)[i,j] \;=\; r_{ij}x_{ij} \sum_{k \in \mathsf{T}_j} \Sigma_{pq}[k,k]$$

$$\sigma_4 := \left(\hat{X}\Sigma_{qp}\hat{R}^\mathsf{T}\right)[i,i] + \left(\hat{X}\Sigma_{qp}\hat{R}^\mathsf{T}\right)[j,j] - 2\left(\hat{X}\Sigma_{qp}\hat{R}^\mathsf{T}\right)[i,j] \;=\; r_{ij}x_{ij} \sum_{k \in \mathsf{T}_j} \Sigma_{qp}[k,k]$$

Covariance of voltages and powers

Summing:

$$\Sigma_v[i,i] - \Sigma_v[i,j] \;=\; \sum_{k=1}^{4} \sigma_k \;=\; \sum_{k \in \mathsf{T}_j} \left(r_{ij}^2 \Sigma_p[k,k] \;+\; x_{ij}^2 \Sigma_q[k,k] \;+\; 2 r_{ij} x_{ij} \Sigma_{pq}[k,k] \right)$$