Power System Analysis

Chapter 7 System operation: estimation and control

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Outline

1. State estimation
2. Voltage control
3. Radial network identification

Network model

State estimation is a key building block for numerous power system applications

- e.g. energy management systems for dispatch of generations and loads, voltage control, …

1. Network $G := (\overline{N}, E)$
   - $\overline{N} := \{0\} \cup N := \{0\} \cup \{1,\ldots,N\}$: buses/nodes/terminals
   - $E \subseteq \overline{N} \times \overline{N}$: lines/branches/links/edges
2. Network state
   - Given: reference angle θ_0
   - State: $x := (\theta, |V|) := (\theta_j, |V_0|, |V_j|, j \in N) \in \mathbb{R}^{2N+1}$
3. State estimation
   - Estimate state $x \in \mathbb{R}^{2N+1}$ from partial and noisy measurements $y \in \mathbb{R}^K$

Network model

Noisy measurements

Partial measurements with additive noise z typically consist of:

1. Voltages $(\theta_j, |V_j|)$ at subset $N_1 \subset \overline{N}$ of buses:

$$y_{2j} \;=\; \theta_j + z_{2j}, \qquad y_{2j+1} \;=\; |V_j| + z_{2j+1}, \qquad j \in N_1$$

2. Real and reactive power injections (p_j, q_j) at subset $N_2 \subset \overline{N}$ of buses:

$$y_{2j} \;=\; p_j + z_{2j}, \qquad y_{2j+1} \;=\; q_j + z_{2j+1}, \qquad j \in N_2$$

where the injections (p_j, q_j) satisfy power flow equations (polar form)

$$p_j \;=\; f_j(x) := \sum_{k:k\sim j} \left(g^s_{jk} + g^m_{jk}\right) |V_j|^2 \;-\; \sum_{k:k\sim j} |V_j|\,|V_k| \left(g^s_{jk} \cos\theta_{jk} + b^s_{jk} \sin\theta_{jk}\right)$$

$$q_j \;=\; g_j(x) := -\sum_{k:k\sim j} \left(b^s_{jk} + b^m_{jk}\right) |V_j|^2 - \sum_{k:k\sim j} |V_j|\,|V_k| \left(g^s_{jk} \sin\theta_{jk} - b^s_{jk} \cos\theta_{jk}\right)$$

Hence: $$y_{2j} \;=\; f_j(x) + z_{2j}, \qquad y_{2j+1} \;=\; g_j(x) + z_{2j+1}, \qquad j \in N_2$$

Network model

Noisy measurements

3. Real and reactive powers (P_{jk}, Q_{jk}) on subset $E_1 \subseteq E$ of lines:

$$y_{2l} \;=\; P_l(x) + z_{2l}, \qquad y_{2l+1} \;=\; Q_l(x) + z_{2l+1}, \qquad l \in E_1$$

where

$$P_{jk}(x) := \left(g_{jk}^s + g_{jk}^m\right)|V_j|^2 \;-\; |V_j||V_k|\left(g_{jk}^s \cos\theta_{jk} + b_{jk}^s \sin\theta_{jk}\right)$$

$$Q_{jk}(x) := -\left(b_{jk}^s + b_{jk}^m\right)|V_j|^2 - |V_j||V_k|\left(g_{jk}^s \sin\theta_{jk} - b_{jk}^s \cos\theta_{jk}\right)$$

State estimation

Given partial and noisy measurements

$$y \;=\; f(x) + z$$

where

- $x \in \mathbb{R}^{2N+1}$: network state
- $z \in \mathbb{R}^K$: additive noise
- $y \in \mathbb{R}^K$: measurement
- $f : \mathbb{R}^{2N+1} \to \mathbb{R}^K$: network model

State estimation computes an estimate $\hat{x}$ of the state x from measurement y by solving:

$$\hat{x} \;:=\; \arg \min_{x \in \mathbb{R}^{2N+1}} \; (y - f(x))^{\mathsf{T}} R^{-1} (y - f(x))$$

where R is positive definite normalization matrix

State estimation

State estimation computes an estimate $\hat{x}$ of the state x from measurement y by solving:

$$\hat{x} \quad := \quad \arg \min_{x \in \mathbb{R}^{2N+1}} (y - f(x))^{\mathsf{T}} R^{-1} (y - f(x))$$

where R is positive definite normalization matrix

- e.g. $R := E(z - Ez)(z - Ez)^{\mathsf{T}}$ is covariance matrix of noise z
- This problem is called least square estimation or nonlinear regression

State estimation

Linear regression

Linearize power flow model $f(x)$ around operating point x_0:

$$y_0 + \Delta y := y = f(x_0) + \frac{\partial f}{\partial x}(x_0)\Delta x + z$$

Assume: x_0, f are known and $y_0 = f(x_0)$

Then

$$\Delta y = F\Delta x + z$$

where $F := \frac{\partial f}{\partial x}(x_0)$ is $K \times (2N+1)$ Jacobian matrix of f at x_0

State estimation becomes linear regression:

$$\hat{\Delta} x \quad := \quad \arg \min_{\Delta x \in \mathbb{R}^{2N+1}} \; (\Delta y - F\Delta x)^{\top} R^{-1} (\Delta y - F\Delta x)$$

State estimation

Linear regression

State estimation becomes linear regression:

$$\hat{\Delta}x \quad := \quad \arg \min_{\Delta x \in \mathbb{R}^{2N+1}} \ (\Delta y - F\Delta x)^\mathsf{T} R^{-1} (\Delta y - F\Delta x)$$

Estimation error:

$$\epsilon^2 \ := \ \min_{\Delta x} \ z^\mathsf{T} R^{-1} z \ = \ \min_{\Delta x} \ (\Delta y - F\Delta x)^\mathsf{T} R^{-1} (\Delta y - F\Delta x)$$

To simplify notation, consider normalized quantities:

$$\Delta \bar{y} \ := \ R^{-1/2} \Delta y, \qquad \bar{F} \ := \ R^{-1/2} F$$

State estimation becomes:

$$\min_{\Delta x} \ z^\mathsf{T} R^{-1} z \ = \ \min_{\Delta x} \ \left\| \, \Delta \bar{y} - \bar{F} \Delta x \, \right\|_2^2$$

State estimation

Solution

State estimation:

$$\min_{\Delta x} \left\| \Delta \bar{y} - \bar{F} \Delta x \right\|_2^2$$

General solution:

$\hat{\Delta} x = \bar{F}^{\dagger} \Delta \bar{y}$ where $\bar{F}^{\dagger}$ is pseudo-inverse of normalized Jacobian $\bar{F} := R^{-1/2} F$

State estimation

Special cases

1. More measurements than state vars $K \geq 2N+1$: When columns of F (and hence $\bar{F}$) are linearly independent, the unique optimal estimate is:

$$\hat{\Delta}x = \left(\bar{F}^\top \bar{F}\right)^{-1} \bar{F}^\top \Delta\bar{y}$$

with minimum estimation error:

$$\epsilon^2 = \left\| \Delta\bar{y} - \bar{F}\hat{\Delta}x \right\|_2^2 = \left\| \Delta\bar{y} \right\|_2^2 - \left\| \left(\bar{F}^\top \bar{F}\right)^{-1/2} \bar{F}^\top \Delta\bar{y} \right\|_2^2$$

The estimated state is:

$$x_0 + \hat{\Delta}x = x_0 + \left(F^\top R^{-1} F\right)^{-1} F^\top R^{-1} \Delta y$$

Redundant measurements enable state estimation

State estimation

Special cases

2. Fewer measurements than state vars $K < 2N + 1$: When rows of F (and hence $\bar{F}$) are linearly independent, the unique optimal estimate is:

$$\hat{\Delta} x \ = \ \bar{F}^{\mathsf{T}} \left(\bar{F}\bar{F}^{\mathsf{T}}\right)^{-1} \Delta \bar{y}$$

with zero estimation error $\epsilon^2 = 0$

There is a subspace of solutions to the linear regression and $\hat{\Delta} x$ is minimum-norm solution.

Typically, insufficient measurement produces poor state estimate

Outline

1. State estimation
2. Voltage control
    - Linear DistFlow model
    - Decentralized control: convergence and optimality
3. Radial network identification

volt/var control

Stabilize voltages on distribution grid by adapting reactive power injections

- e.g., at inverters, capacitor banks

Questions we will study

- How to design simple control schemes?
- What is the dynamic behavior of closed-loop system?
- What is the optimality of closed-loop system?

Design and analysis method

- Use LinDistFlow model due to its analytical properties

volt/var control

Network model

At each bus j, there are

- Fixed and given active and reactive load $\left(p_j^0, q_j^0\right)$
- Possibly a DER (e.g. inverter) with fixed p_j (e.g. PV generation) and controllable q_j

Notation: write $s = (p, q) \in \mathbb{R}^{2N}$ and $v \in \mathbb{R}^N$ at non-reference buses, instead of $(\hat{s}, \hat{v})$

From linear solution theorem:

$$v \;=\; v_0\, \mathbf{1} + 2\left(R(p - p^0) + X(q - q^0)\right)$$

Or

$$v(q) = 2Xq + \tilde{v}$$

where $\tilde{v} := v_0\mathbf{1} + 2R(p - p^0) - 2Xq^0$ independent of the control q

volt/var control

Inverter model

At each bus j, the reactive power q_j is constrained to stay in the intersection of

- Capacity limt $\{q_j : p_j^2 + q_j^2 \leq \sigma^2\}$ which depends on p_j (e.g. PV generation), and
- Power factor limit $-\phi_j \leq \tan^{-1}(q_j/p_j) \leq \phi_j$

Hence q_j must lie in

$$U_j \ := \ U_j(p_j) \left\{ \ q_j \ : \ \underline{q}_j \ \leq \ q_j \ \leq \ \overline{q}_j \ \right\}$$

where $\overline{q}_j := \min \left\{ p_j \tan \phi_j, \ \sqrt{\sigma^2 - p_j^2} \right\}$ and $\underline{q}_j := \max \left\{ -p_j \tan \phi_j, \ -\sqrt{\sigma^2 - p_j^2} \right\}$

volt/var control

Local memoryless control

Let v^{ref} = given vector of reference voltages at buses $j > 0$

Control goal: design $q \in U$ to drive voltages towards v^{ref}

- Local control: $q_j(t+1)$ depends only on $v_j(t)$, not voltages $v_k(t)$ at buses $k \neq j$
- Memoryless control: $q_j(t+1)$ depends only on $v_j(t)$, not on $\left(v_j(s), s < t\right)$

Restrict control law $u_j : \mathbb{R} \to \mathbb{R}$ to depend on voltage error $v_j(t) - v_j^{\text{ref}}$

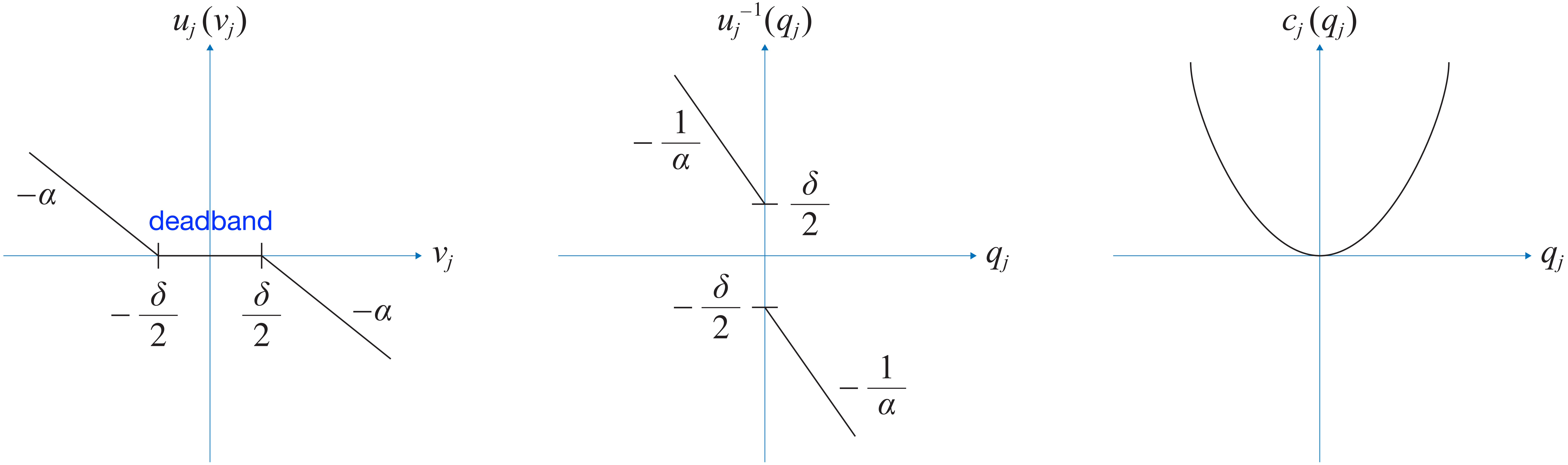
$$q_j(t+1) \;=\; \left[u_j\left(v_j(t) - v_j^{\text{ref}}\right)\right]_{U_j}, \qquad j = 1,\ldots,N$$

i.e. we are to design u_j that map voltage errors $v_j(t) - v_j^{\text{ref}}$ to reactive power settings $q_j(t+1)$

volt/var control

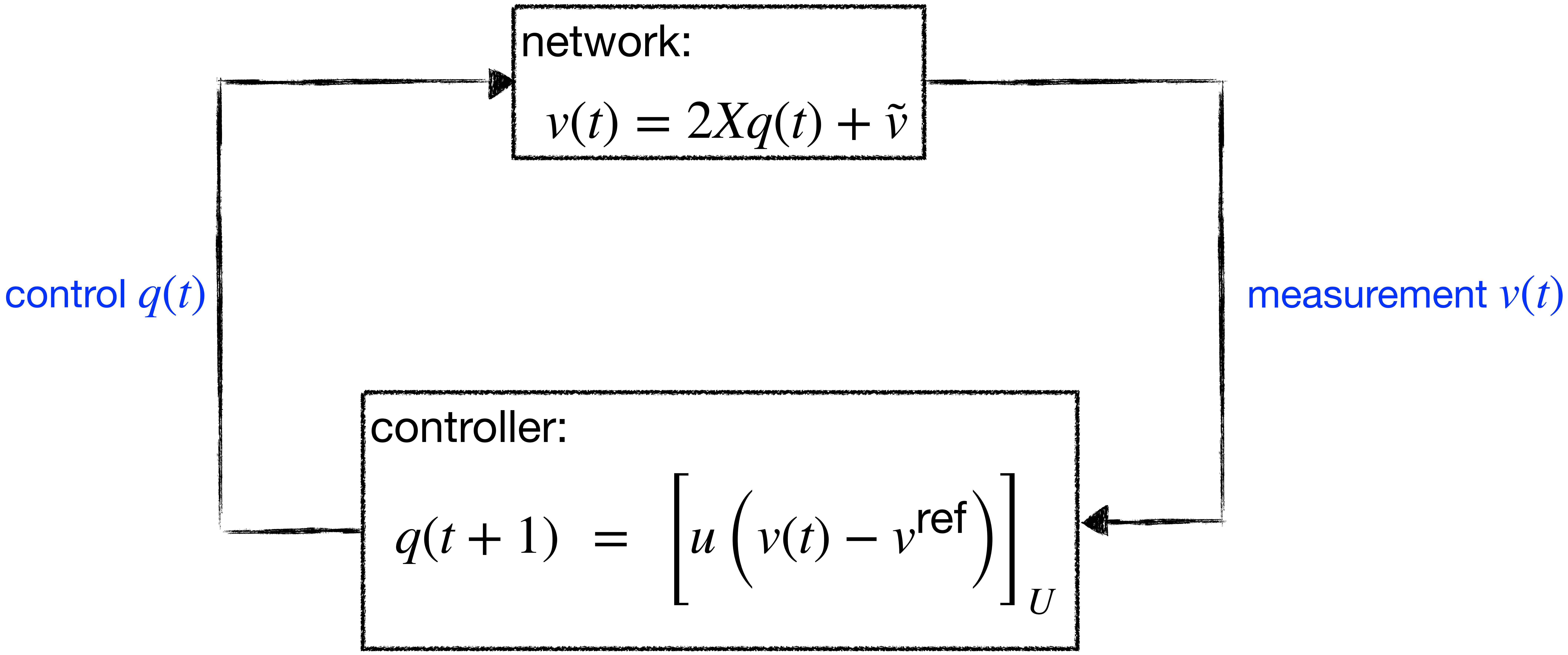
Local memoryless control

Example:



(a) Piecewise linear control $u_j(v_j)$ (b) Inverse $u_j^{-1}(q_j)$ (c) Implied cost $c_j(q_j)$

Closed-loop system



Closed-loop system

Closed-loop system is discrete-time dynamical system:

$$q(t+1) \;=\; \left[u\left(v(q(t)) - v_j^{\text{ref}}\right)\right]_U$$

- $v(q) := 2Xq + \tilde{v}$: maps linearly reactive power control q to network voltage
- $u\left(v - v^{\text{ref}}\right)$: maps voltage error to potential control action
- $[u]_U$: projects potential control action to its feasibility region U

Questions:

- Stability: will $(q(t), v(t))$ converge to an equilibrium point (q^*, v^*) ?
- Optimality: is the equilibrium point (q^*, v^*) optimal, in what sense?

Closed-loop system

Closed-loop system is discrete-time dynamical system:

$$q(t+1) \;=\; \left[u\left(v(q(t)) - v_j^{\text{ref}}\right)\right]_U$$

where $v(q) := 2Xq + \tilde{v}$

Definition:

q^* is an **equilibrium point** if it is a fixed point, i.e., $q^* \;=\; \left[u\left(v(q^*) - v_j^{\text{ref}}\right)\right]_U$

Assumptions:

1. u_j are differentiable; $\exists \alpha_j$ s.t. $\left| u_j'(v_j) \right| \leq \alpha_j$ $\qquad A :=$ diag $\left(\alpha_j, j \in N\right)$
2. u_j are strictly decreasing

Convergence

<u>Theorem</u> [Convergence]

Suppose Assumption 1 holds. If largest singular value $\sigma_{\max}(AX) < 1/2$ then

$A := \text{diag}\left(\alpha_j, j \in N\right)$

1. $\exists$ unique equilibrium point $q^* \in U$
2. $q(t)$ convergest to q^* geometrically, i.e.,

$$\|q(t) - q^*\| \leq \beta^t \|q(0) - q^*\| \quad \to \quad 0$$

for some $\beta \in [0,1)$

Optimality

Theorem [Optimality]

Suppose Assumptions 1 and 2 hold. The unique equilibrium point q^* of the dynamical system is the unique minimizer of

$$\min_{q \in U} \quad \sum_j c_j(q_j) \ + \ q^{\mathsf{T}} X q \ + \ q^{\mathsf{T}}(\tilde{v} - v^{\mathsf{ref}})$$

where $c_j(q_j) \ := -\int_0^{q_j} u_j^{-1}(\hat{q}_j)\, d\hat{q}_j$

Closed-loop behavior

Questions:

- Stability: will $(q(t), v(t))$ converge to an equilibrium point (q^*, v^*) ?
- Optimality: is the equilibrium point (q^*, v^*) optimal, in what sense?

Answer: under assumptions 1 and 2

- $(q(t), v(t))$ converges geometrically to a unique equilibrium point (q^*, v^*)
- The unique equilibrium point (q^*, v^*) minimizes a cost function determined by control law u_j

Reverse engineering: by choosing a control function u_j, we implicitly choose a cost function $c_j\left(q_j\right)$ that the closed-loop equilibrium optimizes

Closed-loop behavior

Questions:

- Stability: will $(q(t), v(t))$ converge to an equilibrium point (q^*, v^*) ?
- Optimality: is the equilibrium point (q^*, v^*) optimal, in what sense?

Answer: under assumptions 1 and 2

- $(q(t), v(t))$ converges geometrically to a unique equilibrium point (q^*, v^*)
- The unique equilibrium point (q^*, v^*) minimizes a cost function determined by control law u_j

Forward engineering: Choose a cost function $c_j\left(q_j\right)$ and derive control functions u_j as distributed algorithm to solve the optimization problem

Convergence proof

Sketch

Mean value theorem $\implies$ $u_j(v_j) - u(\hat{v}_j) = u_j'(w)(u - \hat{u})$ where $w := \lambda u + (1-\lambda)\hat{u}$ for some $\lambda \in [0,1]$

Assumption 1 and MVT

Hence

$$\|u(v) - u(\hat{v})\|_2^2 = \sum_j \left| u_j(v_j) - u_j(\hat{v}_j) \right|^2 \leq \sum_j \left| \alpha_j(v_j - \hat{v}_j) \right|^2 = \| A(v - \hat{v}) \|_2^2$$

Therefore

$$\left\| u\left(v(q) - v^{\text{ref}}\right) - u\left(v(\hat{q}) - v^{\text{ref}}\right) \right\|_2 \leq \| Av(q) - Av(\hat{q}) \|_2$$

Convergence proof

Sketch

Vector-function mean value theorem: if $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable then

$$\|f(y) - f(x)\| \ \leq \ \left\| \frac{\partial f}{\partial x}(z) \right\| \ \|y - x\|$$

for any induced matrix norm $\|\cdot\|$ where $z := \mu x + (1-\mu)y$ for some $\mu \in [0,1]$

Hence

$$\left\| Av(q) - Av(\hat{q}) \right\|_2 \ \leq \ \left\| \frac{\partial Av}{\partial q} \right\|_2 \|q - \hat{q}\|_2 \ \leq \ \|2AX\|_2 \|q - \hat{q}\|_2$$

because $\dfrac{\partial Av}{\partial q}(q) = A\dfrac{\partial v}{\partial q}(q) = 2AX$

Convergence proof

Sketch

Therefore

$$\left\| u\left(v(q) - v^{\text{ref}}\right) - u\left(v(\hat{q}) - v^{\text{ref}}\right) \right\|_2 \leq \|2AX\|_2 \|q - \hat{q}\|_2$$

Since induced matrix norm $\|AX\|_2 = \sigma_{\max}(AX)$, if $\beta = 2\sigma_{\max}(AX) < 1$ then

$$\left\| u\left(v(q) - v^{\text{ref}}\right) - u\left(v(\hat{q}) - v^{\text{ref}}\right) \right\|_2 \leq \beta \|q - \hat{q}\|_2$$

i.e. $u(q)$ is a contraction mapping.

Since projection $[u]_U$ is non-expansive, i.e., $\|[u]_U - [\hat{u}]_U\|_2 \leq \|u - \hat{u}\|_2$, the mapping $\left[u\left(v(q) - v^{\text{ref}}\right)\right]_U$ is a contraction mapping in q

Convergence proof

Sketch

Contraction theorem implies, for the dynamical system

$$q(t+1) \ = \ \left[u\left(v(q(t)) - v_j^{\text{ref}} \right) \right]_U$$

that

- $\exists$ unique fixed point $q*$
- $q(t)$ converges to $q*$ geometrically

Optimality

Theorem [Optimality]

Suppose Assumptions 1 and 2 hold. The unique equilibrium point q^* of the dynamical system is the unique minimizer of

$$\min_{q \in U} \quad \sum_j c_j(q_j) \ + \ q^\mathsf{T} X q \ + \ q^\mathsf{T}(\tilde{v} - v^{\text{ref}})$$

where $c_j(q_j) \ := -\int_0^{q_j} u_j^{-1}(\hat{q}_j)\, d\hat{q}_j$

Optimality proof

Sketch

Assumption 1 implies that there is a unique equilibrium pt q^*

Let $C(q) \ := \ \sum_j c_j(q_j) \ + \ q^\top X q \ + \ q^\top \Delta \tilde{v}$ where $\Delta \tilde{v} := \tilde{v} - v^{\text{ref}}$

Assumption 2 and $X \succ 0$ imply that $C(q)$ is strictly convex and hence, if an optimal q^* exists, it is unique

It thus suffices to show that q^* is the unique equilibrium pt if and only if q^* is the unique minimizer

We will show this in 3 steps:

1. Obtain optimality condition (necessary and sufficient because of convexity)
2. Relate $[\nabla C(q^*)]_j$ to $u_j\left(v_j(q_j^*) - v_j^{\text{ref}}\right)$ and q^*
3. Conclude optimality condition is equivalent to $q^* = \left[u\left(v(q^*) - v^{\text{ref}}\right)\right]_U$

Optimality proof

Sketch

Step 1: By convexity, $q^* \in U$ is optimal iff

$$\left(\nabla C(q^*)\right)^{\mathsf{T}}\left(q - q^*\right) \;\geq\; 0 \qquad \forall q \in U$$

This is equivalent to

$$q_j^* \in \; (\underline{q}_j, \overline{q}_j) \qquad \Longrightarrow \qquad \left[\nabla C(q^*)\right]_j = 0$$

$$q_j^* = \; \underline{q}_j \qquad \Longleftarrow \qquad \left[\nabla C(q^*)\right]_j > 0$$

$$q_j^* = \; \overline{q}_j \qquad \Longleftarrow \qquad \left[\nabla C(q^*)\right]_j < 0$$

Optimality proof

Sketch

Step 2: Evaluate

$$\nabla C(q^*) \;=\; \nabla c(q^*) + 2Xq^* + \Delta\tilde{v} \;=\; \nabla c(q^*) + \left(v(q^*) - v^{\text{ref}}\right)$$

where $\nabla c(q^*) = (c_j'(q_j^*) = -u_j^{-1}(q_j^*), i \in N)$

Hence $[\nabla C(q^*)]_j \;=\; -u_j^{-1}(q^*) + \left(v_j(q^*) - v_j^{\text{ref}}\right)$

Since u_j is strictly decreasing (Assumption 2), we have

$$\left[\nabla C(q^*)\right]_j = 0 \quad \Longleftrightarrow \quad u_j\left(v_j(q_j^*) \;-\; v_j^{\text{ref}}\right) \;=\; q_j^*$$

$$\left[\nabla C(q^*)\right]_j > 0 \quad \Longleftrightarrow \quad u_j\left(v_j(q_j^*) \;-\; v_j^{\text{ref}}\right) \;<\; q_j^*$$

$$\left[\nabla C(q^*)\right]_j < 0 \quad \Longleftrightarrow \quad u_j\left(v_j(q_j^*) \;-\; v_j^{\text{ref}}\right) \;>\; q_j^*$$

Optimality proof

Sketch

Step 3: Use $[\nabla C(q^*)]_j$ to combine the conditions in Steps 1 and 2 into:

$$q_j^* \in (\underline{q}_j, \overline{q}_j) \quad \Longrightarrow \quad \left[\nabla C(q^*)\right]_j = 0 \quad \Longleftrightarrow \quad u_j\left(v_j(q_j^*) - v_j^{\text{ref}}\right) = q_j^*$$

$$q_j^* = \underline{q}_j \quad \Longleftarrow \quad \left[\nabla C(q^*)\right]_j > 0 \quad \Longleftrightarrow \quad u_j\left(v_j(q_j^*) - v_j^{\text{ref}}\right) < \underline{q}_j$$

$$q_j^* = \overline{q}_j \quad \Longleftarrow \quad \left[\nabla C(q^*)\right]_j < 0 \quad \Longleftrightarrow \quad u_j\left(v_j(q_j^*) - v_j^{\text{ref}}\right) > \overline{q}_j$$

But this is equivalent to:

$$q^* = \left[u\left(v(q^*) - v^{\text{ref}}\right)\right]_U$$

i.e. q^* is the unique equilibrium point

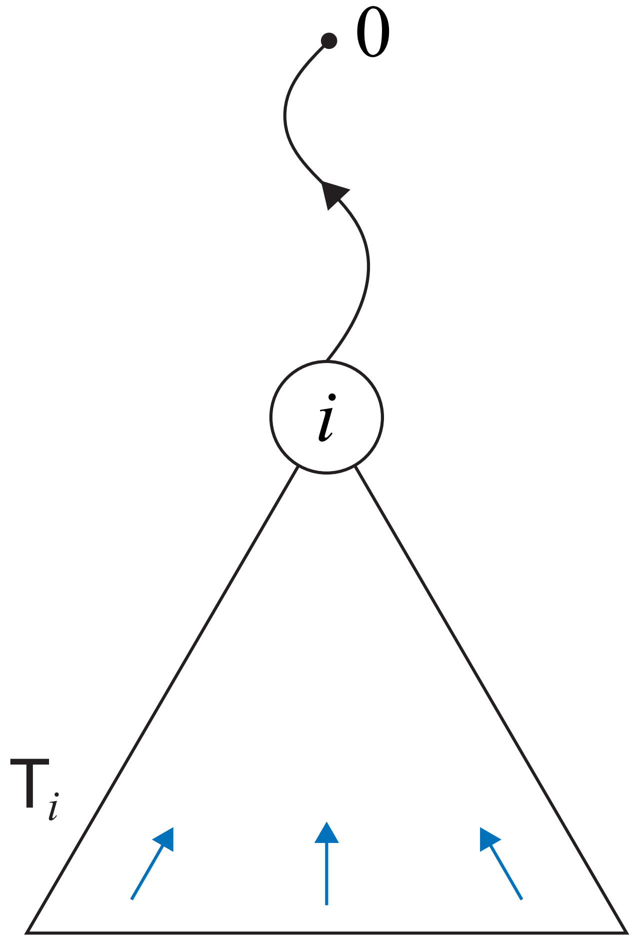
Therefore q^* is the unique equilibrium pt if and only if q^* is the unique minimizer

Outline

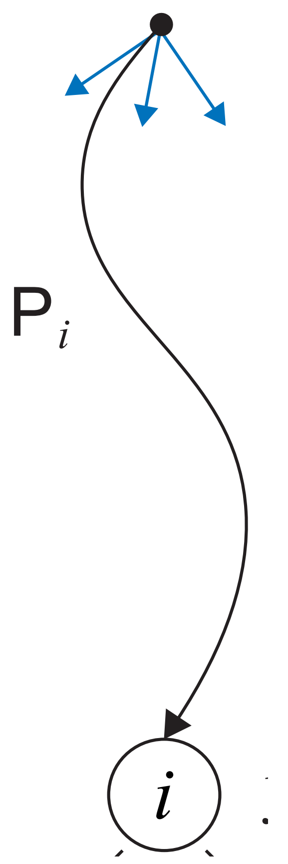
1. State estimation
2. Voltage control
3. Radial network identification
    - Linearized polar-form AC model
    - Covariances of voltage magnitudes

Recall: radial networks

When $y^s_{jk} = y^s_{kj}$ and $y^m_{jk} = y^m_{kj} = 0$



T_i : subtree rooted at bus i



T_i : unique path from 0 to i

<u>Theorem 10</u>

Suppose G is connected, Y is complex symmetric ($y^s_{jk} = y^s_{kj}$) and $y^m_{jk} = y^m_{kj} = 0$.

1. Reduced incidence matrix $\hat{C}$ is nonsingular

$$\left[\hat{C}^{-1}\right]_{lj} = \begin{cases} -1 & l \in \mathsf{P}_j \\ 1 & -l \in \mathsf{P}_j \\ 0 & \text{otherwise} \end{cases}$$

2. Reduced admittance matrix $\hat{Y}$ is nonsingular, and

$$\hat{Z} := \hat{Y}^{-1} = \hat{C}^{-\mathsf{T}} D^s_z \, \hat{C}^{-1}$$

$$\hat{Z}_{jk} = \sum_{l \in \mathsf{P}_j \cap \mathsf{P}_k} z^s_l$$

Apply this result to topology identification problem

Topology identification

1. Distribution grid typically consists of a meshed network with sectionalizing and tie switches on some lines
2. At any time switch are configured s.t. operational network is a spanning tree (substation at its root)
3. System operator knows the meshed network, but may not always know accurately switch status and hence operational network

Goal: Identify operational radial network from measurements of voltage magnitudes

Linearized power flow model

Linearization of polar form

Assumptions: For all $(j, k) \in E$

1. $y_{jk}^s = y_{kj}^s = g_{jk}^s + ib_{jk}^s$; $y_{jk}^m = y_{kj}^m = 0$
2. $g_{jk}^s > 0$ and $b_{jk}^s < 0$

Consider flat voltage profile: $V_j^{\text{flat}} = \mu e^{i\theta} \implies \left(p^{\text{flat}}, q^{\text{flat}}\right) = (0,0)$

- All voltages have same magnitude (e.g. $\mu = 1$ pu) and angle

Let

- $(|\hat{V}|, \hat{\theta})$: perturbation variable around V^{flat} at non-reference buses
- $(\hat{p}, \hat{q})$: perturbation variable around $\left(p^{\text{flat}}, q^{\text{flat}}\right) = (0,0)$ at non-reference buses

Linearized power flow model

Linearization of polar form

Polar form power flow model

$$p_j = \sum_{k:k\sim j} \left(g_{jk}^s + g_{jk}^m\right) |V_j|^2 - \sum_{k:k\sim j} |V_j||V_k| \left(g_{jk}^s \cos\theta_{jk} + b_{jk}^s \sin\theta_{jk}\right)$$

$$q_j = -\sum_{k:k\sim j} \left(b_{jk}^s + b_{jk}^m\right) |V_j|^2 - \sum_{k:k\sim j} |V_j||V_k| \left(g_{jk}^s \sin\theta_{jk} - b_{jk}^s \cos\theta_{jk}\right)$$

Linearized power flow model

Linearization of polar form

Polar form power flow model

$$p_j = \sum_{k:k\sim j} \left(g_{jk}^s + g_{jk}^m\right) |V_j|^2 - \sum_{k:k\sim j} |V_j||V_k| \left(g_{jk}^s \cos\theta_{jk} + b_{jk}^s \sin\theta_{jk}\right)$$

$$q_j = -\sum_{k:k\sim j} \left(b_{jk}^s + b_{jk}^m\right) |V_j|^2 - \sum_{k:k\sim j} |V_j||V_k| \left(g_{jk}^s \sin\theta_{jk} - b_{jk}^s \cos\theta_{jk}\right)$$

Linearize around $\left(V^{\text{flat}}, p^{\text{flat}}, q^{\text{flat}}\right)$ yields a linear model from $|\hat{V}|$ to $(\hat{p}, \hat{q})$ at non-reference buses:

$$|\hat{V}| = \hat{R}\hat{p} + \hat{X}\hat{q} + \hat{v}_0$$

where

$$\hat{R} := \hat{C}^{-T} D_1 \hat{C}^{-1} \succ 0, \quad \hat{X} := -\hat{C}^{-T} D_2 \hat{C}^{-1} \succ 0$$

$\hat{C}$ is reduced incidence matrix and

$$D_g := \text{diag}\left(g_l^s, l \in E\right) \succ 0, \qquad D_b := \text{diag}\left(b_l^s, l \in E\right) \prec 0$$

$$D_1 := \left(D_g + D_b D_g^{-1} D_b\right)^{-1} \succ 0, \qquad D_2 := \left(D_b + D_g D_b^{-1} D_g\right)^{-1} \prec 0$$

Covariance of voltages and powers

Suppose injections (p, q) vary randomly and induce random fluctuations in $|\hat{V}|$

Define covariance and cross-covariance matrices

$$\Sigma_v := E[|\hat{V}| - E(|\hat{V}|)][(|V| - E(|V|)]^{\mathsf{T}}$$

$$\Sigma_p := E[\hat{p} - E\hat{p}][\hat{p} - E\hat{p}]^{\mathsf{T}}, \qquad \Sigma_q := E[\hat{q} - E\hat{q}][\hat{q} - E\hat{q}]^{\mathsf{T}}$$

$$\Sigma_{pq} := E[\hat{p} - E\hat{p}][\hat{q} - E\hat{q}]^{\mathsf{T}}, \qquad \Sigma_{qp} := E[\hat{q} - E\hat{q}][\hat{p} - E\hat{p}]^{\mathsf{T}}$$

Then

$$\Sigma_v \;=\; \hat{R}\Sigma_p\hat{R}^{\mathsf{T}} + \hat{X}\Sigma_q\hat{X}^{\mathsf{T}} + \hat{R}\Sigma_{pq}\hat{X}^{\mathsf{T}} + \hat{X}\Sigma_{qp}\hat{R}^{\mathsf{T}}$$

Covariance of voltages and powers

Assumptions: power injections at same bus are positively correlated, those at different buses are uncorrelated

3. For all $j \in N$: $\Sigma_p[j,j] > 0, \;\; \Sigma_q[j,j] > 0, \;\; \Sigma_{pq}[j,j] = \Sigma_{qp}[j,j] > 0 \; ; \;\; y_{jk}^m = y_{kj}^m = 0$

4. For all $j \neq k$: $\Sigma_p[j,k] = \Sigma_q[j,k] = \Sigma_{pq}[j,k] = \Sigma_{qp}[j,k] = 0$

Covariance of voltages and powers

Assumptions: power injections at same bus are positively correlated, those at different buses are uncorrelated

3. For all $j \in N$: $\Sigma_p[j,j] > 0, \;\; \Sigma_q[j,j] > 0, \;\; \Sigma_{pq}[j,j] = \Sigma_{qp}[j,j] > 0 \,; \;\; y^m_{jk} = y^m_{kj} = 0$

4. For all $j \neq k$: $\Sigma_p[j,k] = \Sigma_q[j,k] = \Sigma_{pq}[j,k] = \Sigma_{qp}[j,k] = 0$

<u>**Theorem**</u>

Under assumptions 1-4:

1. If a non-reference bus $j \in N$ is a descendant of bus i, then var$(|V_j|) >$ var$(|V_i|)$
2. If bus i is a parent of bus j then the variance of $|V_i| - |V_j|$ is given by:

$$E\left((|V_i| - |V_j|) - E(|V_i| - |V_j|)\right)^2 = \sum_{k \in \mathrm{T}_j} \left(r_{ij}^2 \mathrm{var}(p_k) + x_{ij}^2 \mathrm{var}(q_k) + 2 r_{ij} x_{ij} \mathrm{cov}(p_k, q_k) \right)$$

Covariance of voltages and powers

Theorem

Under assumptions 1-4:

1. If a non-reference bus $j \in N$ is a descendant of bus i, then $\text{var}(|V_j|) > \text{var}(|V_i|)$
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$$E\left((|V_i| - |V_j|) - E(|V_i| - |V_j|)\right)^2 = \sum_{k \in \mathsf{T}_j} \left(r_{ij}^2 \text{var}(p_k) + x_{ij}^2 \text{var}(q_k) + 2 r_{ij} x_{ij} \text{cov}(p_k, q_k)\right)$$

Implications

Property 1 identifies a leaf node j as one with max $\text{var}(|V_j|)$

Property 2 identifies j's parent i as one that most closely satisfies the formula

Algorithm

1. Identify a leaf node j among unidentified nodes.
2. Identify j's parent.
3. Remove j from set of unidentified nodes and goto 1

Covariance of voltages and powers

<u>Proof:</u> part 1

Theorem 10 implies

$$\hat{R}_{jk} = \sum_{l \in \mathsf{P}_j \cap \mathsf{P}_k} r_l \ > \ 0, \qquad \hat{X}_{jk} = \sum_{l \in \mathsf{P}_j \cap \mathsf{P}_k} x_l \ > \ 0$$

Hence

$$\hat{R}_{jk} = \hat{R}_{ik} + r_{ij}, \qquad \hat{R}_{ik} = \sum_{l \in P_i} r_l, \qquad \text{if } k \in \mathsf{T}_j$$

$$\hat{R}_{ik} = \hat{R}_{jk}, \qquad \text{if } k \notin \mathsf{T}_j$$

Use these to evaluate the diagonal entries of var$(|V_j|)$ − var$(|V_i|) = \Sigma_v[j,j] - \Sigma_v[i,i]$, for each of the four terms in

$$\Sigma_v = \hat{R}\Sigma_p\hat{R}^\mathsf{T} + \hat{X}\Sigma_q\hat{X}^\mathsf{T} + \hat{R}\Sigma_{pq}\hat{X}^\mathsf{T} + \hat{X}\Sigma_{qp}\hat{R}^\mathsf{T}$$

Covariance of voltages and powers

Due to covariances Σ_p, Σ_q :

$$\left(\hat{R}\Sigma_p\hat{R}^\top\right)[j,j] - \left(\hat{R}\Sigma_p\hat{R}^\top\right)[i,i] = \sum_{k\in\mathsf{T}_j}\Sigma_p[k,k]\left(2\sum_{l\in\mathsf{P}_i} r_l + r_{ij}\right) r_{ij} > 0$$

similarly: $\left(\hat{X}\Sigma_q\hat{X}^\top\right)[j,j] > \left(\hat{X}\Sigma_q\hat{X}^\top\right)[i,i]$

Due to cross-covariances Σ_{pq}, Σ_{qp} :

$$\left(\hat{R}\Sigma_{pq}\hat{X}^\top\right)[j,j] - \left(\hat{R}\Sigma_{pq}\hat{X}^\top\right)[i,i] = \sum_k \Sigma_{pq}[k,k]\left(\hat{R}_{jk}\hat{X}_{jk} - \hat{R}_{ik}\hat{X}_{ik}\right) > 0$$

similarly: $\left(\hat{X}\Sigma_{qp}\hat{R}^\top\right)[j,j] > \left(\hat{X}\Sigma_{qp}\hat{R}^\top\right)[i,i]$

yielding: $\Sigma_v[j,j] > \Sigma_v[i,i]$

Covariance of voltages and powers

Proof: part 2

If bus i is a parent of bus j, then variance of $|V_i| - |V_j|$ is:

$$E\left((|V_i| - E|V_j|) - (|V_i|) - |V_j|)\right)^2 \; = \; \Sigma_v[i,i] + \Sigma_v[j,j] - 2\Sigma_v[i,j]$$

Again use

$$\hat{R}_{jk} = \hat{R}_{ik} + r_{ij}, \qquad \hat{R}_{ik} = \sum_{l \in P_i} r_l, \qquad \text{if } k \in \mathsf{T}_j$$

$$\hat{R}_{ik} = \hat{R}_{jk}, \qquad \text{if } k \notin \mathsf{T}_j$$

to show that the first term of

$$\Sigma_v \; = \; \hat{R}\Sigma_p\hat{R}^\mathsf{T} + \hat{X}\Sigma_q\hat{X}^\mathsf{T} + \hat{R}\Sigma_{pq}\hat{X}^\mathsf{T} + \hat{X}\Sigma_{qp}\hat{R}^\mathsf{T}$$

yields a simple expression:

$$\sigma_1 \; := \; \left(\hat{R}\Sigma_p\hat{R}^\mathsf{T}\right)[i,i] + \left(\hat{R}\Sigma_p\hat{R}^\mathsf{T}\right)[j,j] - 2\left(\hat{R}\Sigma_p\hat{R}^\mathsf{T}\right)[i,j] \; = \; r_{ij}^2 \sum_{k \in \mathsf{T}_j} \Sigma_p[k,k]$$

Covariance of voltages and powers

Similarly, the other terms of

$$\Sigma_v \;=\; \hat{R}\Sigma_p\hat{R}^\top + \hat{X}\Sigma_q\hat{X}^\top + \hat{R}\Sigma_{pq}\hat{X}^\top + \hat{X}\Sigma_{qp}\hat{R}^\top$$

yield

$$\sigma_1 := \left(\hat{R}\Sigma_p\hat{R}^\top\right)[i,i] + \left(\hat{R}\Sigma_p\hat{R}^\top\right)[j,j] - 2\left(\hat{R}\Sigma_p\hat{R}^\top\right)[i,j] \;=\; r_{ij}^2 \sum_{k\in \mathsf{T}_j} \Sigma_p[k,k]$$

$$\sigma_2 := \left(\hat{X}\Sigma_q\hat{X}^\top\right)[i,i] + \left(\hat{X}\Sigma_q\hat{X}^\top\right)[j,j] - 2\left(\hat{X}\Sigma_q\hat{X}^\top\right)[i,j] \;=\; x_{ij}^2 \sum_{k\in \mathsf{T}_j} \Sigma_q[k,k]$$

$$\sigma_3 := \left(\hat{R}\Sigma_{pq}\hat{X}^\top\right)[i,i] + \left(\hat{R}\Sigma_{pq}\hat{X}^\top\right)[j,j] - 2\left(\hat{R}\Sigma_{pq}\hat{X}^\top\right)[i,j] \;=\; r_{ij}x_{ij} \sum_{k\in \mathsf{T}_j} \Sigma_{pq}[k,k]$$

$$\sigma_4 := \left(\hat{X}\Sigma_{qp}\hat{R}^\top\right)[i,i] + \left(\hat{X}\Sigma_{qp}\hat{R}^\top\right)[j,j] - 2\left(\hat{X}\Sigma_{qp}\hat{R}^\top\right)[i,j] \;=\; r_{ij}x_{ij} \sum_{k\in \mathsf{T}_j} \Sigma_{qp}[k,k]$$

Covariance of voltages and powers

Summing:

$$\Sigma_v[i,i] - \Sigma_v[i,j] \;=\; \sum_{k=1}^{4} \sigma_k \;=\; \sum_{k \in \mathsf{T}_j} \left(r_{ij}^2 \Sigma_p[k,k] \;+\; x_{ij}^2 \Sigma_q[k,k] \;+\; 2 r_{ij} x_{ij} \Sigma_{pq}[k,k] \right)$$