Power System Analysis

Chapter 6 System operation: power balance

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Outline

1. Background
    - Overview
    - Basic optimization concepts
2. Unit commitment & real-time dispatch
3. Frequency control
4. Pricing electricity & reserves

Overview

Central challenge

Balance supply & demand second-by-second everywhere on grid

- While satisfying operational constraints, e.g. injection/voltage/line limits
- Unlike usual commodities, electricity cannot (yet) be stored in large quantity

Overview
Traditional approach

Bulk generators generate 79% of electricity in US (2023)

- Fossil (gas, coal): 60%, nuclear: 18.5%
- Hydro: 5.9%, renewables: 15.5%

They are fully dispatchable and centrally controlled

- ISO determines in advance how much each generates when & where

They mostly determine dynamics and stability of entire network

- System frequency, voltages, prices

Overview

Traditional approach

Challenges

- Large startup/shutdown time and cost
- Uncertainty in future demand (depends mostly on weather)
- Contingency events such as generator/transmission outages

Elaborate electricity markets and hierarchical control

- Schedule generators and (large) controllable loads/batteries
- Day-ahead (12-36 hrs in advance): unit commitment
- Real-time (5-15 mins in advance): economic dispatch
- Ancillary services (secs - hours): frequency control, reserves

Overview

Future challenges

Sharply increased uncertainty makes balancing more difficult

- Renewable sources such as wind and solar
- Random large frequent fluctuations in net load, e.g., Duck Curve due to PV
- Contingency events such as generator/transmission outages
- *Response*: real-time feedback control, better monitoring & forecast, stochastic OPF

Low-inertia system

- Bulk generators have large inertia that is bedrock of stability
- They will be replaced by inverter-based resources with low or zero inertia, e.g., PV
- *Response*: dynamics and stability need to be re-thought (both risk and opportunity)

Indispatchable renewable generation resources

- *Response*: More active dynamic feedback control of flexible loads to match fluctuating supply

Hierarchical control

Unit commitment and real-time dispatch

Unit comment

- Computed in day-ahead market 12-36 hrs in advance of energy delivery
- Determine which generators will be online (and their generation levels) for each hour or 1/2 hour over 24-hr horizon
- … assuming generation levels will be optimal given commitment decisions
- … based on forecast of loads and variable generations
- Commitment decisions are binding; generation levels may be advisory

Real-time dispatch

- Computed in real-time market 5-15mins in advance of energy delivery
- Adjust generation or consumption levels to the schedules produced by day-ahead market
- … as uncertainty in generation, consumption, network state is resolved

Hierarchical control

Frequency control

Frequency deviation from nominal indicated power imbalance

- Excess supply accelerates rotating machines in bulk generators —> frequency rise
- Excess demand decelerates rotating machines in bulk generators —> frequency drop

Primary control

- Generators use governors to automatically adjust power in proportion to local frequency deviations
- Rebalance power and stabilizes frequency to a new equilibrium (generally not nominal) in ~30 secs
- Decentralized control

Hierarchical control
Frequency control

Frequency deviation from nominal indicated power imbalance

- Excess supply accelerates rotating machines in bulk generators —> frequency rises
- Excess demand decelerates rotating machines in bulk generators —> frequency drops

Secondary control

- Generators adjust their set points around real-time dispatch values in order to
- … restore frequency to nominal value
- … restore tie-line powers between balancing areas to their scheduled values
- … in a few minutes
- Setpoint adjustment is computed centrally within each balancing area
- … based on real-time measurements of tie-line flows and frequency deviations in the area

Pricing electricity

Security constrained economic dispatch/UC

Real-time dispatch determines not only optimal generation/consumption levels, but also electricity prices

- Prices are locational dependent, called locational marginal prices (LMP) or nodal prices
- They measure both marginal production costs/user utilities and network congestion
- They maximizes social welfare and are incentive compatible

Secure operation

- System operator needs to deal with uncertainties
- … discrete uncertainty: outages of generators, lines, transformers
- … continuous uncertainty: random fluctuations of renewable generations and loads
- Security constrained economic dispatch jointly optimizes energy and reserves
- … can be incorporated as stage-two decisions in security constrained unit commitment

Basic optimization concepts

Many power system applications can be formulated as an optimization problem

Constrained optimization:

$$\min_{x \in \mathbb{R}^n} \; f(x) \quad \text{s.t.} \quad g(x) = 0, \; h(x) \leq 0$$

- Optimization vars: x
- Cost function: $f(x)$
- Constraint functions: $f(x), \; g(x)$
- Feasible set: $X := \{x \in \mathbb{R}^n : g(x) \leq 0, h(x) \leq 0\}$
- Feasible solution: $x \in X$
- (Primal) optimal solution or minimizer: $x^* \in X$ s.t. $f(x^*) \leq f(x) \quad \forall x \in X$

Optimality condition: suppose f is convex

- Unconstrained opt: x^* is optimal $\iff \nabla f(x^*) = 0$
- Constrained opt: KKT condition

Basic optimization concepts

Optimality (KKT) condition

Associate with the constraints dual variable $(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}_+^l$

- Equality constraint $g_j(x) = 0$: dual var $\lambda_j \in \mathbb{R}$
- Inequality constraint $h_j(x) \leq 0$: dual var $\mu_j \in \mathbb{R}_+$

Suppose f, g, h are convex functions

Optimality (KKT) condition: $x^* \in \mathbb{R}^n$ is optimal $\iff$ $\exists$ dual optimum $(\lambda^*, \mu^*) \in \mathbb{R}^m \times \mathbb{R}_+^l$ s.t.

$$
\begin{aligned}
&\text{Stationarity :} && \nabla f(x^*) + \nabla g(x^*)\lambda^* + \nabla h(x^*)\mu^* = 0 \\
&\text{Primal feasibility :} && g(x^*) = 0,\ h(x^*) \leq 0 \\
&\text{Dual feasibility :} && \mu^* \geq 0 \\
&\text{Complementary slackness :} && \mu^{*\mathsf{T}} h(x^*) = 0
\end{aligned}
$$

where $\nabla f(x) := \left(\frac{\partial f}{\partial x_i}, \forall i\right) \in \mathbb{R}^n,\ \nabla g(x) := \left(\frac{\partial g_i}{\partial x_j}, \forall i, j\right) \in \mathbb{R}^{m \times n},\ \ \nabla h(x) := \left(\frac{\partial h_i}{\partial x_j}, \forall i, j\right) \in \mathbb{R}^{l \times n}$

Optimal power flow

We will formulate various control and pricing mechanisms as constrained optimization:

- Equality constraint $g_j(x) = 0$: dual var $\lambda_j \in \mathbb{R}$
- Inequality constraint $h_j(x) \leq 0$: dual var $\mu_j \in \mathbb{R}_+$

Suppose f, g, h are convex functions

Optimality (KKT) condition: $x^* \in \mathbb{R}^n$ is optimal $\iff$ $\exists$ dual optimum $(\lambda^*, \mu^*) \in \mathbb{R}^m \times \mathbb{R}^l_+$ s.t.

$$\begin{aligned}
\text{Stationarity :} \quad & \nabla f(x^*) + \nabla g(x^*)\lambda^* + \nabla h(x^*)\mu^* &= 0 \\
\text{Primal feasibility :} \quad & g(x^*) = 0, \ h(x^*) &\leq 0 \\
\text{Dual feasibility :} \quad & \mu^* &\geq 0 \\
\text{Complementary slackness :} \quad & \mu^{*\mathsf{T}} h(x^*) &= 0
\end{aligned}$$

where $\nabla f(x) := \left(\frac{\partial f}{\partial x_i}, \forall i \right) \in \mathbb{R}^n, \ \nabla g(x) := \left(\frac{\partial g_i}{\partial x_j}, \forall i, j \right) \in \mathbb{R}^{m \times n}, \ \ \nabla h(x) := \left(\frac{\partial h_i}{\partial x_j}, \forall i, j \right) \in \mathbb{R}^{l \times n}$

Optimal power flow

We will formulate various control and pricing mechanisms as constrained optimization:

$$\min_{u,x} \quad f(u,x) \qquad \text{s.t.} \qquad g(u,x) = 0, \quad h(u,x) \leq 0$$

- Optimization vars: control u, network state x
- Cost function: $f(u,x)$
- Constraint functions: $g(u,x)$, $h(u,x)$
- They depend on the application under study

Outline

1. Background
2. Unit commitment & real-time dispatch
   - Unit commitment
   - Real-time dispatch
   - Secure operation
3. Frequency control
4. Pricing electricity & reserves

Unit commitment

Solved by ISO in day-ahead market 12-36 hrs in advance

- Determine which generators will be on (commitment) and their output levels (dispatch)
- For each hour (or half hour) over 24-hour period
- Commitment decisions are binding
- Dispatch decisions may be binding or advisory

Two-stage optimization

- Determine commitment, based on assumption that dispatch will be optimized

Unit commitment
Problem formulation

Model

- Network: graph $G = (\overline{N}, E)$
- Time horizon: $T := \{1,2,\ldots,T\}$, e.g., $t = 1$ hour, $T = 24$

Optimization vars

- Control:
  - Commitment: on/off status $\kappa(t) := \left(\kappa_j(t), j \in \overline{N}\right),\ \kappa_j(t) \in \{0,1\}$
  - Dispatch: real & reactive power injections $u(t) := (u_j(t), j \in \overline{N})$
- Network state:
  - Voltages $V(t) := (V_j(t), j \in \overline{N})$
  - Line flows $S(t) := \left(S_{jk}(t), S_{kj}(t), (j,k) \in E\right)$

Unit commitment

Problem formulation

Capacity limits: injection is bounded if it is turned on

$$\underline{u}_j(t)\kappa_j(t) \quad \le \quad u_j(t) \quad \le \quad \overline{u}_j(t)\kappa_j(t)$$

Startup and shutdown incur costs regardless of injection level

$$d_{jt}(\kappa_j(t-1), \kappa_j(t)) = \begin{cases} \text{startup cost} & \text{if} \quad \kappa_j(t) - \kappa_j(t-1) = 1 \\ \text{shutdown cost} & \text{if} \quad \kappa_j(t) - \kappa_j(t-1) = -1 \\ 0 & \text{if} \quad \kappa_j(t) - \kappa_j(t-1) = 0 \end{cases}$$

Once turned on/off, a bulk generator must stay in the same state for a min amount time:

$$\kappa_j(t) - \kappa_j(t-1) \le \kappa_j^\tau, \qquad \forall \tau \in \{t+1,\ t+\text{up}_j - 1\}$$

$$\kappa_j(t-1) - \kappa_j(t) \le 1 - \kappa_j^\tau, \qquad \forall \tau \in \{t+1,\ t+\text{down}_j - 1\}$$

Unit commitment

Problem formulation

Two-stage optimization

$$\min_{\kappa \in \{0,1\}^{(N+1)T}} \quad \sum_t \sum_j d_{jt}\Big(\kappa_j(t-1), \kappa_j(t)\Big) + f^*(\kappa) \quad \text{s.t.} \quad \text{min up/down time constraints}$$

where $f^*(\kappa)$ is optimal dispatch cost over entire horizon T:

$$\begin{aligned} f^*(\kappa) \quad := \quad \min_{(u,x)} \quad & \sum_t f_t(u(t), x(t); \kappa(t)) \\ \text{s.t.} \quad & g_t(u(t), x(t); \kappa(t)) = 0, \;\; h_t(u(t), x(t); \kappa(t)) \leq 0, \quad t \in T \\ & \tilde{g}(u, x) = 0, \;\; \tilde{h}(u, x) \leq 0 \end{aligned}$$

- f_t : dispatch cost
- Each time t constraint includes injection limits, power
- $\tilde{f}(u, x) = 0, \;\; \tilde{g}(u, x) \leq 0$ can include ramp rate limits

Unit commitment

Problem formulation

UC in practice

- Binary variable makes UC computationally difficult for large networks
- Typically use linear model, e.g., DC power flow, and solve mixed integer linear program

Serious effort underway in R&D community to scale UC solution with AC model

- e.g., ARPA-E Grid Optimization Competition Challenge 2

Real-time dispatch

Solved by ISO in real-time market every 5-15 mins

- Determine injection levels of those units that are online
- Adjustment to dispatch schedule from day-ahead market (unit commitment)

Real-time dispatch

Problem formulation

Model

- Network: graph $G = (\overline{N}, E)$

Optimization vars

- Control:
  - Dispatch: real & reactive power injections $u := (u_j, j \in \overline{N})$
- Network state:
  - Voltages $V := (V_j, j \in \overline{N})$
  - Line flows $S := \left(S_{jk}, S_{kj}, (j,k) \in E\right)$

Real-time dispatch

Problem formulation

Parameters

- Uncontrollable injections $\sigma := \left(\sigma_j, j \in \overline{N}\right)$

Generation cost is quadratic in real power

$$f(u,x) \quad = \quad \sum_{\text{generators } j} \left(a_j \left(\mathrm{Re}(u_j) \right)^2 \ + \ b_j \, \mathrm{Re}(u_j) \right)$$

Real-time dispatch

Constraints

Power flow equations: $S = S(V)$

- Complex form: $S_{jk}(V) \;=\; \left(y_{jk}^{s}\right)^{H}\left(|V_j|^2 - V_j V_k^H\right) \;+\; \left(y_{jk}^{m}\right)^{H}|V_j|^2$
- Polar form:

$$P_{jk}(V) = \left(g_{jk}^{s} + g_{jk}^{m}\right)|V_j|^2 \;-\; |V_j|\,|V_j|\left(g_{jk}^{s}\cos(\theta_j - \theta_k) \;-\; b_{jk}^{s}\sin(\theta_j - \theta_k)\right)$$

$$Q_{jk}(V) = \left(b_{jk}^{s} + b_{jk}^{m}\right)|V_j|^2 \;-\; |V_j|\,|V_k|\left(b_{jk}^{s}\cos(\theta_j - \theta_k) \;+\; g_{jk}^{s}\sin(\theta_j - \theta_k)\right)$$

Power balance: $u_j \;+\; \sigma_j \;=\; \sum_{k:j\sim k} S_{jk}(V)$

Real-time dispatch

Constraints

Injection limits: $\underline{u}_j \;\leq\; u_j \;\leq\; \overline{u}_j$

Voltage limits: $\underline{v}_j \;\leq\; |V_j|^2 \;\leq\; \overline{v}_j$

Line limits: $|S_{jk}(V)| \;\leq\; \overline{S}_{jk}, \quad |S_{kj}(V)| \;\leq\; \overline{S}_{kj}$

Real-time dispatch

$$
\begin{aligned}
\min_{u,x} \quad & f(u, x) \\
\text{s.t.} \quad & u_j \; + \; \sigma_j \;\; = \;\; \sum_{k:j\sim k} S_{jk}(V) \\
& \underline{u}_j \;\le\; u_j \;\le\; \overline{u}_j \\
& \underline{v}_j \;\le\; |V_j|^2 \;\le\; \overline{v}_j \\
& |S_{jk}(V)| \;\le\; \overline{S}_{jk}, \quad |S_{kj}(V)| \;\le\; \overline{S}_{kj}
\end{aligned}
$$

$u^{\text{opt}}(\sigma)$: optimal dispatch driven by σ

Real-time dispatch

Interpretation

- ISO dispatches u_j^{opt} to unit j as generation setpoint (needs incentive compatibility)
- Resulting network state x^{opt} satisfies operational constraints

Economic dispatch in practice

- Real-time market use linear approximation, e.g., DC power flow, instead of AC (nonlinear) power flow equations
- ISO solves linear program for dispatch and wholesale prices
- AC power flow equations are used to verify that operational constraints are satisfied if dispatched
- If not, DC OPF is modified and procedure repeated

Intra-interval imbalance

In theory, power is balanced at all points of network, since $\left(u^{\text{opt}}, x^{\text{opt}}\right)$ satisfies

$$u_j \ + \ \sigma_j \ \ = \ \sum_{k:j\sim k} S_{jk}(V)$$

Intra-interval imbalance, however, arises due to

- Random error $\Delta_1(\xi, t)$
- Discretization error $\Delta_2(t)$
- Prediction error $\Delta_3(\xi, t)$

Intra-interval imbalance

Error model

Uncontrollable injections $\sigma := \left(\sigma(t), t \in \mathbb{R}_+\right)$: continuous-time stochastic process

$u\left(\sigma(\xi, t)\right)$: injection needed to maintain power balance over network

Imbalance:

$$\Delta u(\xi, t) \ := \ \underbrace{u\left(\sigma(\xi, t)\right)}_{\text{injection needed for balance}} - \underbrace{u^{\text{opt}}\left(\hat{m}(n)\right)}_{\text{dispatch on } n\text{th control interval}}, \quad t \in [n\delta, (n+1)\delta), \ n = 0, 1, \ldots$$

- $u\left(\sigma(\xi, t)\right)$: random, continuous
- $u^{\text{opt}}\left(\hat{m}(n)\right)$: fixed for nth interval, based on estimate $\hat{m}(n)$ of σ

Intra-interval imbalance

Error model

Imbalance:

$$\Delta u(\xi, t) \quad = \quad \Delta_1(\xi, t) \; + \; \Delta_2(t) \; + \; \Delta_3(\xi, t)$$

- Random error $\Delta_1(\xi, t)$: tends to have zero mean
- Discretization error $\Delta_2(t)$: time avg over control interval tends to be small
- Prediction error $\Delta_3(\xi, t)$: tends to be small if $\sigma(t)$ is slow-varying

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System security

- System security refers to ability to withstand contingency events
- A contingency event is an outage of a generator, transmission line, or transformer
- Contingency events are rare, but can be catastrophic
- NERC's (North America Electricity Reliability Council) $N-1$ rule the outage of a single piece of equipment should not result in violation of voltage or line limits

System security

Secure operation

- Analyze credible contingencies that may lead to voltage or line limit violations
- Account for these contingencies in optimal commitment and dispatch schedules (security constrained UC/ED)
- Monitor system state in real time and take corrective actions when contingency arises

Optimal dispatch

Recall: OPF without security constraints (base case):

$$\begin{aligned} \min_{(u_0, x_0)} \quad & f_0\left(u_0, x_0\right) \\ \text{s.t.} \quad & g_0\left(u_0, x_0\right) \ = \ 0, \ \ h_0\left(u_0, x_0\right) \ \leq \ 0 \end{aligned}$$

where

- u_0 : dispatch in base case
- x_0 : network state in base case
- $g_0\left(u_0, x_0\right)$: power flow equations, etc.
- $h_0\left(u_0, x_0\right)$: operational constraints, etc.

Security constrained OPF

Preventive approach

Basic idea

- Augment optimal dispatch (OPF) with additional constraints …
- … so that the (new) network state under optimal dispatch u^{opt} will satisfy operational constraints after contingency events
- Dispatch remains unchanged until next update period, even if a contingency occurs in the middle of control interval

Security constrained OPF

Preventive approach

Security constrained OPF (SCOPF)

$$\min_{(u_0, x_0,\ \tilde{x}_k,\ k \geq 1)} \quad f_0\left(u_0, x_0\right)$$

$$\text{s.t.} \quad \begin{aligned} g_0\left(u_0, x_0\right) &= 0, \quad h_0\left(u_0, x_0\right) \leq 0 \quad \text{base case constraints} \\ \tilde{g}_k\left(u_0, \tilde{x}_k\right) &= 0, \quad \tilde{h}_k\left(u_0, \tilde{x}_k\right) \leq 0 \quad \text{constraints after cont. } k \end{aligned}$$

where

- $\tilde{x}_k$: new state under same dispatch u_0 after contingency k
- $\tilde{g}_k\left(u_0, \tilde{x}_k\right)$: power flow equations for post-contingency network
- $\tilde{h}_k\left(u_0, \tilde{x}_k\right)$: (more relaxed) emergency operational constraints after contingency k

Security constrained OPF

Corrective approach

Basic idea

- Compute optimal dispatch not only for base case, but also for each contingency k
- System operator can dispatch a response immediately after contingency without waiting till next dispatch period

Security constrained OPF

Corrective approach

Security constrained OPF (SCOPF)

$$\min_{(u_k, x_k,\ k \geq 0)} \quad \sum_{k \geq 0} w_k\, f_k\left(u_k, x_k\right)$$

$$\text{s.t.} \quad g_k\left(u_k, x_k\right) \ = \ 0, \quad h_k\left(u_k, x_k\right) \ \leq \ 0, \quad k \geq 0$$

$$\|u_k - u_0\| \ \leq \ \rho_k, \quad k \geq 1 \qquad \text{ramp rate limits}$$

where

- $\left(u_k, x_k\right)$: dispatch & state in base case $k = 0$ and after contingency $k \geq 1$
- $\left(g_k, h_k\right)$: power flow equations & operational constraints for $k \geq 0$
- $\|u_k - u_0\|$: ramp rate limits

Outline

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2. Unit commitment & real-time dispatch
3. Frequency control
    - Model and assumptions
    - Primary control
    - Secondary control
4. Pricing electricity & reserves

Frequency control

Overview

Power delivered by thermal generator is determined by mechanical output of turbine

- Mechanical output of turbine controlled by opening or closing of valves that regulate steam or water flow
- If load increases, valves will be opened wider to generate more power to balance

Frequency control
Overview

Power delivered by thermal generator is determined by mechanical output of turbine

- Mechanical output of turbine controlled by opening or closing of valves that regulate steam or water flow
- If load increases, valves will be opened wider to generate more power to balance

Power imbalance $\Longrightarrow$ frequency deviates from nominal

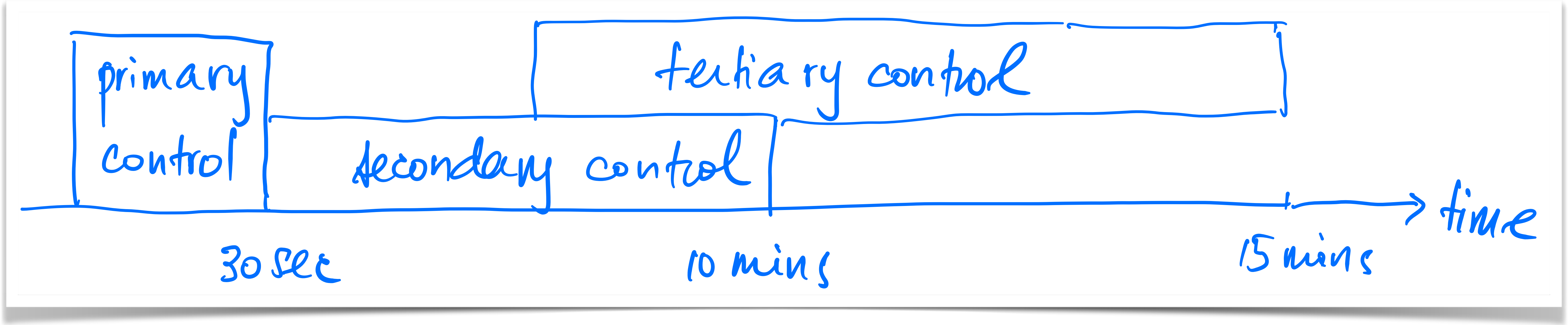
- Excess supply: rotating machines speed up $\Rightarrow$ frequency rises
- Shortage: rotating machines slow down $\Rightarrow$ frequency drops
- If power is not re-balanced, frequency excursion will continue and may disconnect generators to protect them from damage
- Can lead to load shedding (blackout) or even system collapse

Frequency control
Overview

Frequency deviation is global control signal for participating generators and loads

Automatic generation control (AGC) : hierarchical control

- **Primary (droop) control**: stabilize frequency in ~30 secs
    - Uses governor to adjust valve position and control mechanical output of turbine
    - Control proportional to local frequency deviation
    - Decentralized

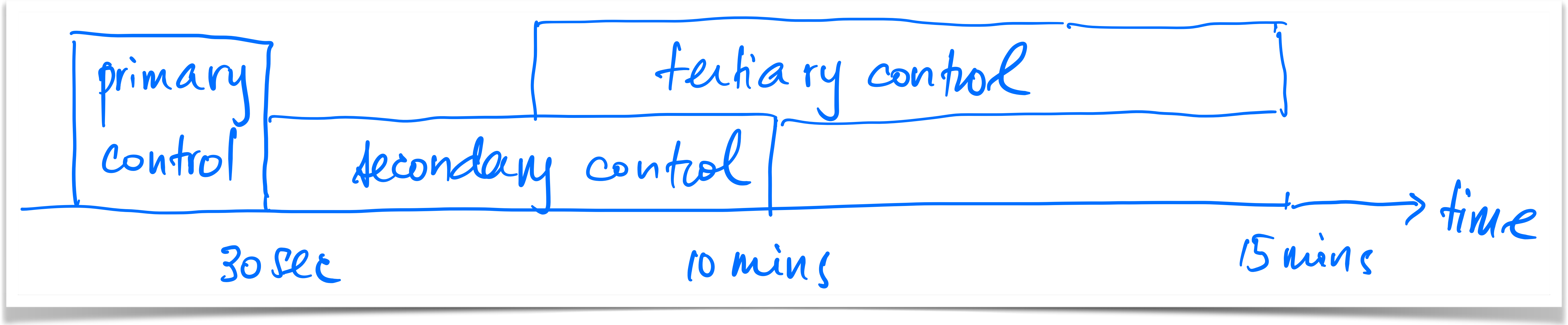


Frequency control
Overview

Frequency deviation is global control signal for participating generators and loads

Automatic generation control (AGC) : hierarchical control

- **Secondary control**: restore nominal frequency within a few mins
    - Adjust generator setpoints around dispatch values
    - Interconnected system: also restore scheduled tie-line flows between areas (need non-local info of tie-line flow deviations)
    - Each area is controlled centrally by an operator

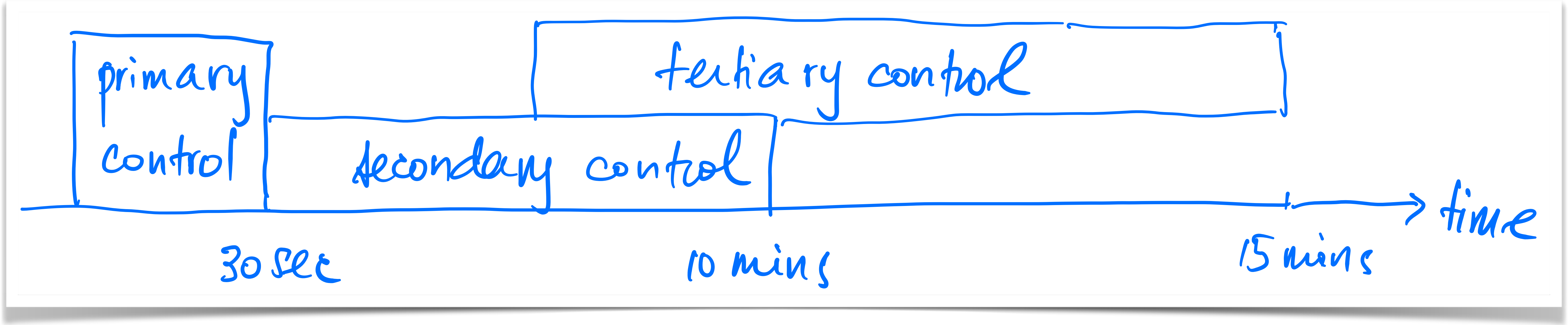


Frequency control
Overview

Frequency deviation is global control signal for participating generators and loads

Automatic generation control (AGC) : hierarchical control

- **Tertiary control**: real-time optimal dispatch every 5-15 mins
    - Determine generator setpoints and schedule inter-area tie-line flows
    - Optimize across areas for economic efficiency
    - Restore reserve capacities of primary & secondary control so that they are available for contingency response



Frequency control

Model

Primary and secondary control model

- Fix control interval n
- Fix random realization ξ of $\sigma(t)$

Assumptions (DC power flow)

- Lossless lines $y_{jk}^s = ib_{jk}$
- Fixed voltage magnitudes (voltage control operates at faster timescale)
- Small angle difference $\sin\left(\theta_{jk}\right) \approx \theta_{jk}$

$\implies$ Linearized dynamic model on

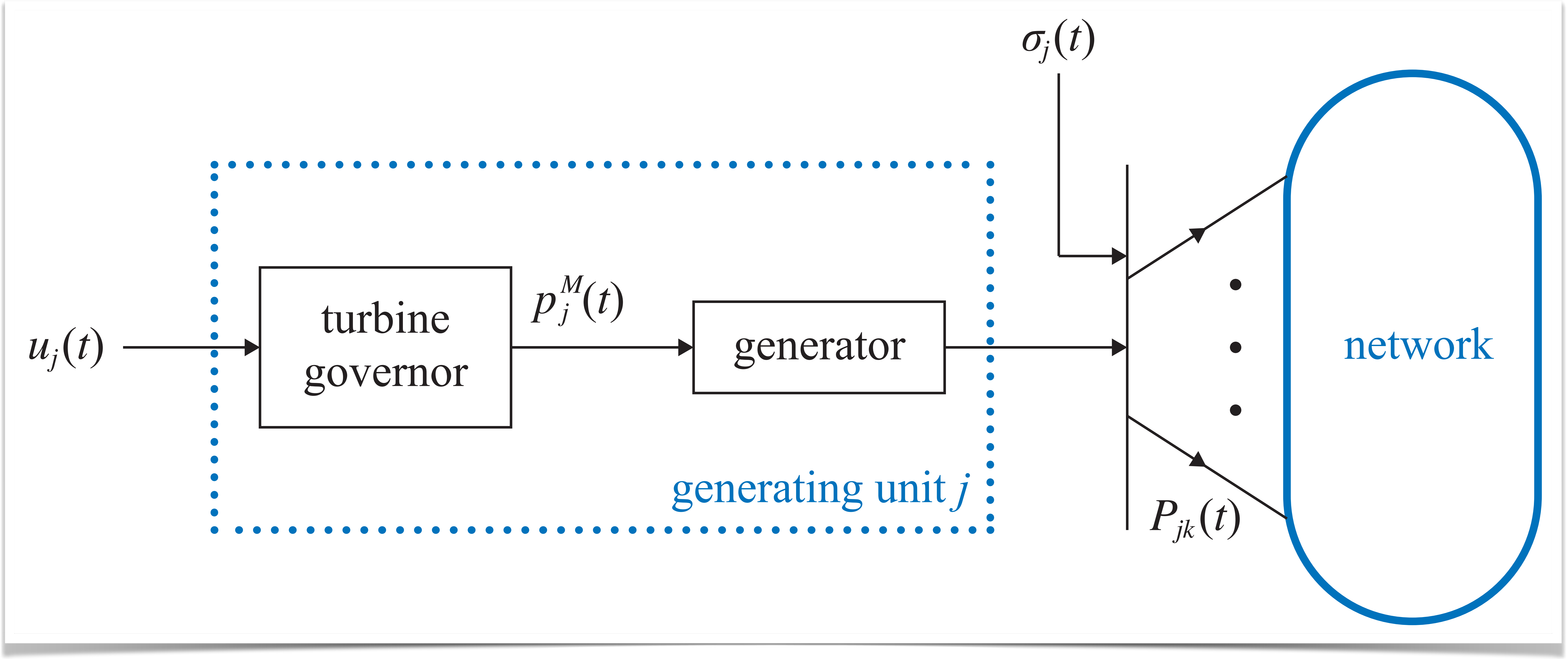
- How real power control voltage angles & local frequencies (derivatives)

Frequency control

Model

Linearized around operating point, defined by

$$u_j^0 \; + \; \sigma_j^0 \;\; = \;\; \sum_{k:j\sim k} P_{jk}^0$$



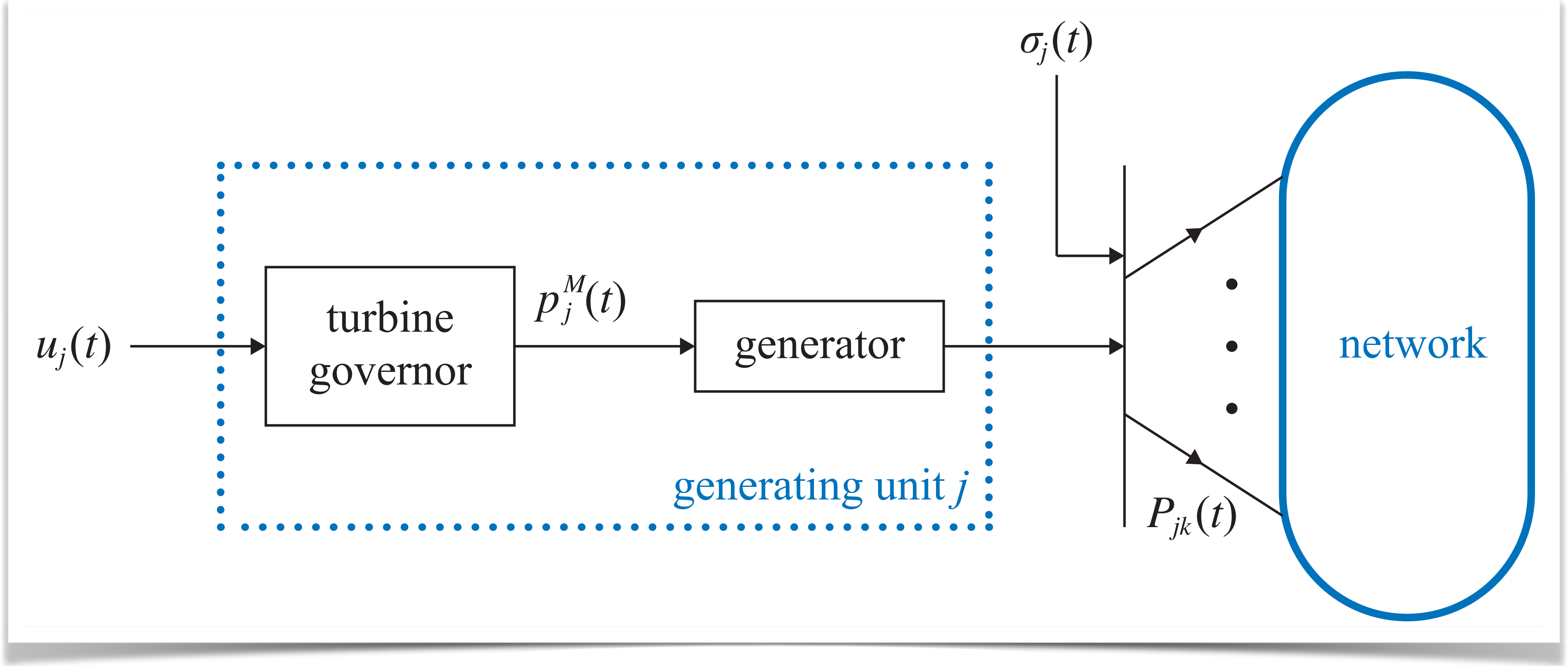
Primary frequency control

Turbine-governor model

2nd order model with droop control

$$T_{gj}\,\dot{a}_j = -\,a_j(t) \;+\; u_j(t) - \frac{\Delta\omega_j(t)}{R_j}$$

$$T_{tj}\,\dot{p}_j^M = -\,p_j^M(t) \;+\; a_j(t)$$



where

- $a_j(t)$: valve position of turbine-governor
- $p_j^M(t)$: mechanical power output of turbine
- $u_j(t)$: generator setpoint (operating point u_j^0 is from tertiary control)
- $\Delta\omega_j(t) = \Delta\dot{\theta}_j(t)$: frequency deviation from operating-point frequency ω^0

Primary frequency control

Turbine-governor model

Linearized around operating point

$$T_{gj}\,\Delta\dot{a}_j = -\,\Delta a_j(t) \;+\; \Delta u_j(t) \;-\; \frac{\Delta\omega_j(t)}{R_j}$$

$$T_{tj}\,\Delta\dot{p}_j^M = -\,\Delta p_j^M(t) \;+\; \Delta a_j(t)$$

incremental vars:

- $\Delta a_j(t) \;:=\; a_j(t) - a_j^0$: deviation of valve position of turbine-governor
- $\Delta p_j^M(t) \;:=\; p_j^M(t) - P_j^{M0}$: deviation of mechanical power output of turbine
- $\Delta u_j(t) \;:=\; u_j(t) - u_j^0$: adjustment to dispatched setpoint

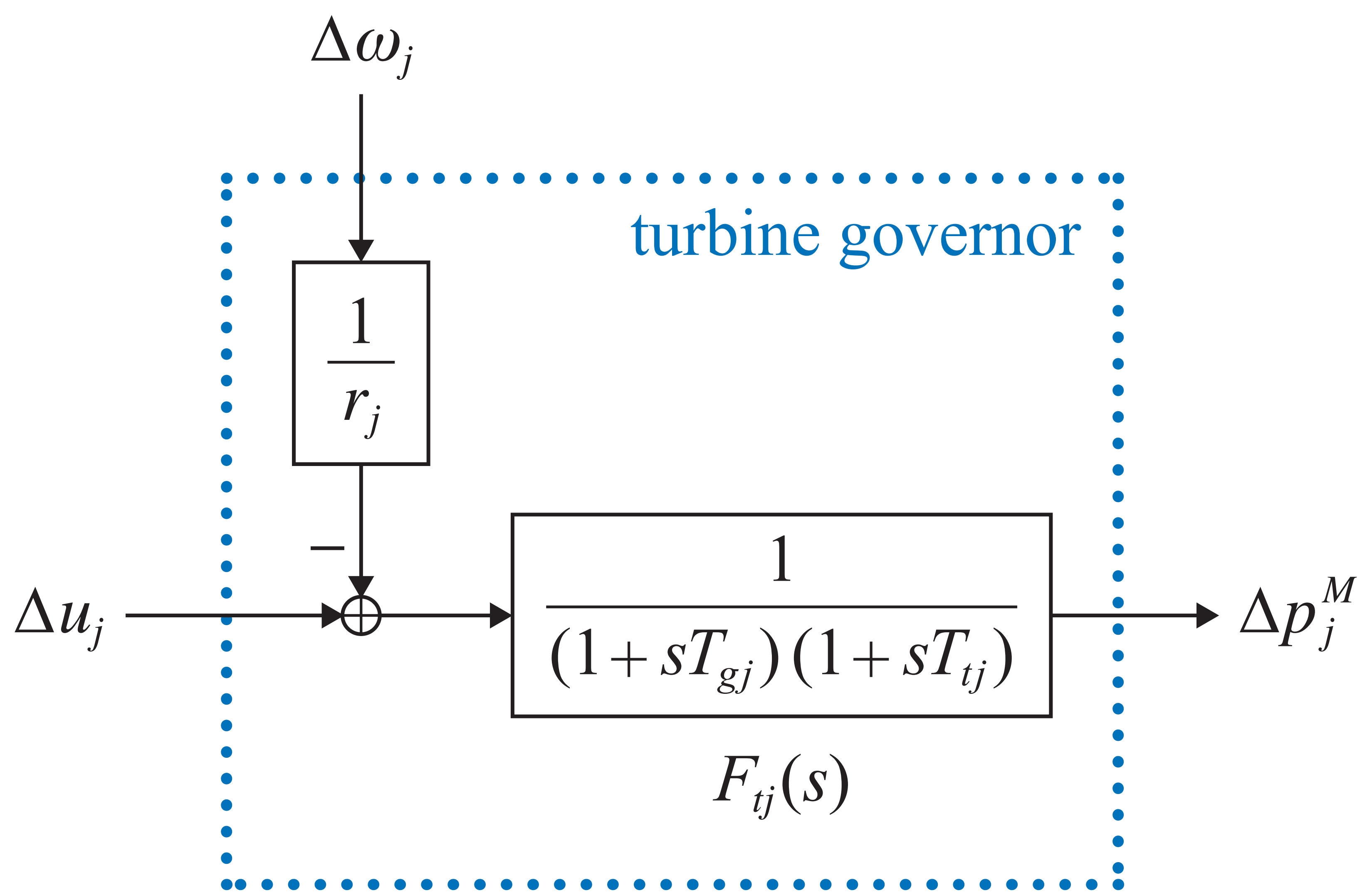
Primary frequency control

Turbine-governor model

Linearized around operating point

$$T_{gj}\,\Delta\dot{a}_j = -\,\Delta a_j(t) \;+\; \Delta u_j(t) \;-\; \frac{\Delta\omega_j(t)}{R_j}$$

$$T_{tj}\,\Delta\dot{p}_j^M = -\,\Delta p_j^M(t) \;+\; \Delta a_j(t)$$



Primary frequency control

Turbine-governor model

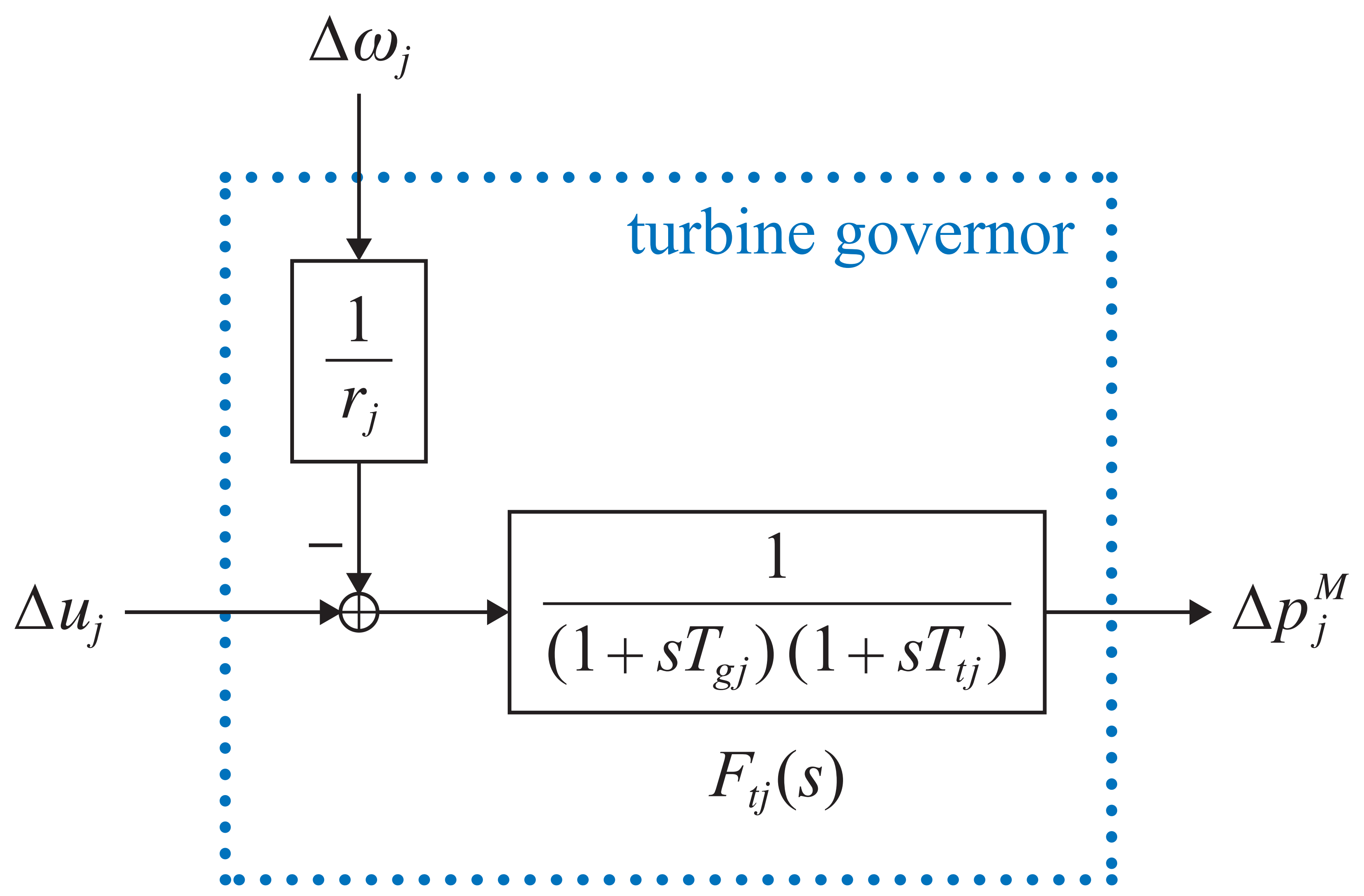
Linearized around operating point

$$T_{gj}\,\Delta\dot{a}_j = -\,\Delta a_j(t) \;+\; \Delta u_j(t) \;-\; \frac{\Delta\omega_j(t)}{R_j}$$

$$T_{tj}\,\Delta\dot{p}_j^M = -\,\Delta p_j^M(t) \;+\; \Delta a_j(t)$$

For primary control, $\Delta u_j(t) = \Delta u_j$ is constant

- $\Delta u_j(t)$ is adjusted by secondary control on a slower timescale



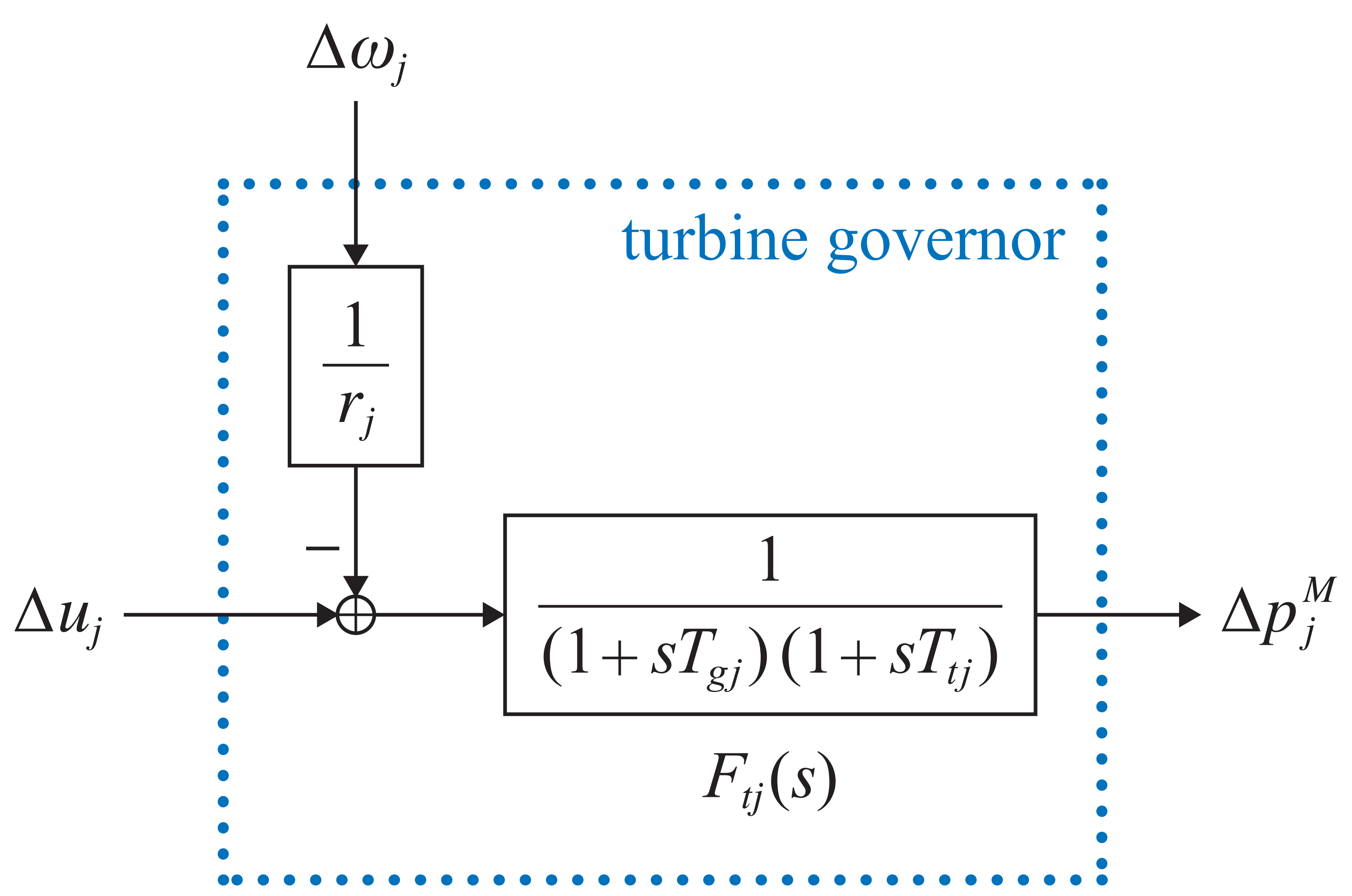
Primary frequency control

Turbine-governor model

Linearized around operating point

$$T_{gj}\,\Delta\dot{a}_j = -\,\Delta a_j(t) \;+\; \Delta u_j(t) \;-\; \frac{\Delta\omega_j(t)}{R_j}$$

$$T_{tj}\,\Delta\dot{p}_j^M = -\,\Delta p_j^M(t) \;+\; \Delta a_j(t)$$



Equilibrium of turbine-governor (primary control):

$$\Delta\dot{a}_j(t) \;=\; \Delta\dot{p}_j^M \;=\; 0$$

Therefore

$$\Delta p_j^{M*} \;=\; \Delta a_j^* \;=\; \Delta u_j \;-\; \frac{1}{R_j}\Delta\omega_j^*,$$

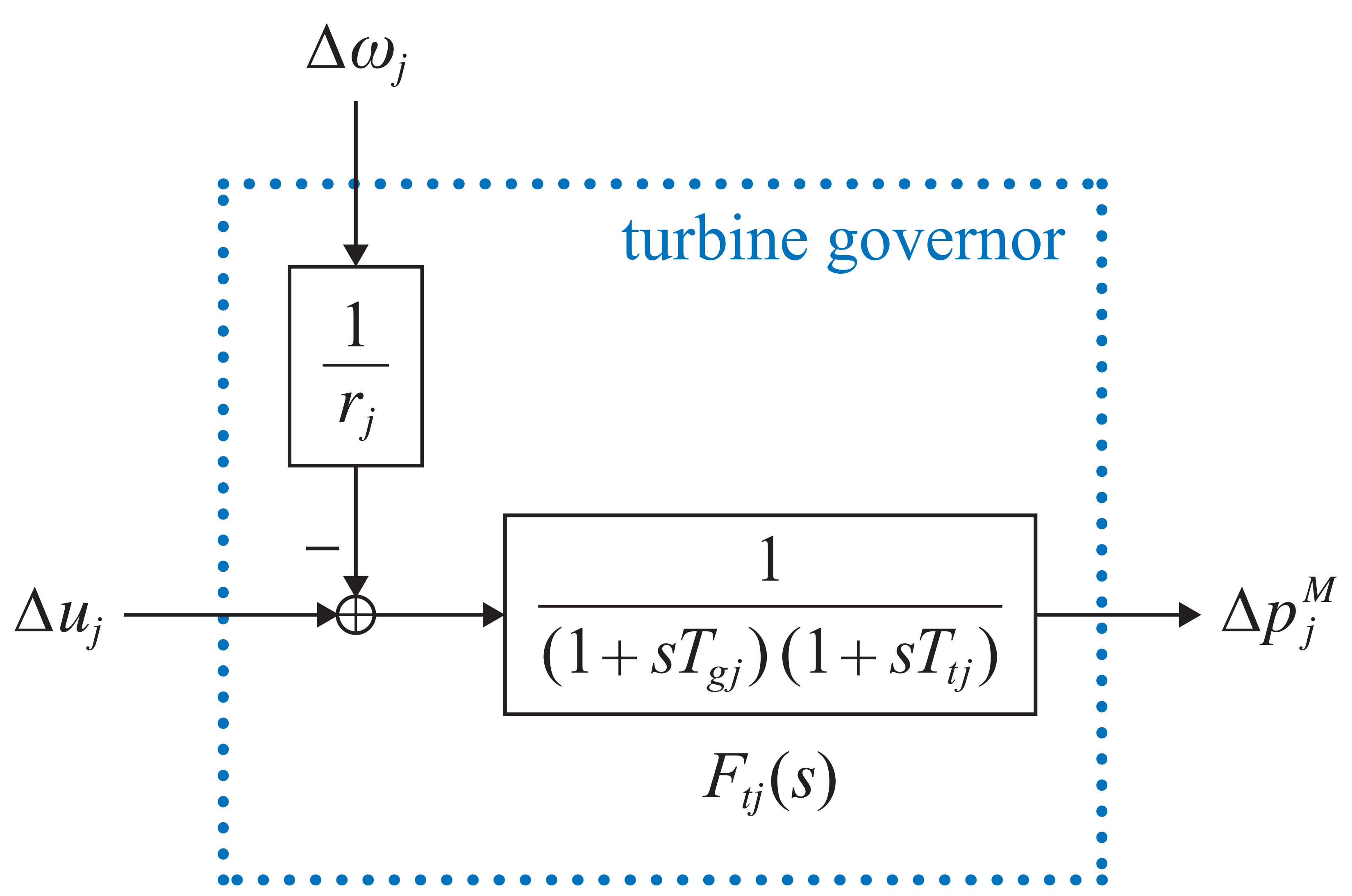
Primary frequency control

Turbine-governor model

Linearized around operating point

$$T_{gj}\,\Delta\dot{a}_j = -\,\Delta a_j(t) \;+\; \Delta u_j(t) \;-\; \frac{\Delta\omega_j(t)}{R_j}$$

$$T_{tj}\,\Delta\dot{p}_j^M = -\,\Delta p_j^M(t) \;+\; \Delta a_j(t)$$



Equilibrium of turbine-governor (primary control):

- Frequency deviation $\Delta\omega_j^* \neq 0$
- Incremental mechanical power Δp_j^{M*} depends on $\Delta\omega_j^*$

Primary frequency control

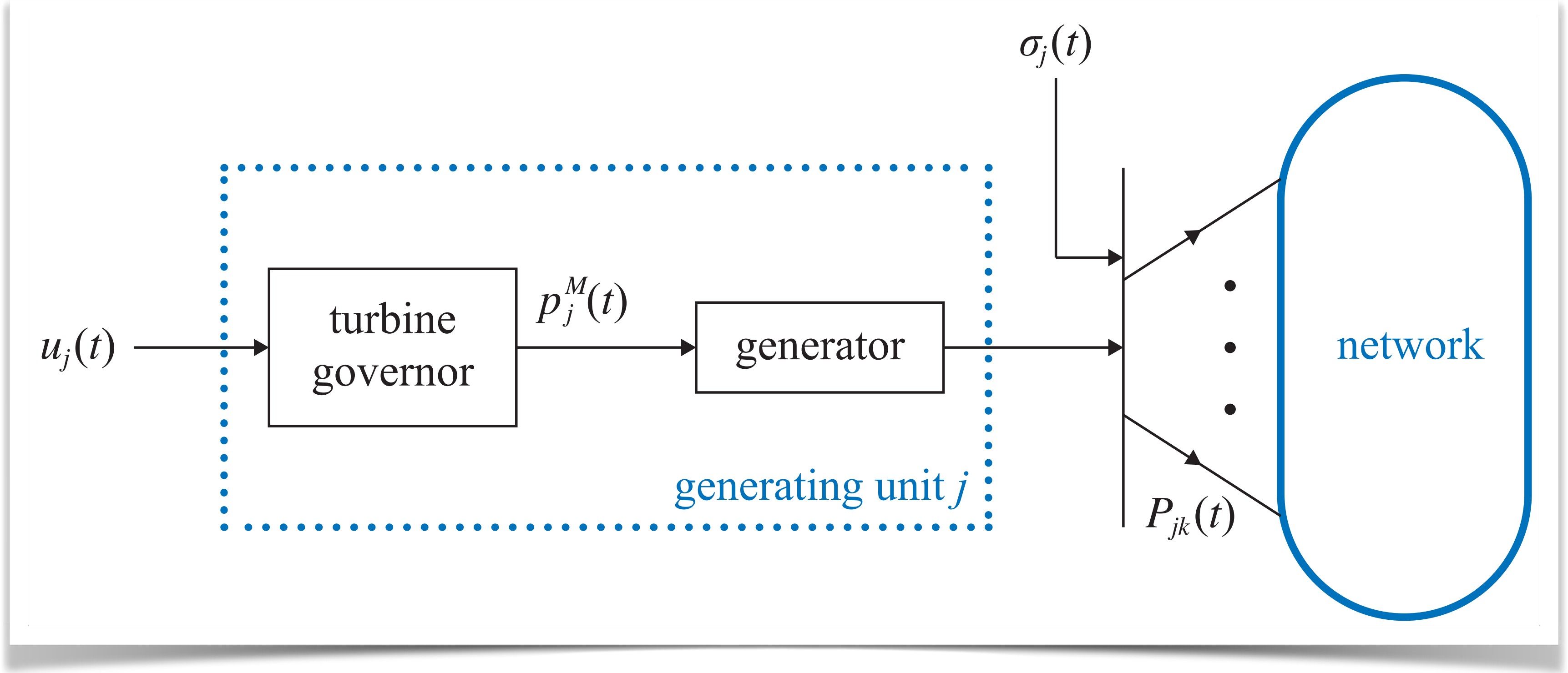
Generator model

$$\Delta\dot{\theta}_j = \Delta\omega_j(t)$$

$$M_j\Delta\dot{\omega}_j + D_j\Delta\omega_j(t) = \Delta p_j^M(t) + \Delta\sigma_j(t) - \sum_{k:j\sim k}\Delta P_{jk}(t)$$

where

- $\Delta\theta_j(t) := \theta_j(t) - \theta_j^0$: incremental angle relative to rotating frame of ω^0
- $\Delta\sigma_j(t)$: deviation of uncontrollable injection from its forecast σ_j^0
- $\Delta P_{jk}(t) := P_{jk}(t) - P_{jk}^0$: line flow deviation



Primary frequency control

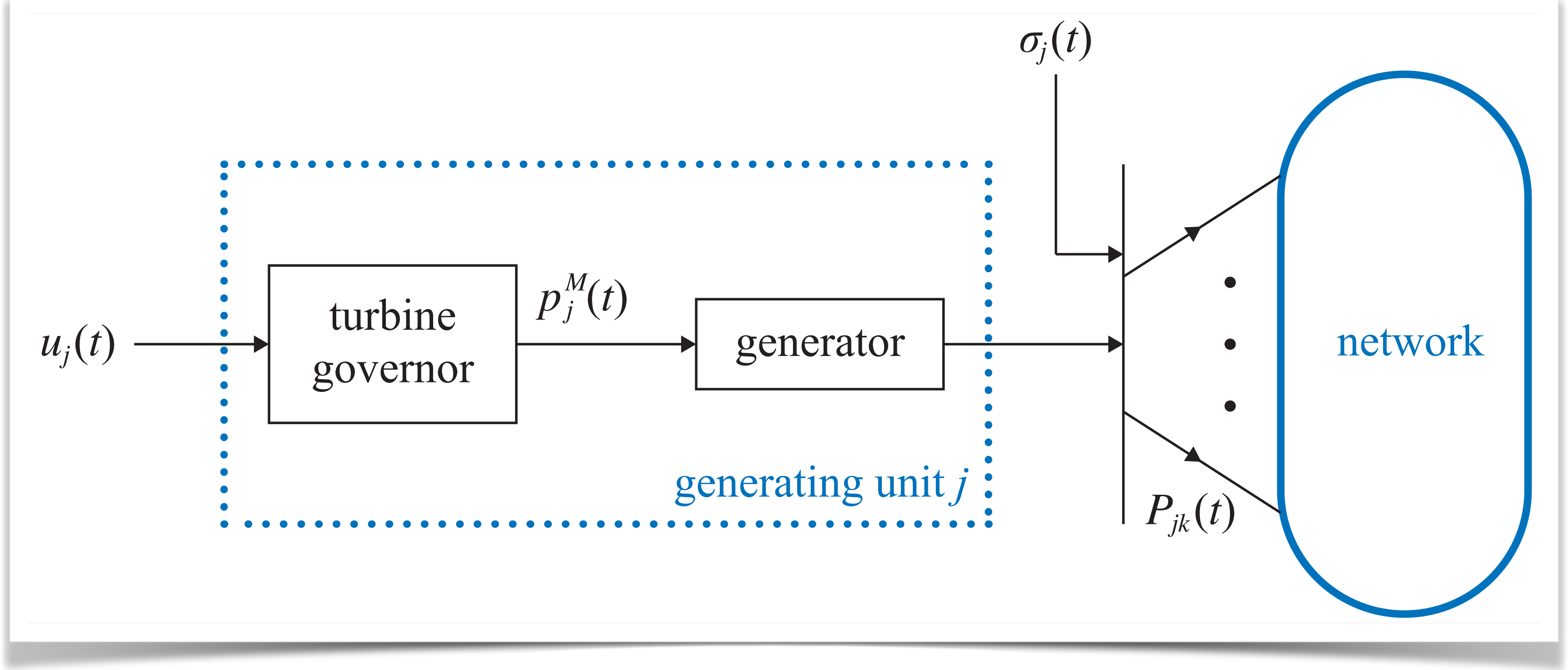
Generator model

$$\Delta\dot{\theta}_j = \Delta\omega_j(t)$$

$$M_j\Delta\dot{\omega}_j + D_j\Delta\omega_j(t) = \Delta p_j^M(t) + \Delta\sigma_j(t) - \sum_{k:j\sim k} \Delta P_{jk}(t)$$

where

- M_j : inertia constant of synchronous machine
- D_j : damping and frequency-sensitive load



Primary frequency control

Generator model

Model for instantaneous line flow

$$P_{jk}(t) \;=\; |V_j||V_k|\left(-b_{jk}\right)\sin\left(\theta_j(t)-\theta_k(t)\right)$$

Primary frequency control

Generator model

Model for instantaneous line flow

$$P_{jk}(t) \;=\; |V_j||V_k|\left(-b_{jk}\right)\sin\left(\theta_j(t) - \theta_k(t)\right)$$

Linear approximation

$$P_{jk}(t) \;=\; \underbrace{|V_j||V_k|\left(-b_{jk}\right)\sin\left(\theta_j^0 - \theta_k^0\right)}_{P_{jk}^0} + \; T_{jk}\left(\Delta\theta_j(t) - \Delta\theta_k(t)\right)$$

Primary frequency control

Generator model

Model for instantaneous line flow

$$P_{jk}(t) \; = \; |V_j||V_k|\left(-b_{jk}\right)\sin\left(\theta_j(t) - \theta_k(t)\right)$$

Linear approximation

$$P_{jk}(t) \quad = \quad \underbrace{|V_j||V_k|\left(-b_{jk}\right)\sin\left(\theta_j^0 - \theta_k^0\right)}_{P_{jk}^0} + \; T_{jk}\left(\Delta\theta_j(t) - \Delta\theta_k(t)\right)$$

Linearized model

$$\Delta P_{jk}(t) \; = \; T_{jk}\left(\Delta\theta_j(t) - \Delta\theta_k(t)\right)$$

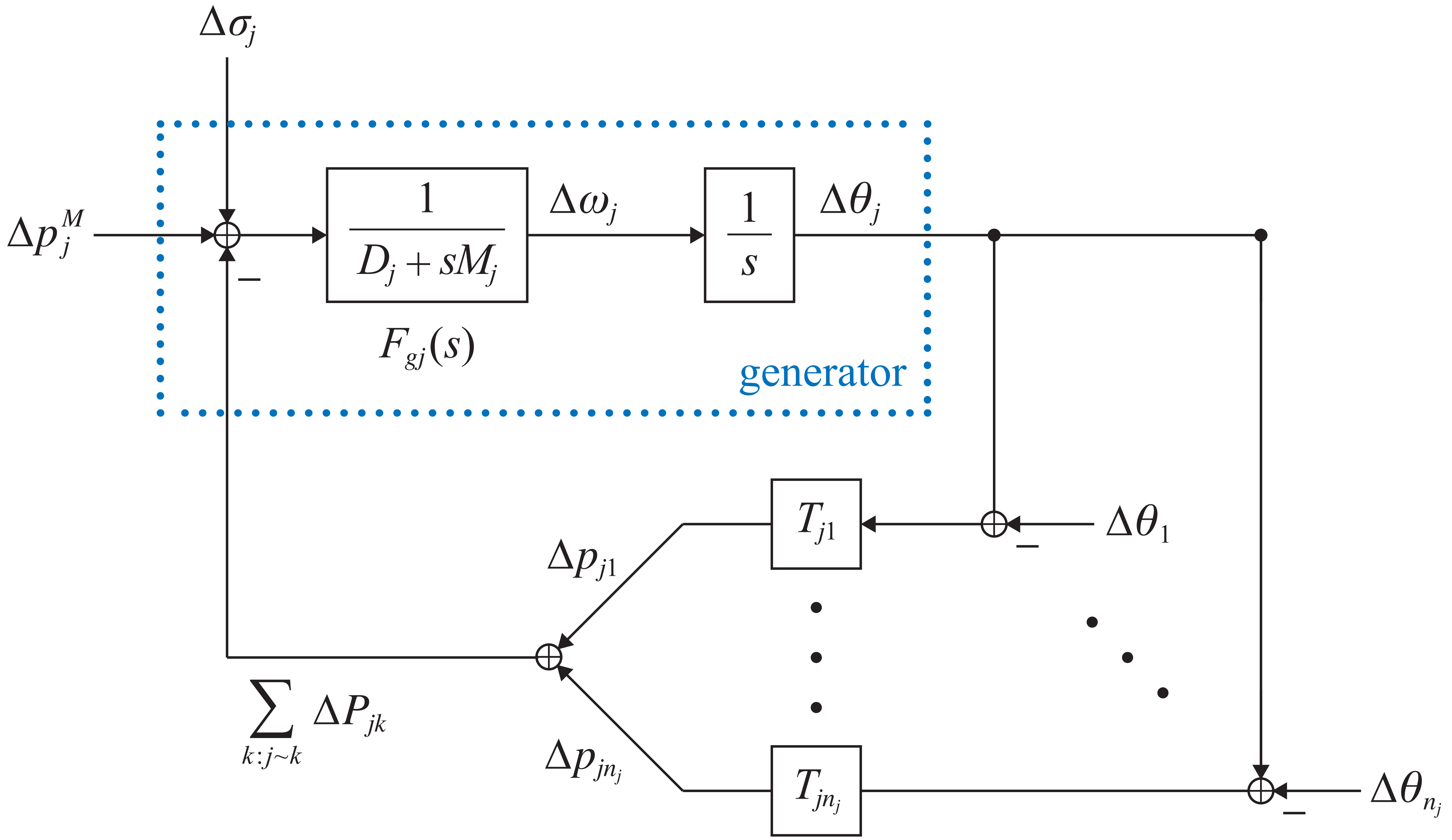
where $T_{jk} := |V_j||V_k|\left(-b_{jk}\right)\cos\left(\theta_j^0 - \theta_k^0\right)$

Primary frequency control

Generator model

$$\Delta\dot{\theta}_j = \Delta\omega_j(t)$$

$$M_j\Delta\dot{\omega}_j + D_j\Delta\omega_j(t) = \Delta p_j^M(t) + \Delta\sigma_j(t) - \sum_{k:j\sim k}\Delta P_{jk}(t)$$



Primary frequency control

Turbine-governor-generator model

$$T_{gj}\,\Delta\dot{a}_j = -\,\Delta a_j(t) \;+\; \Delta u_j(t) \;-\; \frac{\Delta\omega_j(t)}{R_j}$$

$$T_{tj}\,\Delta\dot{p}_j^M = -\,\Delta p_j^M(t) \;+\; \Delta a_j(t)$$

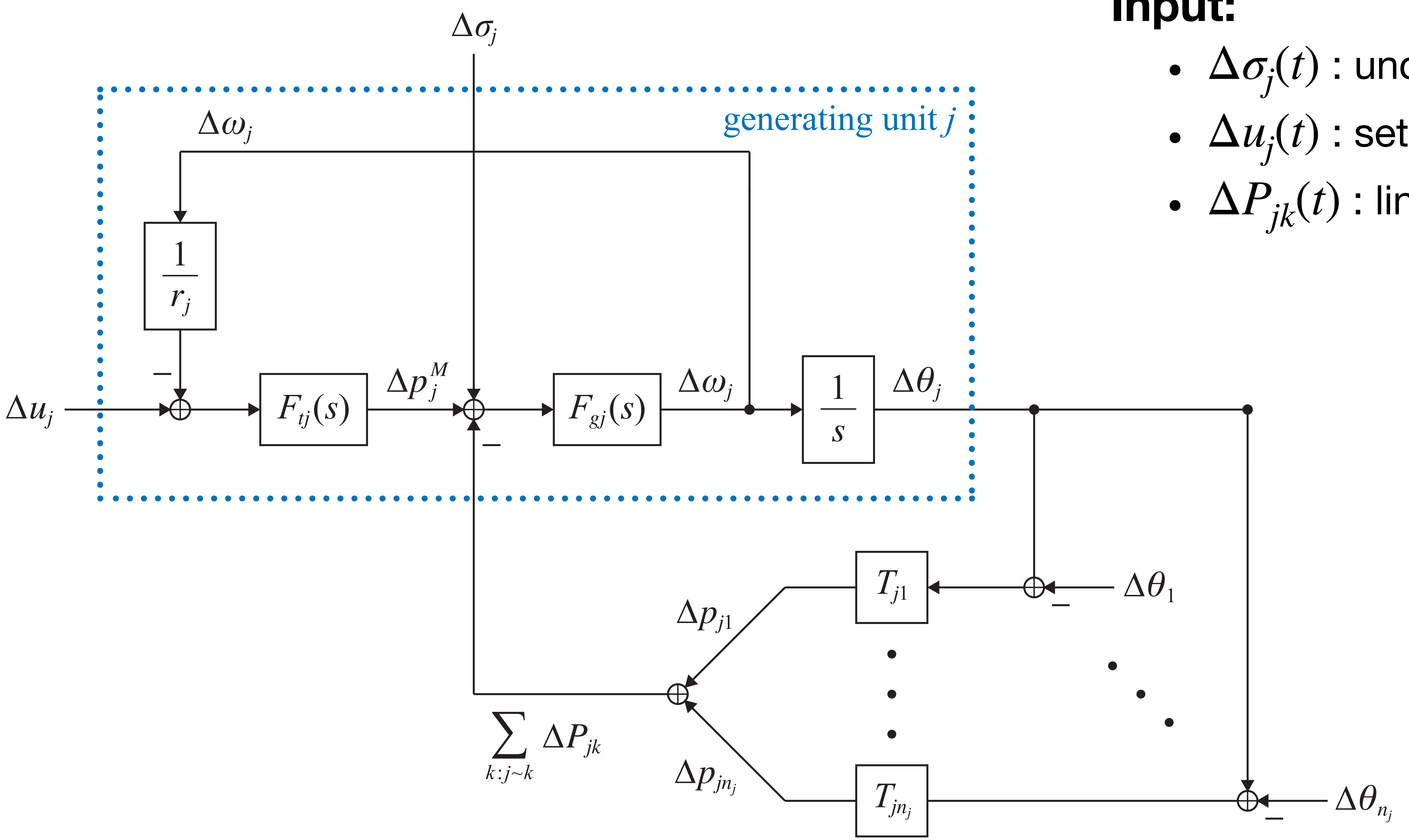
$$M_j\Delta\dot{\omega}_j \;+\; D_j\Delta\omega_j(t) = \;\Delta p_j^M(t) \;+\; \Delta\sigma_j(t) \;-\; \sum_{k:j\sim k}\Delta P_{jk}(t)$$

$$\Delta P_{jk}(t) = \;T_{jk}\left(\Delta\theta_j(t) - \Delta\theta_k(t)\right)$$

$$\Delta\dot{\theta}_j = \;\Delta\omega_j(t)$$

Primary frequency control

Turbine-governor-generator model



Input:

- $\Delta\sigma_j(t)$: uncontrollable injection
- $\Delta u_j(t)$: setpoint adjusted by secondary control
- $\Delta P_{jk}(t)$: line flows to other areas

Primary frequency control

Turbine-governor-generator model

$$T_{gj}\,\Delta\dot{a}_j = -\,\Delta a_j(t) \;+\; \Delta u_j(t) \;-\; \frac{\Delta\omega_j(t)}{R_j}$$

$$T_{tj}\,\Delta\dot{p}_j^M = -\,\Delta p_j^M(t) \;+\; \Delta a_j(t)$$

$$M_j\Delta\dot{\omega}_j \;+\; D_j\Delta\omega_j(t) = \Delta p_j^M(t) \;+\; \Delta\sigma_j(t) \;-\; \sum_{k:j\sim k}\Delta P_{jk}(t)$$

$$\Delta P_{jk}(t) = T_{jk}\left(\Delta\theta_j(t) - \Delta\theta_k(t)\right)$$

$$\Delta\dot{\theta}_j = \Delta\omega_j(t)$$

Equilibrium of primary control: $\Delta\dot{\omega}_j \;=\; \Delta\dot{a}_j \;=\; \Delta\dot{p}_j^M \;=\; 0$ (does not require $\Delta\dot{\theta} = 0$)

Primary frequency control

Equilibrium

Bus-by-line incidence matrix C :

$$C_{jl} \;:=\; \begin{cases} 1 & \text{if } \; l = j \rightarrow k \text{ for some bus } k \\ -1 & \text{if } \; l = i \rightarrow j \text{ for some bus } i \\ 0 & \text{otherwise} \end{cases}$$

Stiffness matrix: $T \;:=\; \text{diag}(T_{jk}, (j,k) \in E)$

Laplacian matrix: $L \;:=\; CTC^T$ and its pseudo-inverse $L^\dagger$

Primary frequency control

Equilibrium

Theorem

Let $x^* \; := \; \left(\Delta\omega^*, \Delta P^*, \Delta\theta^*, \Delta a^*, \Delta p^{M*}\right)$ be an equilibrium driven by step change $\Delta\sigma$ and constant setpoint Δu

Primary frequency control

Equilibrium

Theorem

Let $x^* \;:=\; \left(\Delta\omega^*, \Delta P^*, \Delta\theta^*, \Delta a^*, \Delta p^{M*}\right)$ be an equilibrium driven by step change $\Delta\sigma$ and constant setpoint Δu

1. Local frequency deviations converge to

$$\Delta\omega_j^* \quad = \quad \Delta\omega^* \quad := \quad \frac{\sum_k \left(\Delta u_k + \Delta\sigma_k\right)}{\sum_k \left(D_k + 1/R_k\right)}$$

Primary frequency control

Equilibrium

Theorem

Let $x^* \;:=\; \left(\Delta\omega^*, \Delta P^*, \Delta\theta^*, \Delta a^*, \Delta p^{M*}\right)$ be an equilibrium driven by step change $\Delta\sigma$ and constant setpoint Δu

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2. Line flow deviations converge to

$$\Delta P^* \quad = \quad TC^T L^\dagger \left(\Delta u \;+\; \Delta\sigma \;-\; \Delta\omega^* d\right)$$

where $d \;:=\; (D_j + 1/R_j, j \in \overline{N})$

Primary frequency control

Equilibrium

Theorem

Let $x^* \ := \ \left(\Delta\omega^*, \Delta P^*, \Delta\theta^*, \Delta a^*, \Delta p^{M*}\right)$ be an equilibrium driven by step change $\Delta\sigma$ and constant setpoint Δu

1. Local frequency deviations converge to

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2. Line flow deviations converge to

$$\Delta P^* \quad = \quad TC^T L^\dagger \left(\Delta u \ + \ \Delta\sigma \ - \ \Delta\omega^* d\right)$$

where $d \ := \ (D_j + 1/R_j, j \in \overline{N})$

Secondary control:

- Adjusting Δu to drive $\Delta\omega_j^*$ and ΔP_{jk}^* to 0

Primary frequency control

Example: interconnected system

Model

- $N+1$ areas each modeled as a bus
- $\Delta u_j = 0$ for all j
- Step change: at time 0, $\sigma_j(t)$ changes from 0 to a constant value $\Delta\sigma_j$
- Suppose $\Delta\sigma_j$ are iid random variables with mean $\Delta\bar{\sigma}_j$ and variance v_j^2

Compare the mean & variance of equilibrium frequency deviation $\Delta\omega_j^*$:

- Case 1: the areas (buses) are not connected and operate independently.
- Case 2: the areas (buses) are connected into a network

Primary frequency control

Example: interconnected system

Case 1: independent operation

$$\Delta\omega_j^* \quad = \quad \frac{\Delta\sigma_j}{d_j} \qquad \text{where } d_j := D_j + 1/R_j$$

with $\quad E\Delta\omega_j^* \;=\; \dfrac{\Delta\bar{\sigma}_j}{d_j}, \quad \text{var}\left(\Delta\omega_j^*\right) \;=\; \dfrac{v_j^2}{d_j^2}$

Case 2: interconnected system

$$\Delta\omega^* \quad = \quad \frac{\sum_j \Delta\sigma_j}{\sum_j d_j} \quad = \quad \frac{1}{N+1}\sum_j \frac{\Delta\sigma_j}{\hat{d}} \qquad \text{where } \hat{d}_j := \frac{1}{N+1}\sum_j d_j$$

with $\quad E\Delta\omega^* \;=\; \dfrac{\Delta\hat{\sigma}}{\hat{d}}, \quad \text{var}(\Delta\omega^*) \;=\; \dfrac{1}{N+1}\,\dfrac{\hat{v}^2}{\hat{d}^2} \qquad$ where $\Delta\hat{\sigma}$, $\hat{v}^2$ are avgerages

Frequency control

Model

Linearized around operating point, defined by

$$u_j^0 \ + \ \sigma_j^0 \ = \ \sum_{k:j\sim k} P_{jk}^0$$

Incremental variables (full list)

- $\Delta u_j(t) \ := \ u_j(t) - u_j^0$: adjustment to dispatched setpoint
- $\Delta \theta_j(t) \ := \ \theta_j(t) - \theta_j^0$: incremental angle relative to rotating frame of ω^0
- $\Delta \omega_j(t) = \Delta \dot{\theta}_j(t)$: frequency deviation from operating-point frequency ω^0
- $\Delta P_{jk}(t) \ := \ P_{jk}(t) - P_{jk}^0$: line flow deviation
- $\Delta p_j^M(t) \ := \ p_j^M(t) - P_j^{M0}$: deviation of mechanical power output of turbine
- $\Delta a_j(t) \ := \ a_j(t) - a_j^0$: deviation of valve position of turbine-governor

Outline

1. Background
2. Unit commitment & real-time dispatch
3. Frequency control
   - Model and assumptions
   - Primary control
   - Secondary control
4. Pricing electricity & reserves

Secondary frequency control

Objectives

1. Restore frequency to nominal value
   - Drive $\Delta\omega* = 0$
2. Restore tie-line flows to scheduled values (scheduled by tertiary control)
   - Drive $\Delta P* = 0$ (each bus represents a control area)

Secondary frequency control

Objectives

At equilibrium of primary control :

$$\Delta\omega_j^* \quad = \quad \Delta\omega^* \quad := \quad \frac{\sum_k \left(\Delta u_k + \Delta\sigma_k\right)}{\sum_k \left(D_k + 1/R_k\right)}$$

$$\Delta P^* \quad = \quad TC^T L^\dagger \left(\Delta u \; + \; \Delta\sigma \; - \; \Delta\omega^* d\right)$$

Therefore, need to adjust setpoints $\Delta u(t)$

- $\Delta\omega_j^* = 0$ if $\sum_k \left(\Delta u_k + \Delta\sigma_k\right) = 0$
- $\Delta P_{jk}^* = 0$ if $\Delta u_j + \Delta\sigma_j = 0$

Secondary frequency control

Area control error (ACE)

$$\text{ACE}_j(t) \quad := \quad \sum_{k:j\sim k} \Delta P_{jk}(t) \quad + \quad \beta_j \Delta\omega_j(t)$$

Setpoint adjustment

$$\Delta \dot{u}_j \quad = -\gamma_j \left(\sum_{k:j\sim k} \Delta P_{jk}(t) \quad + \quad \beta_j \Delta\omega_j(t) \right)$$

Implementation

- Real-time measurements of $P_{jk}(t)$ with neighboring areas k are sent to system operator
- System operator centrally computes $\Delta \dot{u}_j$ and dispatch setpoint adjustments $\alpha_{ji}\Delta u_j(t)$ to participating generators i in areal j ($\alpha_{ji} \geq 0$ with $\sum_i \alpha_{ji} = 1$ are called participation factors)

Secondary frequency control

Overall (primary & secondary) model

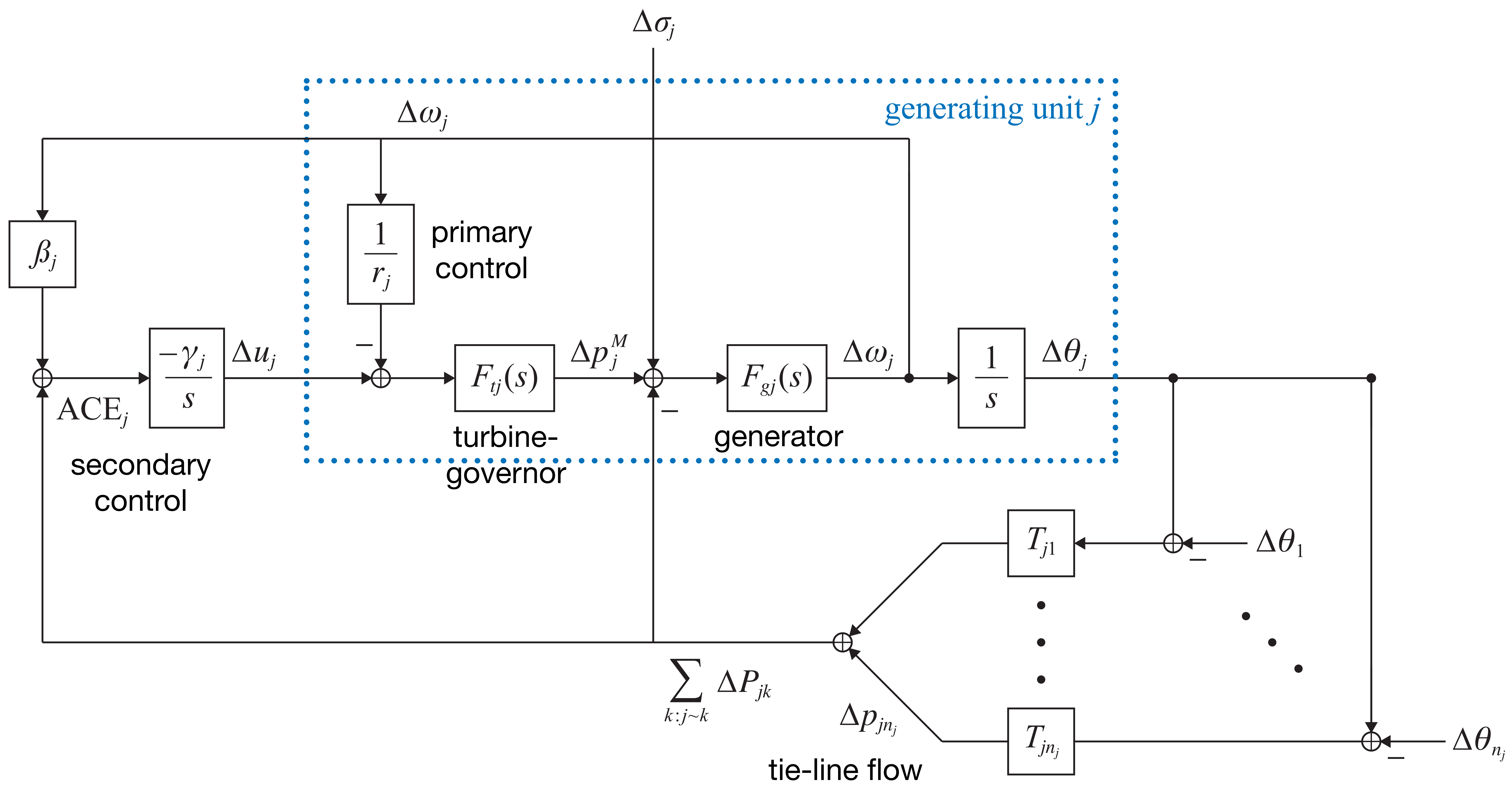
$$\left.\begin{aligned} T_g \Delta\dot{a} &= -\Delta a(t) + \Delta u(t) - R^{-1}\Delta\omega(t) \\ T_t \Delta\dot{p}^M &= -\Delta p^M(t) + \Delta a(t) \end{aligned}\right\} \quad \text{turbine-governor}$$

$$\left.\begin{aligned} M\Delta\dot{\omega} + D\Delta\omega(t) &= \Delta p^M(t) + \Delta\sigma(t) - C\Delta P(t) \\ \Delta P(t) &= TC^T\Delta\theta(t) \\ \Delta\dot{\theta} &= \Delta\omega(t) \\ \Delta\dot{u} &= -\Gamma\left(C\Delta P(t) + B\Delta\omega(t)\right) \end{aligned}\right\} \quad \text{generator}$$

Equilibrium of secondary control: $\Delta\dot{u} = \Delta\dot{\omega} = \Delta\dot{a} = \Delta\dot{p}^M = 0$
(does not req $\Delta\dot{\theta} = 0$)

Secondary frequency control

Overall (primary & secondary) model



Secondary frequency control

Equilibrium

Theorem

Let $x^* \; := \; \left(\Delta u^*, \Delta\omega^*, \Delta P^*, \Delta\theta^*, \Delta a^*, \Delta p^{M*}\right)$ be an equilibrium driven by step change $\Delta\sigma$

1. Frequencies are restored to ω^0 : $\Delta\omega^* = 0$

Secondary frequency control

Equilibrium

Theorem

Let $x^* := \left(\Delta u^*, \Delta\omega^*, \Delta P^*, \Delta\theta^*, \Delta a^*, \Delta p^{M*}\right)$ be an equilibrium driven by step change $\Delta\sigma$

1. Frequencies are restored to ω^0 : $\Delta\omega^* = 0$
2. Line flow are restored to P^0 : $\Delta P^* = 0$

Secondary frequency control

Equilibrium

Theorem

Let $x^* \ := \ \left(\Delta u^*, \Delta\omega^*, \Delta P^*, \Delta\theta^*, \Delta a^*, \Delta p^{M*}\right)$ be an equilibrium driven by step change $\Delta\sigma$

1. Frequencies are restored to ω^0 : $\Delta\omega^* = 0$
2. Line flow are restored to P^0 : $\Delta P^* = 0$
3. Disturbances are compensated for locally at each bus (i.e., in each area) : $\Delta u_j^* + \Delta\sigma_j = 0$

Outline

1. Background
2. Unit commitment & real-time dispatch
3. Frequency control
   - Model and assumptions
   - Primary control
   - Secondary control
4. Pricing electricity & reserves
   - DC power flow model
   - Economic dispatch and LMP
   - LMP properties
   - Security constrained economic dispatch

DC power flow model

Goal: to illustrate the use of DC power flow model that is widely used for market applications

Setup

1. A connected network $G := (\overline{N}, E)$ with $N + 1$ buses and M lines modeled by DC power flow
2. p : real power injections p_j at buses j
    - generator: $p_j > 0$, incurs cost $f_j(p_j)$
    - load: $p_j < 0$, enjoys utility $-f_j(p_j)$
    - capacity limits: $p^{\min} \leq p \leq p^{\max}$
    - supply = demand: $\mathbf{1}^\mathsf{T} p = 0$
3. $P := BC^\mathsf{T} L^\dagger p =: S^\mathsf{T} p$: line power flows
    - $B := \text{diag}\left(b_l, l \in E\right) \succ 0$, C : incidence matrix, $L^\dagger$: pseudo-inverse of Laplacian $L := CBC^\mathsf{T}$
    - line limits: $P^{\min} \leq P = BC^\mathsf{T} L^\dagger p \leq P^{\max}$
    - $S = \left(\partial P / \partial p\right)^\mathsf{T}$ shift factor that maps line vars (e.g. line congestion prices) to nodal var (e.g. nodal congestion prices)

Economic dispatch

LMP

$$\min_{p^{\min} \le p \le p^{\max}} \quad \sum_{j \in \overline{N}} f_j(p_j)$$

$$\text{subject to} \quad \mathbf{1}^\mathsf{T} p = 0 \qquad [\gamma]$$

$$P^{\min} \le S^\mathsf{T} p \le P^{\max} \qquad [\kappa^-, \kappa^+]$$

- p : primal variable
- Associated with each constraint is a Lagrange multiplier: $\gamma \in \mathbb{R}, \ \kappa^- \in \mathbb{R}^M_+, \ \kappa^+ \in \mathbb{R}^M_+$
- Given an optimal dispatch p^* and optimal Lagrange multiplier $\left(\gamma^*, \kappa^{-*}, \kappa^{+*}\right)$, define locational marginal price (LMP):

$$\lambda^* := \gamma^* \mathbf{1} + L^\dagger C B \kappa^* = \gamma^* \mathbf{1} + S\kappa^*$$

where $\kappa^* := \kappa^{-*} - \kappa^{+*}$

Economic dispatch

Settlement rule

Locational marginal price (LMP):

$$\lambda^* := \gamma^* \mathbf{1} + S\kappa^*$$

Settlement rule

- System operator (SO) solves economic dispatch to obtain optimal dispatch p^* and $\left(\gamma^*, \kappa^{-*}, \kappa^{+*}\right)$, and compute LMP λ^*
- Generator that generates $p_j > 0$: is paid $\lambda_j^* p_j$
- Load that consumes $-p_j > 0$: pays $-\lambda_j^* p_j$
- Some markets allow participants to choose their own p_j, some markets dispatch binding p_j^* (e.g. many US markets)

Optimality condition

Assume: cost functions f_j are convex and optimal value of ED is finite

- Optimal Lagrange multiplier $\left(\gamma^*, \kappa^{-*}, \kappa^{+*}\right)$ and hence LMP λ^* exist; moreover strong duality holds
- p^* is an optimal dispatch if and only if p^* and $\left(\gamma^*, \kappa^{-*}, \kappa^{+*}\right)$ satisfy the KKT condition:
  - Primal feasibility: $p^{\min} \leq p^* \leq p^{\max}, \quad \mathbf{1}^\mathsf{T} p^* = 0, \quad P^{\min} \leq S^\mathsf{T} p^* \leq P^{\max}$
  - Dual feasibility: $\kappa^{-*} \geq 0, \quad \kappa^{+*} \geq 0$
  - Stationarity: $$f_j'(p_j^*) \begin{cases} = \lambda_j^* \text{ if} & p_j^{\min} < p_j^* < p_j^{\max} \\ > \lambda_j^* \text{ only if} & p_j^* = p_j^{\min} \\ < \lambda_j^* \text{ only if} & p_j^* = p_j^{\max} \end{cases}$$ marginal unit
  - Complementary slackness:

$$\left(\kappa^{-*}\right)^\mathsf{T} \left(P^{\min} - S^\mathsf{T} p^*\right) = 0, \quad \left(\kappa^{+*}\right)^\mathsf{T} \left(S^\mathsf{T} p^* - P^{\max}\right) = 0$$

LMP properties

We study properties of optimal dispatch p^* and LMP λ^*

- Competitive equilibrium
- Nodal and line congestion price κ^*
- Revenue adequacy
- Price reference bus

These properties are consequences of DC power flow equation and KKT condition

Competitive equilibrium

An important justification for pricing according to LMP is that optimal dispatch and LMP (p^*, λ^*) is a competitive equilibrium:

- Market clearing: supply = demand, $\mathbf{1}^\mathsf{T} p^* = 0$
- Power flows satisfy line limits: $P^{\min} \leq S^\mathsf{T} p \leq P^{\max}$
- Welfare optimization: p^* solves economic dispatch
- Incentive compatibility: individually optimal p_j^* that solve

$$\max_{p_j^{\min} \leq p_j \leq p_j^{\max}} \quad \lambda_j^* p_j \; - \; f_j(p_j)$$

  turn out to be socially optimal

LMP consists of Lagrange multipliers associated with non-local constraints (only) that couple individual decisions p_j

- It prices externalities of unit j's decisions and aligns individual optimality with social optimality

Nodal and line congestion prices

LMP: $\lambda^* := \gamma^* \mathbf{1} + S\kappa^*$

Energy price γ^*

- Same prices $\lambda_j^* = \gamma^*$ at all buses j if no congestion ($P^{\min} < S^\mathsf{T} p < P^{\max} \;\Rightarrow\; \kappa^* = 0$)
- In general, energy price $\gamma^* = \dfrac{1}{N+1}\mathbf{1}\lambda^*$, the average LMP (system λ)

Nodal and line congestion prices

LMP: $\lambda^* := \gamma^* \mathbf{1} + S\kappa^*$

Line congestion price κ^*

- Interpret $\kappa^* := \kappa^{-*} - \kappa^{+*}$ as line congestion prices, for two reasons
- κ_l^* : shadow price of line capacities $(P_l^{\min}, P_l^{\max})$ at l because (Envelop Theorem)

$$\frac{\partial f^*}{\partial P_l^{\min}}(P^{\min}, P^{\max}) = \kappa_l^{-*}$$

$$\frac{\partial f^*}{\partial P_l^{\max}}(P^{\min}, P^{\max}) = -\kappa_l^{+*}$$

i.e., each unit of additional capacities reduces optimal cost f^* by $(\kappa^{-*}, \kappa^{+*}) \geq 0$

- $-\kappa_l^* P_l \geq 0$: cost of carrying P_l on line l (due to complementary slackness)

Nodal and line congestion prices

LMP: $\lambda^* := \gamma^* \mathbf{1} + S\kappa^*$

Nodal congestion price $c^* := S\kappa^*$

- c_j^* : marginal cost of serving 1 additional load at node j
- Main observation : $S = \left(\dfrac{\partial P}{\partial p}\right)^{\mathsf{T}}$ because $P = S^{\mathsf{T}} p$
  - $\dfrac{\partial P_l}{\partial p_j} \Delta p_j = S_{jl} \Delta p_j$ = increase in power flow at line l due to additional injection Δp_j at node j
  - $\therefore\ -\kappa_l^* \left(S_{jl} \Delta p_j\right)$ = increase in congestion cost at line l due to additional injection Δp_j at node j
  - $\therefore - \sum_l S_{jl} \kappa_l^* \Delta p_j$ = increase in congestion cost over network due to additional injection Δp_j at node j
  - $\therefore\ c_j^* := \sum_l S_{j} \kappa_l^*$ = increase in congestion cost over network due to 1 additional unit of load at node j

Nodal and line congestion prices

Negative price

LMP: $\lambda_j^* := \gamma^* + c_j^*$

- Since the nodal congestion price c_j^* can be positive or negative, λ_j^* can be negative
- Negative λ_j^* are not uncommon in practice, e.g., in CAISO market during daytime when there is a lot of solar generation
- Negative λ_j^* can be due to congestion or nonzero generation limit $p_j^{\min} > 0$

Revenue adequacy

System operator collects payment $\lambda_j^*(-p_j^*)$ from load j and pays $\lambda_j^* p_j^*$ to generator j

The residue is merchandizing surplus

$$\text{MS} := -\sum_j \lambda_j^* p_j^* \ = -\left(\lambda^*\right)^{\mathsf{T}} p^*$$

- Substituting $\lambda^* := \gamma\mathbf{1} + S\kappa^*$ and complementary slackness yield

$$\text{MS} \ = \ \left(\kappa^{+*}\right)^{\mathsf{T}} P^{\max} \ + \ \left(\kappa^{-*}\right)^{\mathsf{T}}(-P^{\min}) \ \geq \ 0$$

  i.e., SO will not run cash negative. This is called revenue adequate

- MS > 0 if and only if there is congestion ($\kappa^{-*} > 0$ or $\kappa^{+*} > 0$)

Price reference bus

Summary

Price reference (slack) bus r : injections p_{-r} at non-price-reference buses can be arbitrarily chosen and always by p_r at bus r, so that $p_r = -\mathbf{1}^{\mathsf{T}} p_{-r}$

May be different from angle reference bus 0 where $\theta_0 := 0$

Can write everything in terms of injections p_{-r} and shift factor S_r at non-price-ref buses only

- DC power flow equations, economic dispatch (DC OPF), LMP λ^*

Optimal dispatch p^*, LMP λ^* , and line flows P do **not** depend on choice of r

- Lagrange multiplier γ^* does

Disadvantages of designating a price reference bus r

- Somewhat arbitrary (typically a bus where there is large generator that is rarely bottlenecked)
- Reduced Laplacian matrix $L_r := C_{-r} B C_{-0}^{sfT}$ is not principal submatrix of L, hence may not be symmetric nor nonsingular (unless $r = 0$)
- Reduced shift factor $S_r := L_r^{-1} C_{-0} B$ depends on r (when L_r is nonsingular)
- Seems unnecessary (can express DC power flow, economic dispatch, and LMP in terms of $L^\dagger$)

Price reference bus

In terms of p_{-r} and S_r

Partition node-by-line incidence matrix

$$C =: \begin{bmatrix} c_0^\mathsf{T} \\ C_{-0} \end{bmatrix}, \qquad C =: \begin{bmatrix} C_{-r} \\ c_r^\mathsf{T} \end{bmatrix}$$

DC power flow equations become

$$\begin{bmatrix} p_{-r} \\ p_r \end{bmatrix} = \begin{bmatrix} C_{-r} \\ c_r^\mathsf{T} \end{bmatrix} P, \qquad P = B \begin{bmatrix} c_0 C_{-0}^\mathsf{T} \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_{-0} \end{bmatrix}$$

leading to

$$P = \left(B C_{-0}^\mathsf{T} L_r^{-1} \right) p_{-r} =: S_r^\mathsf{T} p_{-r}$$

Economic dispatch becomes:

$$\min_{p^{\min} \le p \le p^{\max}} \sum_{j \in \overline{N}} f_j(p_j) \qquad \text{s.t.} \quad \mathbf{1}^\mathsf{T} p = 0 \qquad [\gamma]$$

$$P^{\min} \le S_r^\mathsf{T} p_{-r} \le P^{\max} \qquad [\kappa^-, \kappa^+]$$

Price reference bus

In terms of p_{-r} and S_r

LMP: $\lambda^* \;=\; \gamma^* \;+\; \begin{bmatrix} S_r\kappa^* \\ 0 \end{bmatrix}$

where $S_r := L_r^{-\mathsf{T}} C_{-0} B$ and $\kappa^* := \kappa^{-*} - \kappa^{+*}$

Theorem

Suppose cost functions f_j are convex (and hence differentiable), so that KKT is N&S optimality condition.

Fix p^* and let

$$\tilde{\gamma}^* \;=\; \gamma^* - s_r^{\mathsf{T}}\kappa^*, \qquad \tilde{\kappa}^{-*} \;=\; \kappa^{-*}, \qquad \tilde{\kappa}^{+*} \;=\; \kappa^{+*}$$

1. $\tilde{\lambda} := \tilde{\gamma}^* \mathbf{1} + S\kappa^* = \lambda^*$
2. $(p^*, \tilde{\lambda}^*)$ is primal-dual optimal for original ED iff (p^*, λ^*) is primal-dual optimal for reduced ED
3. $P^* = S^{\mathsf{T}} p = S_r^{\mathsf{T}} p_{-r}$

Outline

1. Background
2. Unit commitment & real-time dispatch
3. Frequency control
   - Model and assumptions
   - Primary control
   - Secondary control
4. Pricing electricity & reserves
   - DC power flow model
   - Economic dispatch and LMP
   - LMP properties
   - Security constrained economic dispatch

Secure operation

System operator needs to deal with uncertainties

- … discrete uncertainty: outages of generators, lines, transformers
- … continuous uncertainty: random fluctuations of renewable generations and loads

Security constrained economic dispatch jointly optimizes energy and reserves

- Can be formulated as two-stage optimization with recourse

Dispatch & reserve decisions

Uncertain generation and demand take one of K values $(g_k, d_k) \in \mathbb{R}_+^{2(N+1)}$ with probability $w_k > 0$

- $\sum_k w_k = 1$

First-stage decision before (g, d) is realized

- Dispatch p_j and reserve capacities $(r_j^{\min}, r_j^{\max})$
- Capacity constraints: $p_j^{\min} \le p_j + r_j^{\min} \le p_j + r_j^{\max} \le p_j^{\max}$

Second-stage decision after (g, d) is realized

- Adjustment r_{kj} if (g_k, d_k) is realized so that actual injection is $p_j + r_{kj}$
- Capacity constraints: $r_j^{\min} \le r_{kj} \le r_j^{\max}$

System reserve requirement

System-wide reliability requirements on reserve capacities $(r_j^{\min}, r_j^{\max})$ imposed by SO

$$h_k(r^{\min}, r^{\max}) \ := \ \sum_j h_{kj}(r_j^{\min}, r_j^{\max}) \ \geq \ 0$$

- In general, h_{kj} can be positive or negative

For example: total reserve must cover outage of largest generator

- $\sum_{j \neq j_k} r_j^{\min} \geq \max_j p_i^{\max}$ where $j_k := \arg\max_j p_j^{\max}$
- Can be expressed as: $h_{kj}(r_j^{\min}, r_j^{\max}) = r_j^{\min} - \alpha_j p_{j_k}^{\max}$ for $j \neq j_k$ with $\alpha_j \geq 0$ and $\sum_{j \neq j_k} \alpha_j = 1$

Security constrained ED

SCED can be formulated as two-stage optimization with recourse: 1st-stage problem

$$\min_{p,r^{\min},r^{\max}} \quad \sum_{k=1}^{K} w_k \, Q_k(p, r^{\min}, r^{\max})$$

$$\text{s.t.} \qquad p^{\min} \leq p + r^{\min}, \;\; p + r^{\max} \leq p^{\max} \qquad [\alpha^-, \alpha^+]$$

$$h_k\left(r^{\min}, r^{\max}\right) := \sum_j h_{kj}\left(r_j^{\min}, r_j^{\max}\right) \;\geq\; 0 \qquad [\mu_k]$$

- 1st-stage constraints (energy+reserve capacity, rsystem eliability requirement) do not involve uncertainty
- Min expected 2nd-stage dispatch cost Q_k
- For each scenario k, Q_k solves economic dispatch after uncertainty is realized

Security constrained ED

2nd-stage problem: Q_k solves economic dispatch in scenario k after uncertainty is realized

$$
\begin{aligned}
Q_k(p, r^{\min}, r^{\max}) \;:=\; \min_{r_k} \;& f_k(p + r_k) \;:=\; \sum_j f_{kj}(p_j + r_{kj}) \\
\text{s.t.} \;& \mathbf{1}^\mathsf{T}(p + r_k + g_k - d_k) \;=\; 0 && [\gamma_k] \\
& P^{\min} \;\le\; S^\mathsf{T}(p + r_k + g_k - d_k) \;\le\; P^{\max} && [\kappa_k^-, \kappa_k^+] \\
& r^{\min} \;\le\; r_k \;\le\; r^{\max} && [\beta_k^-, \beta_k^+]
\end{aligned}
$$

- 2nd-stage problem optimizes reserve decisions r_k in response to (g_k, d_k), given 1st-stage decision $(p_j, r_j^{\min}, r_j^{\max})$
- Same as economic dispatch capacity constraint, power balance, and line limits

LMP

Since 2nd-stage problem is separable in k, SCED is equivalent to single-stage optimization

$$\min_{\substack{p,\, r^{\min},\, r^{\max} \\ (r_k,\, k \geq 1)}} \quad \sum_k w_k f_k(p + r_k) := \sum_k w_k \sum_j f_{kj}(p_j + r_{kj})$$

s.t. 1st and 2nd-stage constraints

Define LMP for each scenario k:

$$\lambda_k^* \ := \ \gamma_k^* \mathbf{1} \ + \ S\kappa_k^*$$

where $\kappa_k^* := \kappa_k^{-*} - \kappa_k^{+*}$

Let ξ^* denote an optimal dual variable (vector)

Optimality condition

Assume f_{kj}, h_{kj} are convex, optimal cost is finite, and Slater condition ($p_j^{\min} < p_j^{\max}$) is satisfied

Then

1. LMP exists
2. A feasible (x^*, ξ^*) is (primal-dual) optimal if and only if
   - Stationarity:

$$w_k \nabla f_k(p^* + r_k^*) \;=\; \lambda_k^* + \beta_k^*, \quad \sum_k \mu_k^* \nabla h_k\left(r^{\min *}, r^{\max *}\right) \;=\; 0, \quad \alpha^* \;=\; \sum_k \beta_k^*$$

   - Complementary slackness: for decentralized constraints

$$\left(\alpha^{-*}\right)^{\mathsf{T}}\left(p^{\min} - p^* - r^{\min *}\right) \;=\; 0, \qquad \left(\alpha^{+*}\right)^{\mathsf{T}}\left(p^* + r^{\max *} - p^{\max}\right) \;=\; 0$$

$$\left(\beta_k^{-*}\right)^{\mathsf{T}}\left(r^{\min *} - r_k^*\right) \;=\; 0, \qquad \left(\beta_k^{+*}\right)^{\mathsf{T}}\left(r_k^* - r^{\max *}\right) \;=\; 0$$

Optimality condition

Assume f_{kj}, h_{kj} are convex, optimal cost is finite, and Slater condition ($p_j^{\min} < p_j^{\max}$) is satisfied

Then

1. LMP exists
2. A feasible (x^*, ξ^*) is (primal-dual) optimal if and only if
   - Stationarity:

$$w_k \nabla f_k(p^* + r_k^*) \;=\; \lambda_k^* + \beta_k^*, \quad \sum_k \mu_k^* \nabla h_k\left(r^{\min *}, r^{\max *}\right) \;=\; 0, \quad \alpha^* \;=\; \sum_k \beta_k^*$$

   - Complementary slackness: for coupled constraints

$$\mu_k^* h_k\left(r^{\min *}, r^{\max *}\right) \;=\; 0$$

$$\left(\kappa^{-*}\right)^{\mathsf{T}} \left(P^{\min} - S^{\mathsf{T}}(p^* + r_k^* + g_k - d_k)\right) \;=\; 0$$

$$\left(\kappa^{+*}\right)^{\mathsf{T}} \left(S^{\mathsf{T}}(p^* + r_k^* + g_k - d_k) - P^{\max}\right) \;=\; 0$$

ICRA settlement rule

1. Energy prices (scenario-dependent LMP) λ_k^*/w_k
    - In scenario k, unit j that provides energy $p + r_{kj}$ is paid $\lambda_{kj}^* \left(p + r_{kj}\right)/w_k$
2. Reserve payment $\sum_k \mu_k^* \, h_{kj}\left(r_j^{\min}, r_j^{\max}\right)$
    - Regardless of scenario at delivery time, unit j that provides reserve capacities $\left(r_j^{\min}, r_j^{\max}\right)$ is paid $\displaystyle\sum_k \mu_k^* \, h_{kj}\left(r_j^{\min}, r_j^{\max}\right)$

ICRA settlement rule

1. Incentive compatible in expectation

   - Unit j prefers $x_j^* := \left(p_j^*, r_j^{\min *}, r_j^{\max *}, r_{kj}^* \ \forall k\right)$ that max its expected (energy & reserve) surplus

$$\max_{x_j} \quad \sum_k w_k\Big(\lambda_{kj}^*(p_j + r_{kj})/w_k - f_{kj}(p_j + r_{kj})\Big) + \sum_k \mu_k^* h_{kj}(r_j^{\min}, r_j^{\max})$$

$$\text{s.t.} \quad p_j^{\min} \le p_j + r_j^{\min} \le p_j + r_j^{\max} \le p_j^{\max}, \quad r_j^{\min} \le r_{kj} \le r_j^{\max}$$

   - Settlement rule is incentive compatible in expectation if socially optimal x^* also max every unit j's expected surplus

2. Revenue adequate

   - Merchandizing surplus in scenario k is:

$$\text{MS}_k \ := -\sum_j \frac{1}{w_k} \lambda_{kj}^*(p_j^* + r_{kj}^*) - \sum_i \sum_j \mu_i^* \, h_{ij}\left(r_j^{\min}, r_j^{\max}\right)$$

   - Settlement rule is revenue adequate in each scenario k if $\text{MS}_k \ge 0$

ICRA settlement rule

3. Reserve payment balance

    - Settlement rule is reserve payment balance if

$$\sum_j \sum_k \mu_k^* \, h_{kj}\left(r_j^{\min}, r_j^{\max}\right) \;=\; 0$$

    - i.e., units that need more reliability exactly compensate those that can provide more reliability

ICRA settlement rule

Theorem

Suppose cost functions f_{kj} and reserve requirement functions h_{kj} are convex and differentiable, the 2-stage problem has a finite optimal value. Then the settlement rule is

1. Incentive compatible in expectation (and in each scenario)
2. Revenue adequate in each scenario k, i.e., $\text{MS}_k \geq 0$
3. Balanced reserved payment, i.e., $\sum_j \sum_k \mu_k^* h_{kj}\left(r_j^{\min}, r_j^{\max}\right) = 0$

Summary

Central challenge: balance supply & demand second-by-second

- While satisfying operational constraints, e.g. injection/voltage/line limits
- Unlike usual commodities, electricity cannot (yet) be stored in large quantity

This is achieved through a complex set of control and pricing mechanisms that operate in concert across multiple timescales

- Slow timescale mechanisms (minutes and up) can be formulated as OPF problems
- Fast timescales (seconds to minutes) can be formulated as feedback control problems