Power System Analysis

Chapter 9 Optimal power flow

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Outline

- 1. Bus injection model
- 2. Branch flow model
- 3. NP-hardness
- 4. Global optimality

Outline

- 1. Bus injection model
	- Single-phase devices
	- Single-phase OPF
	- Single-phase OPF as QCQP
- 2. Branch flow model
- 3. NP-hardness
- 4. Global optimality

Single-phase OPF

Optimal power flow (OPF) is fundamental because it underlies numerous power system applications

• Unit commitment, optimal dispatch, state estimation, contingency analysis, voltage control, …

OPF is a constrained optimization problem

min $c(u, x)$ *u*,*x* subject to $f(u, x) = 0$, $g(u, x) \le 0$

- Control u : generation commitment, generation set points, transformer taps, EV charging levels, inverter reactive power, …
- Network state x : voltages, line currents, power flows, ...
- Cost function $c(u, x)$: generation cost, voltage deviation, power loss, user disutility, ...
- Equality constraint $f(u, x) = 0$: power flow equations, ...
- Inequality constraint $g(u, x) \leq 0$: operation constraints, e.g., generation/consumption limits, voltage limits, line limits, security constraints, …

Single-phase devices

Voltage source $V_j \in \mathbb{C}$

- \bullet *Ideal* voltage source: terminal voltage V_j = internal voltage
- \bullet \quad V_j is variable if the source is controllable, or fixed and given otherwise

Current source $I_j \in \mathbb{C}$

- \bullet $\,$ *Ideal* current source: terminal current I_j = internal current
- I_j is variable if the source is controllable, or given otherwise

Power source $s_j \in \mathbb{C}$

- \bullet *Ideal* power source: terminal power s_j = internal power
- s_j is variable if the source is controllable, or given otherwise

Impedance $z_j \in \mathbb{C}$

- Impedance z_j : constrains its terminal voltage & current $V_j = z_j I_j$
- Nodal vars at each bus $j : s_j = V_j \overline{I}_j$
- Nodal vars at different buses :
	- Current balance: $I = YV$
	- Power balance: $s_j = f_j(V)$

Single-phase OPF Assumptions

 N etwork: $G := (N, E)$ with $N+1$ buses in $N := \{0, 1, \ldots, N\}$ and M lines in E

- Line $(j, k) \in E$: characterized by $\left(y_{jk}^s, y_{jk}^m\right) \in \mathbb{C}^2$ and $\left(y_{kj}^s, y_{kj}^m\right) \in \mathbb{C}^2$
- Special case: $y_{jk}^{s} = y_{kj}^{s}$; $y_{jk}^{m} = y_{kj}^{m} = 0$

Assume WLOG

- Single-phase devices: voltage sources and power sources only
- **•** Each bus has a single device with (s_j, V_j)

Formulate the simplest OPF to study general computational properties

Optimization variable: $(s, C) := (s_j, V_j, j \in N)$

• Represents voltage sources V_j and power sources s_j only

j

Cost function *C*(*s*, *V*)

• **Full cost**:
$$
C(s, V) := \sum_{j: \text{gens}} c_j \text{Re}(s_j)
$$

• Total real power loss: $C(s, V) := \sum_{i=1}^{n}$ Re(*sj*)

Power flow equations in BIM

• Equality constraints on (*V*,*s*)

$$
s_j = \sum_{k:j\sim k} S_{jk}(V) := \sum_{k:j\sim k} \bar{y}_{jk}^s \left(|V_j|^2 - V_j \bar{V}_k \right) + \bar{y}_{jj}^m |V_j|^2, \qquad j \in \overline{N}
$$

where $y_{jj}^m := \sum_{k:j\sim k} y_{jk}^m$

• Derivation:

$$
I_{jk}(V) := y_{jk}^{s}(V_j - V_k) + y_{jk}^{m} V_j
$$

\n
$$
S_{jk}(V) := V_j \overline{I}_{jk}(V) := \overline{y}_{jk}^{s} \left(|V_j|^2 - V_j \overline{V}_k \right) + \overline{y}_{jk}^{m} |V_j|^2
$$

- Can also use polar form and Cartesian form
- Nonlinear and global equality constraints, resulting in nonconvexity of OPF

Operational constraints

- Injection limits (e.g. gen. or load capacity limits): $s_j^{\min} \leq s_j \leq s_j^{\max}$
- Voltage limits: $v_j^{\min} \le |V_j|^2 \le v_j^{\max}$
- Line limits: $|I_{jk}(V)|^2 \le \ell_{jk}^{\max}, |I_{kj}(V)|^2 \le \ell_{kj}^{\max}$

$$
\left| y_{jk}^{s}(V_j - V_k) + y_{jk}^{m} V_j \right|^2 \le \ell_{jk}^{\max}, \qquad (j,k) \in E
$$

$$
\left| y_{kj}^{s}(V_k - V_j) + y_{kj}^{m} V_k \right|^2 \le \ell_{kj}^{\max}, \qquad (j,k) \in E
$$

Line limits can also be on line powers $\left(\,S_{jk}(V),S_{kj}(V)\,\right)$ or apparent powers $\left(\,\left.\,\left|\,S_{jk}(V)\,\right|,\,\left|\,S_{kj}(V)\,\right|\,\right)\,\right)$

OPF in BIM

min $C(s, V)$ (s, V) subject to $f(s, V) = 0$ $g(s, V) \leq 0$ power flow equations operational constraints

- Does not need assumption $y_{jk}^s = y_{kj}^s$
- Can accommodate single-phase transformers with *complex* turns ratios
- Can allow voltages or power injections be fixed and given; e.g., $s_j^{\text{min}} = s_j^{\text{max}}$
- ... or unconstrained, e.g., $s_0^{\min} := -\infty i\infty$, $s_0^{\max} := \infty + i\infty$

Single-phase OPF

- 1. Other devices
	- Can include other devices such as current sources, impedances, capacity taps
	- Allow multiple devices connected to same bus
- 2. $\,$ Can formulate OPF in terms of V only
	- Use power flow equations to express injections $s_j(V)$ as functions of V
	- Eliminate s_j and power flow equations (equality constraints)

Next: explain each in turn

Single-phase OPF Including other devices

Examples

- Current source (controllable): variable I_j with local constraints $|I_j|^2 \le I_j^{\max}, s_j = V_j \overline{I_j}$
- Impedance z_j : imposes additional constraint $s_j = |V_j|^2 / \bar{z}_j$
- Capacitor tap (controllable): variable y_j with local constraints $y_j^{\min} \le y_j \le y_j^{\max}$, $s_j = \bar{y}_j |V_j|^2$ • Multiple devices: injection variables s_{jk} with local constraints $s_{jk}^{\min} \leq s_{jk} \leq s_{jk}^{\max}$, $s_j = \sum_j s_{jk}$ *k*

Including other devices at bus j imposes additional local constraints

- Additional optimization var u_j may be introduced
- Equality constraints relating $\left(s_j, V_j\right)$ and u_j (if present) : $f_j\left(u_j, s_j, V_j\right) = 0$
- Inequality (operational) constraints (e.g., capacity limits): $g_j(u_j) \leq 0$

Single-phase OPF In terms of *V* **only**

Equality constraints (BIM in complex form)

• Expresses s_j in terms of voltages V

$$
s_j(V) \ = \ \sum_{k:j \sim k} S_{jk}(V) \ \ := \ \ \sum_{k:j \sim k} \bar{y}_{jk}^s \left(\ | \ V_j \|^2 - V_j \bar{V}_k \right) \ + \ \bar{y}_{jj}^m \left| \ V_j \right|^2, \qquad j \in \overline{N}
$$

 C ost $C(V) := C(s(V), V)$ expressed as function of V

• Fuel cost:

$$
C(V) := \sum_{j: \text{gens}} c_j \operatorname{Re}(s_j(V)) = \sum_{j: \text{gens}} c_j \operatorname{Re}\left(\sum_{k: j \sim k} \bar{y}_{jk}^s \left(|V_j|^2 - V_j \bar{V}_k\right) + \bar{y}_{jj}^m |V_j|^2\right)
$$

• Total real power loss:

$$
C(V) := \sum_j \text{Re}(s_j(V))
$$

Single-phase OPF Operational constraints

Injection limits (e.g. generation or load capacity limits) $s^{\min}_j \leq \ s_j(V) \leq \ s^{\max}_j$:

$$
s_j^{\min} \le \sum_{k:j \sim k} \bar{y}_{jk}^s \left(|V_j|^2 - V_j \bar{V}_k \right) + \bar{y}_{jj}^m |V_j|^2 \le s_j^{\max}, \quad j \in \overline{N}
$$

• Or in polar form:

$$
p_j^{\min} \leq \sum_{k:k \sim j} \left(g_{jk}^s + g_{jk}^m \right) |V_j|^2 - \sum_{k:k \sim j} |V_j| |V_k| \left(g_{jk}^s \cos \theta_{jk} + b_{jk}^s \sin \theta_{jk} \right) \leq p_j^{\max}
$$

$$
q_j^{\min} \leq - \sum_{k:k \sim j} \left(b_{jk}^s + b_{jk}^m \right) |V_j|^2 - \sum_{k:k \sim j} |V_j| |V_k| \left(g_{jk}^s \sin \theta_{jk} - b_{jk}^s \cos \theta_{jk} \right) \leq q_j^{\max}
$$

Single-phase OPF Operational constraints

Voltage limits (same as before):

$$
v_j^{\min} \le |V_j|^2 \le v_j^{\max}, \quad j \in \overline{N}
$$

Line limits (same as before):

$$
\left| y_{jk}^{s}(V_j - V_k) + y_{jk}^{m} V_j \right|^2 \le \ell_{jk}^{\max}, \qquad (j,k) \in E
$$

$$
\left| y_{kj}^{s}(V_k - V_j) + y_{kj}^{m} V_k \right|^2 \le \ell_{kj}^{\max}, \qquad (j,k) \in E
$$

- Line limits can also be on line powers $\left(S_{jk}(V), S_{kj}(V)\right)$ or apparent powers $\left(\left\lfloor S_{jk}(V)\right\rfloor, \left\lfloor S_{kj}(V)\right\rfloor\right)$

Single-phase OPF In terms of *V* **only**

Feasible set

 $V:=\left\{V\in\mathbb{C}^{N+1}\mid V\right\}$ satisfies operational constraints $\right\}$

OPF in BIM

min *C*(*V*) *V*∈

- Does not need assumption $y_{jk}^s = y_{kj}^s$
- Can accommodate single-phase transformers with *complex* turns ratios

Single-phase OPF In terms of *V* **only**

Feasible set

 $V:=\left\{V\in\mathbb{C}^{N+1}\mid V\right\}$ satisfies operational constraints $\right\}$

OPF in BIM

min *C*(*V*) *V*∈

We will mostly study this simple OPF Can express it as a QCQP

Outline

1. Bus injection model

- Single-phase devices
- Single-phase OPF
- Single-phase OPF as QCQP
- 2. Branch flow model
- 3. NP-hardness
- 4. Global optimality
- 5. Techniques for scalability: case study

OPF as QCQP QCQP

Quadratically constrained quadratic program:

min *x*∈ℂ*ⁿ* $x^{\text{H}}C_0x$

- s.t. $x^{\text{H}}C_{l}x \leq b_{l}$, $l = 1,..., L$
- $C_l: n \times n$ Hermitian matrix $\Rightarrow x^{\mathsf{H}} C_l x \in \mathbb{R}$
- $b_l \in \mathbb{R}$
- Homogeneous QCQP : all monomials are of degree 2

OPF as QCQP QCQP

Inhomogeneous QCQP

$$
\min_{x \in \mathbb{C}^n} x^{H} C_0 x + (c_0^{H} x + x^{H} c_0)
$$
\ns.t.

\n
$$
x^{H} C_l x + (c_l^{H} x + x^{H} c_l) \leq b_l, \quad l = 1, \dots, L
$$

Homogenization: introduce scalar var $t\in\mathbb{C}$

• Set
$$
x := \hat{x}\overline{t}
$$
 and require $|t|^2 = 1$ (i.e., $t = e^{i\theta}$ for some θ). Then

$$
x^{H}C_{l}x + c_{l}^{H}x + x^{H}c_{l} = \hat{x}^{H}C_{l}\hat{x} + c_{l}^{H}(\hat{x}\bar{t}) + (\hat{x}\bar{t})^{H}c_{l} = \begin{bmatrix} \hat{x}^{H} & t^{H} \end{bmatrix} \begin{bmatrix} C_{l} & c_{l} \\ c_{l}^{H} & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ t \end{bmatrix}
$$

OPF as QCQP QCQP

Equivalent homogeneous QCQP

$$
\min_{\hat{x} \in \mathbb{C}^n, t \in \mathbb{C}} \qquad [\hat{x}^{\mathsf{H}} \quad t^{\mathsf{H}}] \begin{bmatrix} C_0 & c_0 \\ c_0^{\mathsf{H}} & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ t \end{bmatrix}
$$
\n
$$
\text{s.t.} \qquad [\hat{x}^{\mathsf{H}} \quad t^{\mathsf{H}}] \begin{bmatrix} C_l & c_l \\ c_l^{\mathsf{H}} & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ t \end{bmatrix} \leq b_l, \qquad l = 1, \dots, L
$$
\n
$$
[\hat{x}^{\mathsf{H}} \quad t^{\mathsf{H}}] \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ t \end{bmatrix} = 1
$$

• If $(\hat{x}^{\text{opt}}, t^{\text{opt}})$ is optimal for homogeneous QCQP, then product $x^{\text{opt}} := \hat{x}^{\text{opt}} t^{\text{opt}}$ is optimal for original inhomogeneous QCQP $(\hat{x}^{\text{opt}}, t^{\text{opt}})$ is optimal for homogeneous QCQP, then product $x^{\text{opt}} := \hat{x}^{\text{opt}} t^{\text{opt}}$

Steven Low OPF Bus injection model

OPF as QCQP Equivalent real QCQP

Even though OPF is often formulated in $\mathbb C$, it is converted to $\mathbb R$ before being solved iteratively

QCQP

min *x*∈ℂ*ⁿ* $x^{\text{H}}C_0x$ s.t. $x^{\text{H}}C_{l}x \leq b_{l}$, $l = 1,..., L$

- $C_l: n \times n$ complex Hermitian matrix
- $b_l \in \mathbb{R}$

Equivalent to:

$$
\min_{(x_r, x_i) \in \mathbb{R}^{2n}} \qquad \begin{bmatrix} x_r \\ x_i \end{bmatrix}^\mathsf{T} \begin{bmatrix} C_{0r} & -C_{0i} \\ C_{0i} & C_{0r} \end{bmatrix} \begin{bmatrix} x_r \\ x_i \end{bmatrix}
$$
\n
$$
\text{s.t.} \qquad \begin{bmatrix} x_r \\ x_i \end{bmatrix}^\mathsf{T} \begin{bmatrix} C_{lr} & -C_{li} \\ C_{li} & C_{lr} \end{bmatrix} \begin{bmatrix} x_r \\ x_i \end{bmatrix} \le b_l, \quad l = 1, \dots, L
$$

• $2n \times 2n$ real symmetric matrices

OPF as QCQP

To write OPF as QCQP:

- Assume cost function $C(V) = V^{H}C_{0}V$ can be written as a quadratic form
- Need to rewrite operational constraints in terms of quadratic forms

OPF as QCQP

 $\textbf{Injection limits} \ \ s_j^{\min} \leq s_j(V) \leq s_j^{\max}$

$$
s_j(V) = V_j I_j^H = (e_j^H V) (e_j^H I)^H = e_j^H V V^H Y^H e_j
$$

$$
s_j(V) = \text{tr}(e_j^H V V^H Y^H e_j) = \text{tr}((Y^H e_j e_j^H) V V^H) =: V^H Y_j^H V
$$

OPF as QCQP

 $\textbf{Injection limits} \ \ s_j^{\min} \leq s_j(V) \leq s_j^{\max}$

$$
s_j(V) = V_j I_j^H = (e_j^H V) (e_j^H I)^H = e_j^H V V^H Y^H e_j
$$

$$
s_j(V) = \text{tr}(e_j^H V V^H Y^H e_j) = \text{tr}((Y^H e_j e_j^H) V V^H) =: V^H Y_j^H V
$$

- Y_j is not Hermitian so $V^\mathsf{H} Y_j^\mathsf{H} V$ is generally complex
- **.** Define $\Phi_j :=$ 1 $\frac{1}{2} (Y_j^H + Y_j), \qquad \Psi_j :=$ 1 $\overline{2i}$ $(Y_j^H - Y_j)$

• Then
$$
\text{Re}(s_j) = V^{\text{H}} \Phi_j V
$$
, $\text{Im}(s_j) = V^{\text{H}} \Psi_j V$

Hence
$$
s_j^{\min} \le s_j(V) \le s_j^{\max}
$$
 is equivalent to:
\n $p_j^{\min} \le V^{\text{H}} \Phi_j V \le p_j^{\max}$, $q_j^{\min} \le V^{\text{H}} \Psi_j V \le q_j^{\max}$

Steven Low OPF Bus injection model

OPF as QCQP Voltage limits

 V oltage magnitude is: $\mid V_j\!\mid^2 = V^\mathsf{H} E_j V$ where $E_j := e_j e_j^\mathsf{T}$

 H ence voltage limits are: $v^{\min}_j \leq V^{\mathsf{H}} E_j V \leq v^{\max}_j$

OPF as QCQP Line limits

Write I_{jk} in terms of voltage vector V_{\cdot}

$$
I_{jk} = y_{jk}^{s}(V_j - V_k) + y_{jk}^{m}V_j = \left(y_{jk}^{s}(e_j - e_k)^{\top} + y_{jk}^{m}e_j^{\top}\right)V
$$

Hence current limit is: $\quad |I_{jk}|^2 = V^{\mathsf{H}} \hat{Y}_{jk} V \ \leq \ \mathscr{C}^{\max}_{jk} \ \ \text{where}$

$$
\hat{Y}_{jk} := \left(\bar{y}_{jk}^s (e_j - e_k) + \bar{y}_{jk}^m e_j \right) \left(y_{jk}^s (e_j - e_k)^\top + y_{jk}^m e_j^\top \right)
$$

OPF as QCQP Simplest formulation

 $\min_{V \subset \subset N+1} V^{\mathsf{H}} C_0 V$ *V*∈ℂ*N*+1 s.t. $p_j^{\min} \leq V^{\text{H}} \Phi_j V \leq p_j^{\max}$ $j \in \overline{N}$ $q_j^{\min} \leq V^{\text{H}} \Psi_j V \leq q_j^{\max}, \qquad j \in \overline{N}$ $v_j^{\min} \leq V^{\text{H}} E_j V \leq v_j^{\max}, \qquad j \in \overline{N}$ $V^{\text{H}} \hat{Y}_{jk} V \leq \ell_{jk}^{\max}, \qquad (j, k) \in E$ $V^{\text{H}} \hat{Y}_{kj} V \leq \ell_{kj}^{\max}, \qquad (j, k) \in E$

Outline

1. Bus injection model

- 2. Branch flow model
	- Radial network
- 3. NP-hardness
- 4. Global optimality

Radial network Assumptions: DistFlow model

Radial network

• BFM most useful for modeling distribution systems which are mostly radial (and unbalanced)

$$
z_{jk}^s = z_{kj}^s
$$
 or equivalently
$$
y_{jk}^s = y_{kj}^s
$$

• Does not apply to 3-phase transformers in ΔY or $Y\Delta$ configuration or their per-phase equivalent with complex gains

 $y_{jk}^{m} = y_{kj}^{m} = 0$

• Reasonable assumption for distribution line where $|y_{jk}^m|, |y_{kj}^m| \ll |y_{jk}^s|$

Includes only voltage sources and power sources

- Optimization variables are voltages (squared magnitudes) v_j and power injections s_j respectively
- Can include current sources or an impedances with additional vars and constraints.

DistFlow model

Power flow equations

• All lines point away from bus 0 (root)

$$
\sum_{k:j\to k} S_{jk} = S_{ij} - z_{ij}^{s} \mathcal{C}_{ij} + s_j, \qquad j \in \overline{N}
$$

$$
v_j - v_k = 2 \operatorname{Re} \left(\bar{z}_{jk}^{s} S_{jk} \right) - |z_{jk}^{s}|^2 \mathcal{C}_{jk}, \qquad j \to k \in E
$$

$$
v_j \mathcal{C}_{jk} = |S_{jk}|^2, \qquad j \to k \in E
$$

Operational constraints

$$
s_j^{\min} \le s_j \le s_j^{\max}
$$

$$
v_j^{\min} \le v_j \le v_j^{\max}
$$

$$
\ell_{jk} \le \ell_{jk}^{\max}
$$

Single-phase OPF DistFlow model

Feasible set

 $\mathbb{X}_{\text{df}} := \{x := (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies PF equations } \& \text{ operational constraints}\}\$

OPF in BFM

 $\text{min} \quad C(x) \qquad \text{s.t.} \qquad x \in \mathbb{X}_{\text{df}}$ *x*

Single-phase OPF Equivalence

Recall for BIM:

- Feasible set: $\mathbb{V}:=\left\{V\!\in\mathbb{C}^{N+1}\mid V\right.$ satisfies operational constraints $\right\}$
- OPF: min *V*∈ *C*(*V*)

OPF in BFM is equivalent to OPF in BIM:

- Feasible sets \mathbb{X}_{df} and $\mathbb {V}$ are equivalent (Ch 5)
- \ldots provided cost functions $C(x)$ and $C(V)$ are the same

General radial network

Does not assume $z_{jk}^s = z_{kj}^s$ nor $y_{jk}^m = y_{kj}^m = 0$

Need branch quantities in both directions

$$
\bullet \ \mathcal{C} := \left(\mathcal{C}_{jk}, \mathcal{C}_{kj}, (j,k) \in E\right), \ S := \left(S_{jk}, S_{kj}, (j,k) \in E\right)
$$

$$
\bullet \ \alpha_{jk} := 1 + z_{jk}^s \ y_{jk}^m, \quad \alpha_{kj} := 1 + z_{kj}^s \ y_{kj}^m
$$

BFM for general radial network

$$
s_j = \sum_{k:j \sim k} S_{jk}, \qquad j \in \overline{N}
$$

$$
|\alpha_{jk}|^2 v_j - v_k = 2 \operatorname{Re} \left(\alpha_{jk} \bar{z}_{jk}^s S_{jk} \right) - |z_{jk}|^2 \ell_{jk}, \qquad (j, k) \in E
$$

$$
|\alpha_{kj}|^2 v_k - v_j = 2 \operatorname{Re} \left(\alpha_{kj} \bar{z}_{kj}^s S_{kj} \right) - |z_{kj}^s|^2 \ell_{kj}, \qquad (j,k) \in E
$$

$$
\left| S_{jk} \right|^2 = v_j \mathcal{C}_{jk}, \qquad \left| S_{kj} \right|^2 = v_k \mathcal{C}_{kj}, \qquad (j,k) \in E
$$

$$
\bar{\alpha}_{jk}v_j - \bar{z}_{jk}^s S_{jk} = \left(\bar{\alpha}_{kj}v_k - \bar{z}_{kj}^s S_{kj}\right)^{\mathsf{H}}, \qquad (j,k) \in E
$$

Single-phase OPF General radial network

Operational constraints (same as before but line limits in both directions)

 $s_j^{\min} \leq s_j \leq s_j^{\max}, \quad v_j^{\min} \leq v_j \leq v_j^{\max}, \quad \mathcal{C}_{jk} \leq \mathcal{C}_{jk}^{\max}, \quad \mathcal{C}_{kj} \leq \mathcal{C}_{kj}^{\max}$

Feasible set

 $\mathbb{X}_{\text{tree}} := \{x := (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies PF equations } \& \text{ operational constraints}\}\$

OPF in BFM

 $\textsf{min} \;\; C(x) \qquad \textsf{s.t.} \qquad x \in \mathbb{X}_{\textsf{tree}}$ *x*

OPF in BFM is equivalent to OPF in BIM:

- Feasible sets $\mathbb{X}_{\mathsf{tree}}$ and $\mathbb {V}$ are equivalent (Ch 5)
- \ldots provided cost functions $C(x)$ and $C(V)$ are the same

Outline

- 1. Bus injection model
- 2. Branch flow model

3. NP-hardness

- OPF feasibility
- OPF is NP-hard
- 4. Global optimality

OPF feasibility Tree network

 Star network (N, E) with $N+1$ buses and $M=N$ lines

- $y_{jk}^s = y_{kj}^s$ and $y_{jk}^m = y_{kj}^m = 0$
- Fixed voltage magnitudes $\mid V_j\mid := 1$ pu
- Fixed and given injections $(p_j,q_j),\,j\in N_L\subset N$
- \bullet Dispatchable generation (p_j, q_j) with $p_j \geq 0,~j \in N_G \subset N$
- Line limits: |*θ^j* − *θ^k* | ≤ *θ* ∈ (0,*π*/2], (*j*, *k*) ∈ *E*

Each instance of OPF feasibility problem is specified by

- Tree network $(N_G \cup N_L, E)$
- Line admittances (*gjk*, *bjk*, (*j*, *k*) ∈ *E*)
- Line limits $\theta \in (0, \pi/2]$
- Fixed injections $(p_j, q_j, j \in N_L)$

OPF feasibility Tree network

Find

- \bullet Real power generations $\left(\mathop{p_{j^{\prime}}}{j\in N_{G}}\right)\geq0$
- Voltage angles $(\theta_j, j \in N)$
- Line flows (*Pjk*, *Qjk*, (*j*, *k*) ∈ *E*)

that satisfy the polar form power flow equation and line limits:

OPF feasibility:

\n
$$
p_{j} = \sum_{k:j \sim k} P_{jk}, \qquad q_{j} = \sum_{k:j \sim k} Q_{jk}, \qquad j \in N_{L}
$$
\n
$$
p_{j} \geq 0, \qquad j \in N_{G}
$$
\n
$$
P_{jk} = g_{jk}(1 - \cos \theta_{jk}) - b_{jk} \sin \theta_{jk}, \qquad (j, k) \in E
$$
\n
$$
Q_{jk} = -b_{jk}(1 - \cos \theta_{jk}) - g_{jk} \sin \theta_{jk}, \qquad (j, k) \in E
$$
\n
$$
|\theta_{j} - \theta_{k}| \leq \overline{\theta}, \qquad (j, k) \in E
$$

NP-hardness P and NP

Let

- Σ : finite set of symbols
- Σ^* : set of all finite strings of symbols in Σ
- $L \subseteq \Sigma^*$: language over Σ

Deterministic Turing machine (DTM): computation model that takes an input $\sigma \in \Sigma^*$, performs computation (read, write, state transition), and either halts in "yes" or "no" state, or does not halt

Given DTM M , time complexity function $c_M: \mathbb{N}_+ \rightarrow \mathbb{N}_+$:

 $c_M(n) := \max\{m : \exists \sigma \in \Sigma^* \text{ with } |\sigma| = n \text{ s.t. } M \text{ takes } m \text{ steps to halt on } \sigma\}$

 M is called a polynomial time DTM if \exists a polynomial p s.t. $c_M(n) \leq p(n)$ for all n

Language recognized by (DTM or NDTM) M is

 $L_M := \{\sigma \in \Sigma^* : M \text{ halts on } \sigma \text{ in } \text{``yes'' state}\}$

NP-hardness P and NP

The class P of languages is

 $P := \{L \subseteq \Sigma^* : \exists$ polynomial time DTM *M* for which $L = L_M\}$

Informally: P consists of all language over Σ that are recognized by a DTM in polynomial time

While P captures "solvability" of a problem, NP captures "verifiability"

• It is difficult (NP-complete) to find a cycle in an arbitrary graph that visits every node exactly once, but easy to verify if a candidate is a solution

Given NDTM M , time complexity function $c_M: \mathbb{N}_+ \rightarrow \mathbb{N}_+$:

 $c_M(n) := \max\{m : \exists \sigma \in \Sigma^* \text{ with } |\sigma| = n \text{ s.t. } M \text{ takes } m \text{ steps to halt on } \sigma \text{ in } \text{"yes" state}\}\$

 M is called a polynomial time NDTM if \exists a polynomial p s.t. $c_M(n) \leq p(n)$ for all n

The class NP of languages is

 $NP := \{L \subseteq \Sigma^* : \exists$ polynomial time NDTM *M* for which $L = L_M\}$ $\qquad \qquad \vert P \subseteq NP$

Informally: NP consists of all language recognized by a NDTM (or verifiable by a DTM) in polynomial time

NP-hardness NP-hard and NP-complete

A function $f\colon \Sigma_1^*\to \Sigma_2^*$ is a language $L_j:=\{(\sigma,f(\sigma)):\sigma\in \Sigma_1^*\}\subseteq \Sigma_1^*\times \Sigma_2^*$ DTM M computes f if $L_M=L_f$

A polynomial reduction from $L_1 \subseteq \Sigma_1^*$ to $L_2 \subseteq \Sigma_2^*$ is a function $f \colon \Sigma_1^* \to \Sigma_2^*$ which can be computed by a polynomial time DTM s.t.

 $\sigma \in L_1 \iff f(\sigma) \in L_2, \quad \sigma \in \Sigma_1$

A language L is NP-hard if for every $L' \in$ NP there exists a polynomial reduction from L' to L

It is NP-complete if L is NP-hard and $L \in$ NP

• NP-complete languages are in a sense the "hardest" languages in NP

NP-hardness Decision problems

A decision problem is a problem whose solution is either "yes" or "no"

• It is defined by a set of finite instances, e.g. specified in terms of sets, graphs, functions, real numbers

Let Π be a decision problem (or its instances) that can be "encoded" into a language problem over some alphabet Σ

• Informally, an encoding is $\sigma:\Pi\to\Sigma^*$ that maps each instance $y\in\Pi$ to a string $\sigma(y)\in\Sigma^*$

Let $Y \subseteq \Pi$ be the subset of instances whose solutions are "yes"

• We will refer to Y either as a set of problem instances or simply a problem by itself

Let $L_Y\coloneqq\{\sigma(y):y\in Y\}$ be the language defined by instances in Y

• Solution of instance $y \in \Pi$ is "yes" if and only if $y \in Y$ if and only if $\sigma(y) \in L_{Y}$

Hardness properties of Y are then defined in terms of hardness properties of its encoding L_{Y}

- e.g. Y is in P if $L_Y \in$ P, $\ Y$ is NP-complete if L_Y is NP-complete
- OPF feasibility problem is such a decision problem

NP-hardness

Theorem

OPF feasibility problem on a tree network is NP-hard

Remarks:

- OPF feasibility is not proved to be in NP, because solution can be irrational
- Proved by polynomial reduction of NP-complete subset sum problem to OPF feasibility
- OPF feasibility can be proved to be strongly NP-hard by polynomial reduction of strongly NP-complete one-in-three 3SAT problem to OPF feasibility

NP-hardness is worst-case result

- Subclasses of OPF cane polynomial time solvable
- e.g., those satisfied sufficient conditions for exact relaxations or global optimaiity

Outline

- 1. Bus injection model
- 2. Branch flow model
- 3. NP-hardness
- 4. Global optimality
	- Convex relaxation
	- Lyapunov-lik condition for global optimality
	- Application to OPF on radial network

ing, if for any relaxed point, there exists a path connecting it ing, if for any relaxed point, there exists a path connecting it to the non-convex feasible set and the path satisfies to the non-convex feasible set and the path satisfies

- *•* along the path the cost is non-increasing, *•* along the path the cost is non-increasing,
- along the path the 'distance' to the non-convex feasible set is non-increasing, set is non-increasing, Setup and Preliminaries

**Optimize in the function (Definition 10). The second part is on
Optimilize surf with a catalog in a problem does have** then the problem must have exact relaxation and no spurious local optima simultaneously. Here the 'distance' can be any properly constructed function, as we will define later as a Lyapunov-like function (Definition 10). The second part is on if it needs that if in which says the necessary conditions and if a problem does have it a problem of exact relaxation and no spurious local optima simultaneously, then there must exist such Lyapunov-like function and paths nen inere must exist such Lyapuno
satisfying the requirements above. then the problem must have exact relaxation and no spurious local optima simultaneously. Here the 'distance' can be any properly constructed function, as we will define later as a Lyapunov-like function (Definition 10). The second part is on the necessary condition, which says that it a problem does have exact relaxation and no spurious local optima simultaneously, then there must exist such Lyapunov-like function and paths taneously eou ed *L*iuncti rung I on $\mathsf{I} \mathsf{L}$ Π, \mathbf{w} l a no st ıst suc ment (eductor relaxation and no spurious
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e second part is on s.t. *x x x* (e) via and its c Tocal opuma
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Consider

Nonconvex optimization Plans it condition fluid to the find to convex while it on e onstruct them. We then derive reatain rules problem from Convex primitives propeats with known Exapunov-like function and
Convex relaxation letes with known Lyapunov-like function and paths. This process allows as to rease and extend known Though Lyapunov-like functions and paths are guaranteed to exist, for specific problems it could still be difficult to find ric Lyapunov-like function and paths of a new problem from
the Lyapunov-like function and paths of a new problem from a superintic
primitives: problems with known Lyapunov₁ results resolted problem shanges and grows. Probally's wo punkil satisfying the requirements above.
satisfying the requirements above... Though Lyapunov-like functions and paths are guaranteed t to existe for specific problems it could still be difficult to find ov-like le propi *i x* and paths are guarar

- \bullet X : noticing approach approach to two specific problems Optimal Rowarn Flowdandh Igunatan SDP. Checked ork proves the first
- *X* : compact and ispurious what dean bee screeked to priore for OPF
• *X* : compact and ispurious a uper set X_{he}eked *a priori*) explain the
- $f: \mathbb{R}^n$ to have no composition that it helps explain that it helps the mention of \mathbb{R}^n widespread empirical experience that local algorithms for OPF

Optimal sofutions exist for both problems P1 and P2

C. Background for Power Systems

C. *Recker cumb fagplications at the main motivation of this work*, Optsmale Bowler aplowcaGORE)and anconempotiblations of dhiel workle Optimal power sy ktom (OPF) i proposed problem OPF is class area optimizationy proble i First hat opininiti za s[19], a oPF orst a whis st do openlinear ophysical rhawn and inperational runn straints. bJe cis t&navintife.drcphysicanyax,smddYBplaaddinalts AStrfarmulmtion khown Ob 60 non-ethore, there ip nature in efficient algorithm [9], [20], [21]. Therefore, there is no known efficient algorithm
The necessary condition is based upon some stronger assumptions so the second part is not the exact converse of the first part. 1The necessary condition is based upon some stronger assumptions so the

II. PRELIMINARIES II. PRELIMINARIES

moets of the set C of complex numbers. For any
ergen regard is a Banach space. In this paper, we will use $\mathbb K$ to denote the set $\mathbb R$ of numbers or the set $\mathcal E$ of complex numbers. For any interger *n*, K*ⁿ* is a Banach space. $\frac{1}{2}$ and $\frac{1}{2}$ or the set \mathbb{C} of complex numbers. For any interger *n*, K*ⁿ* is a Banach space.

Consider a (potentially non-convex) optimization pro

f(**x**) figure that to mid and there *X* is a nonempty compact subset of \mathbb{R}^n , not necessary figure \mathbb{R}^n and convex while \mathbb{R}^n and compact and convex while \mathbb{R}^n . and paths are suaranteed. Here $\mathcal X$ is a nonempty compact subset of $\mathbb K^n$, not nece
vild still be difficult to find Here $\mathcal X$ is a nonempty compact subset of $\mathbb K^n$, not neges

 \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{Y} \leq $f(x)$ $f(x)$ efficiently represented $x \in \mathcal{X}$ *with* $\|x - x^{\text{lo}}\|$

Definition 2 (Strong Exactness). We say the relaxation ${\bf D}$ eficition 2 e s§trong Exagtness b pMe a olay o the ofl a xitio p . *and hence globally optimal, for* (1)*.* exact *with respect to* (1) *if any optimal point of* (2) *is feasible, and hence globally optimal, for* (1)*.*

Unless otherwise specified, ψ will always use the exdandets referent issues pestified, exactness all alerginition the ein particular that such 2 trisng x exact thess. *Definition 2*, if imiparticular that, if (2) is exact, then $\forall \hat{x} \in \hat{\mathcal{X}} \setminus \mathcal{X}$, $f(x)$ \overrightarrow{D} **Einition** $\overrightarrow{3}$. A path in $S \subseteq \mathbb{K}^n$ connecting point a to *b is a continuous function h* : [0*,* 1] ! *S such that h*(0) = *a* Definition 3. *^A* path *in ^S* ✓ ^K*ⁿ connecting point ^a to point b* a b *d* b b *h* t \overline{t} *nlous function* $h : [0,1] \rightarrow S$ *such that* $h(0)$ *and* ψ ⁶($\frac{1}{2}$) ψ are in the separator *h* in *h* in *h* in *h*^{*i*} in *h*^{*i*} *h a* path by the corresponding functio the remainder of the part by the corresponding function the *Kemainder nd fon didner are equivalent*:

Lemma 1. *The following are equivalent:*

us to study conditions, sufficient or necessary, for problems to simultaneously have exact relaxation and no spurious local to simultaneously have exact relaxation and no spurious local optima. Those conditions can also help us to study the local optima. Those conditions can also help us to study the local optimality from properties of its relaxation, instead of its optimality from properties of its relaxation, instead of its landscape. landscape.

Exact the non-convex feasible set and the path satisfies Our conditions have two parts. The first part, which also Our conditions have two parts. The first part, which also appeared in [18], is on the sufficient condition. Roughly speak-appeared in [18], is on the sufficient condition. Roughly speaking, if for any relaxed point, there exists a path connecting it ing, if for any relaxed point, there exists a path connecting it • **along the path the cost in m-increasing,** to the non-convex feasible set and the path satisfies

• along the path the 'distance' to the non-convex feasible set is non-increasing,
set is non-increasing. **•** a busine path the cost is non-increasing set is non-increasing, Setup and Preliminaries

Definition 1. $x^* \in \mathbb{X}$ superly constructed function in the Party depite large $\lim_{x \to a} \sum f(x)$ for all $2. x^* \in \mathbb{X}$ is a global optimum, which says the line of the process days all 3. P2 is $\frac{1}{3}$ is $\frac{1}{3}$ is $\frac{1}{3}$ is $\frac{1}{3}$ is feasible (and hence optimal) for P1 $x^* \in \mathcal{X}$ and \mathcal{X} are \mathcal{X} and \mathcal{X} are \mathcal{X} and \mathcal{X} and \mathcal{X} are \mathcal{X} and \mathcal{X} are \mathcal{X} for all $\|x - \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X}$ for all $\|x - \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X} \mathcal{X}$ $x^* \in X$ **f** \overline{x} and \overline{x} conduction of the \overline{x} of \overline{x} *f*(*x*) \overline{x} and \overline{x} is \overline{x} \overline{x} *x** then the problem must have exact relaxation and no spurious local optima simultaneously. Here the 'distance' can be any properly constructed function, as we will define later as a Lyspienov-like function (Definition 110). The second functis on the necessary condition, which says that if a problem does have exact a calculation and ino spurious local optima simultaneously, then there must exist such Lyapunov-like function and paths nen Inere must exist such Lyapuno
satisfyiwerthe requirement ordured Though Lyapunov-like functions and paths are guaranteed Though Lyapunov-like functions and paths are guaranteed. to exist for specific we diems it could still be difficult to find be construct them. We then derive exitin rules for convex, while a
The protect them. We then derive retain rules problem from convex, while *X*
The protect of *X* file Lyapunov-like function and paths of a new problem from
the Lyapunov-like function and paths of a new problem from and *substitute*
primitives problems with known Lyapunov-like function and primitives problems with known _t by pounov-like function and h Riem This herpfoble all emanges Gn cyfow and nextend we napply results resolted problem shanges and grows. Probally s, we print it tbe wero prosed approach that two pspecific wore ble mes Optimal Rowern Flowdanch low at an an SDP. cheuk work *prive*r the OPF knowaveconodisjon i dthat oean dpaima; c*ked it hetips i* exfotai*t* DPFe twiltaspread empirival looperientienaand alt alglorithmplain OIPF widespread empirical experience that local algorithms for OPF and paths are guaranteed. Here $\mathcal X$ is a nonempty compact subset of $\mathbb K^n$, not necessary
uld still be difficult to find Here $\mathcal X$ is a nonempty compact subset of $\mathbb K^n$ not necessary
uld still be difficult to find then the problem must have exact relaxation and no spurious local optima simultaneously. Here the 'distance' can be any properly constructed function, as we will define later as a Lyapunov-like function (Definition 10). The second part is on the necessary condition, which says that if a problem does have exact relaxation and no spurious local optima simultaneously, then there must exist such Lyapunov-like function and paths satisfying the requirements above. taneously eou ed *L*iuncti rung **LOIDLLLI** .ion, wh a no st ıst suc **Preligient** ov-like (eductor relaxation and no spurious
lere the cdistance, can be any define later as a
 x sdond pattas on problem does have ally lis Tocal **exp**endari
Assumov-like fun \sharp **I** *X* ^{*i*} **OI PZ IS IE** *B* s and paths are guarant

problems often work extremely well.

C. Background for Power Systems

C. *Recker crutted applications at a mate in motivation of this work*, Optimale Bowler aplowcaGORE) and anconemport blations of dheid workle Optimal power syktem(OPF) tiproposed probleh, OPF is anclass of local algorithms. In this paper, we show that a known of local algorithms. In this paper, we show that a known sufficient condition for relaxation exactness is also suffic for local optima to be globally optimal. To the best of authors' knowledge, this is the first analytical result of its kind, authors' knowledge, this is the first analytical result of its kind, and we hope that the approaches proposed in this paper help derive more sufficient conditions along this direction. help derive more sufficient conditions along this direction.

II. PRELIMINARIES II. PRELIMINARIES

interger *n*, Kanach space.
interger *n*, K_n is a Banach space. In this paper, we will use $\mathbb K$ to denote the set $\mathbb R$ of μ umbers or the set $\mathcal C$ of complex numbers. For any if

Consider a (potentially non-convex) optimization problem Consider a (potentially non-convex) optimization problem

 supp $\text{sup$ Here *X* is a nonempty compact subset of \mathbb{K}^n , not necessarily degree *X* is a nonempty compact subset of \mathbb{K}^n not necessary subject to $x \in \mathcal{X}$.

superset of *X* . The cost function *fire* and convex and superset of *X*ⁱ. The convex superset of *X*ⁱ. The convex superset of *X*ⁱ. The convex and $\frac{1}{2}$ and $\frac{1}{2}$ $\frac{1}{2}$ and $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ and $\frac{1}{2}$ $\frac{1}{2}$ and $\frac{1}{2}$ $\frac{1}{2}$ and $\frac{1$ continuous over *X*². We do not relaxation *X*² to be relaxation *X*² to Definition 1. *A poin*
 Definition-d existsoin x x is cal optimum in <i>x is cal optimum is called a y x is call optimum $\begin{array}{c} \mathbf{Defin} \mathbf{F} \mathbf$ \mathcal{X} \mathcal{X} \mathcal{X} \leq $f(x)$ *for* efficiently represented. $x \in \mathcal{X}$ *with* $\|x - x^{\text{lo}}\|$

Definition 2 (Strong Exactness). We say the relaxation (2) ${\bf D}$ eficition 2 e s§trong Exagtness b pMe a olay o the v fl a 2 $\overline{\mu}$ io $\overline{\mu}$ a3 *and hence globally optimal, for* (1)*.* exact *with respect to* (1) *if any optimal point of* (2) *is feasible, and hence globally optimal, for* (1)*.*

Unless otherwise specified, we will always use the $\frac{1}{2}$ exdantess referent issues pestified, exactness all alenginition them *exactation that* such a *x* in gxext thess. Definition 2 if \hat{p} imiparticular that, if (2) is exact, then $\forall \hat{x} \in \hat{\mathcal{X}} \setminus \mathcal{X}$, $f(\overline{\hat{x}})$ \overrightarrow{D} **Definition** $\overrightarrow{3}$. A path in $S \subseteq \mathbb{K}^n$ connecting point a to point

Path

Definition

- 1. A path in $Y \subseteq \mathbb{R}^n$ connecting a to be b in Y is a continuous function $h:[0,1]\rightarrow Y$ s.t. $h(0)=a$ and $h(1)=b$
- 2. An arbitrary set $\{h_i : i \in I\}$ of paths in Y is called
	- uniformly bounded if \exists finite H s.t. $||h_i(t)||_{\infty} \leq H$ for all $t \in [0,1]$ and $i \in I$
	- uniformly equicontinuous if for any $\epsilon > 0$, $\exists \delta > 0$ s.t. $||h_i(t_2) - h_i(t_1)||_{\infty} \leq \epsilon$ for all $i \in I$ whenever $|t_2 - t_1| < \delta$

Example: If all paths in $\{h_i : i \in I\}$ are linear, then $\{h_i : i \in I\}$ is both uniformly bounded and uniformly equicontinuous

Lyapunov-like function

Definition

A Lyapunov-like function associated with problems P1 and P2 is a continuous function $V: \hat{X} \to \mathbb{R}_+$ *s.t.* $V(x) = 0$ if $x \in X$ and $V(x) > 0$ if $x \in \hat{X} \setminus X$ associated with problems P1 and P2 is a continuous function *V* and $V(x) > 0$ if $x \in \hat{X} \setminus X$

Global optimality Optimality conditions

1. There is a Lyapunov-like function V and, for every infeasible point $x \in \hat{X} \backslash X$, \exists path h_x s.t.

(a) $h_x(0) = x$, $h_x(1) \in X$, $f(h_x(1)) < f(x)$

(b) Both $f(h_x(t))$ and $V(h_x(t))$ are nonincreasing for $t \in [0,1]$

Every infeasible pt x can be brought back to X with a lower cost

Nonincreasing cost or certificate along path to feasibility

- 2. The set $\{h_x : x \in \hat{X} \backslash x\}$ of paths in 1 is uniformly bounded and uniformly equicontinuous
- 3. At least one of the following holds:
	- (d) All local optima of P1 are isolated (i.e., every local optimum has a neighborhood with no other local optimum)

(e) For
$$
\{h_x : x \in \hat{X} \setminus x\}
$$
 in 1, $\exists \alpha > 0$ s.t. for all $x \in \hat{X} \setminus X$ and all $0 \le s < t \le 1$,

 $f(h_x(s)) - f(h_x(t)) \ge \alpha ||h_x(s) - h_x(t)||$

Cost must decrease sufficiently along path to feasibility

for some norm $\|\cdot\|$

Global optimality

Theorem [Sufficiency]

Suppose conditions 1, 2, 3 hold.

- 1. The convex relaxation P2 is exact wrt P1
- 2. Every local optimum of P1 is a global optimum

Moreover if condition 3(a) holds, then the optimal point is unique

Remarks

- Exactness \iff existence of $\{h_x : x \in \hat{X} \backslash x\}$ that satisfies condition 1
- Other conditions are to prove that there is no spurious local optimum

Global optimality

 A set $y\subseteq\mathbb{R}^n$ is semianalytic if every $x\in\mathbb{R}^n$ has a neighborhood U s.t. \check{Y} an U can be represented as a finite Boolean combination of sets $\{x:g(x)=0\}$ and $\mathbb{C} x$ coldition \mathcal{Q} must be the exists $k>0$ some analytic functions g,h (usually satisfied by engineering problems) $f(h_x(t)) - f(t)$ ation of sets $\{x : g(x) = 0\}$ and $\{x \in X \setminus X, \forall 0 \leq t \leq s \leq 1 \text{ we have }$ $\in \mathbb{R}^n$ has a neighborhood U s.t. $\widecheck{\mathtt{M}}$ n $\widecheck{\mathbb{W}}$ can be

Theorem [Necessity]

Suppose X is semianalytic and f is analytic. If

- 1. The convex relaxation P2 is exact wrt P1, and
- 2. Every local optimum of P1 is a global optimum

then ∃ Lyapunov-like function V and a family of paths $\{h_x : x \in \hat{X} \backslash x\}$ that satisfy cond 1 and 2

Now we are in a good position to discuss sor that rule out pseudo local optima and therefore g any local optimum must be a global optimum.

Corollary 3. If all local optima of (1) are i *Condition (C) implies that any local optimum of (optimum.*

 Caltech Here, local optima being isolated means any local Calech this case the optimum can be proved to be also uniquely and a local conditions use the optimum can be proved to be also of (1) has an open neighborhood which contains α optimum. The proof is straightforward as by definition is the proof is straightforward as by definition is the $\frac{1}{100}$ local optimum could not be pseudo local optimu

function α is the monotonicity of $f(h_x(t))$ in C $\frac{1}{\sqrt{1-\frac{1$ Another way to eliminate pseudo local optima

In Condition (C'), $\|\cdot\|$ could be any norm

 $f(h_x(t)) - f(h_x(s)) \geq k \|h_x(t) - h$

Lyapunov-like optimality condition Comparison with Lyapunov stability

Consider the dynamical system

 $\dot{x} = f(x(t)),$ $t \ge 0, x(0) = x_0$

Let x^* be an equilibrium point where $f(x^*) = 0$

Lyapunov stability theory

- 1. Lyapunov function $V(x)$ is a continuously differentiable function s.t. $V(x) > V(x^*)$ and $\dot{V}(x) < 0$ for all $x \neq x^*$ in \mathbb{R}^n
- 2. $\ V$ certifies stability of x^* : x^* is globally asymptotically stable if a Lyapunov function $V\!(x)$ exists

Lapunov-like optimality condition

- 1. V certifies global optimality of a local optimum $x^* \in X$
- 2. No dynamics to specify path : no requirement on differentiability of V , but
- 3. Need to construct both V and paths $\{h_x : x \in \hat{X} \backslash x\}$ (no general method known)

Application to OPF Recall: OPF in DistFlow model

DistFlow equations (radial network):

$$
\sum_{k:j \to k} S_{jk} = S_{ij} - z_{ij}^{s} \ell_{ij} + s_j, \qquad j \in \overline{N}
$$

$$
v_j - v_k = 2 \operatorname{Re} \left(\bar{z}_{jk}^{s} S_{jk} \right) - |z_{jk}^{s}|^2 \ell_{jk}, \qquad j \to k \in E
$$

$$
v_j \ell_{jk} = |S_{jk}|^2, \qquad j \to k \in E
$$

Nonconvex constraint

Operational constraints:

$$
s_j^{\min} \le s_j \le s_j^{\max}, \quad v_j^{\min} \le v_j \le v_j^{\max}, \quad \mathcal{C}_{jk} \le \mathcal{C}_{jk}^{\max}
$$

Feasible set

 $X := \{x := (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies DistFlow equations } \& \text{ operational constraints} \}$

Application to OPF Convex relaxation

Replace

$$
v_j \mathcal{C}_{jk} = |S_{jk}|^2, \qquad j \to k \in E
$$

by

 $v_j \mathcal{C}_{jk} \geq |\mathit{S}_{jk}|^2, \qquad j \rightarrow k \in E$ Convex second-order cone (SOC) constraint

Convex superset

 $\hat{X} := \{x : x$ satisfies constraints with SOC replacement}

Consider

Optimality conditions OPF in DistFlow model

- 4. $\,X$ is nonempty compact, \hat{X} is compact, cost function f is convex and continuous
- 5. Cost function $f(x) = f(p, q, v, \ell)$ is independent of $S = (P, Q)$, continuously differentiable in ℓ with $\nabla f(x) \geq 0$. Moreover $\exists c > 0$ s.t. $\frac{f}{\sqrt{2}}(x) \geq 0$ for all $l \in E$ and all 6. No lower bounds on injections: $s^{\text{min}}_j = -\infty - i\infty$ ∂*f* $\partial \ell_l$ $f(x) \geq 0$ for all $l \in E$ and all $x \in \hat{X}$ demand large enough not to pose constraints

usually satisfied

7. $z_{jk} =: (r_{jk}, x_{jk}) > 0$ and line limit satisfies $|z_{jk}|^2 e^{max} \leq v_j^{min}$

Remarks

• Differentiability is not necessary and can be replaced by subgradient (which always exist since f is convex)

Global optimality OPF in DistFlow model

Theorem

Suppose conditions 4-7 hold on radial network.

- 1. Convex relaxation P2 is exact wrt P1
- 2. Every local optimum of P1 is a global optimum

Remarks

• Exactness is proved in Ch 11 on Semidefinite relaxations of OPF in BFM

Global optimality Construction: *V*

Proof requires construction of Lyapunov-like function V and family of paths $\{h_x : x \in \hat{X} \backslash x\}$

Lyapunov-like function:

$$
V(x) := \sum_{j \to k \in E} \left(v_j \mathcal{C}_{jk} - |S_{jk}|^2 \right)
$$

•
$$
V(x) \ge 0
$$
 for all $x \in \hat{X}$, with "=" iff $x \in X$

Global optimality Construction: h_x

Define quadratic function

$$
\phi_{jk}(a) := \frac{|z_{jk}|^2}{4} a^2 + \left(v_j - \text{Re}\left(\bar{z}_{jk} S_{jk}\right)\right) a + \left(|S_{jk}|^2 - v_j \ell_{jk}\right)
$$

Define $\Delta_{jk} :=$ positive root of $\phi_{jk}(a) = 0$ if $v_j\ell_{jk} > |\c{S_{jk}}|^2$, or $\Delta_{jk} := 0$ otherwise

For infeasible pt $x\in \hat{X}\backslash X$, define path $h_x(t)\;:=\; \Big(\,\tilde{s}(t),\tilde{v}(t),\tilde{\mathcal{E}}(t),\tilde{S}(t)\,\Big) \;=\; x-tA\Delta(x)\;$ for $t\in [0,1]$:

$$
\tilde{s}_j(t) := s_j - \frac{t}{2} \sum_{i:i \to j} z_{ij} \Delta_{ij} - \frac{t}{2} \sum_{k:j \to k} z_{jk} \Delta_{jk}, \qquad j \in \overline{N}
$$

$$
\tilde{v}_j(t) := v_j, \qquad j \in N
$$

$$
\tilde{\mathcal{E}}_{jk}(t) := \mathcal{E}_{jk} - t\Delta_{jk}, \qquad j \to k \in E
$$

$$
\tilde{S}_{jk}(t) := S_{jk} - \frac{t}{2} z_{jk} \Delta_{jk}, \qquad j \to k \in E
$$

Global optimality Proof idea

Prove the Lyapunov-like function V and family of paths $\{h_x : x \in \hat{X} \backslash x\}$ defined above satisfy conditions 1, 2, 3