Power System Analysis

Chapter 9 Optimal power flow

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Outline

- 1. Bus injection model
- 2. Branch flow model
- 3. NP-hardness
- 4. Global optimality

Outline

- 1. Bus injection model
 - Single-phase devices
 - Single-phase OPF
 - Single-phase OPF as QCQP
- 2. Branch flow model
- 3. NP-hardness
- 4. Global optimality

Single-phase OPF

Optimal power flow (OPF) is fundamental because it underlies numerous power system applications

• Unit commitment, optimal dispatch, state estimation, contingency analysis, voltage control, ...

OPF is a constrained optimization problem

 $\min_{u,x} c(u,x) \qquad \text{subject to} \qquad f(u,x) = 0, \ g(u,x) \le 0$

- Control *u* : generation commitment, generation set points, transformer taps, EV charging levels, inverter reactive power, ...
- Network state *x* : voltages, line currents, power flows, ...
- Cost function c(u, x): generation cost, voltage deviation, power loss, user disutility, ...
- Equality constraint f(u, x) = 0: power flow equations, ...
- Inequality constraint $g(u, x) \le 0$: operation constraints, e.g., generation/consumption limits, voltage limits, line limits, security constraints, ...

Single-phase devices

Voltage source $V_i \in \mathbb{C}$

- Ideal voltage source: terminal voltage V_i = internal voltage
- V_i is variable if the source is controllable, or fixed and given otherwise

Current source $I_j \in \mathbb{C}$

- *Ideal* current source: terminal current I_i = internal current
- I_i is variable if the source is controllable, or given otherwise

Power source $s_i \in \mathbb{C}$

- *Ideal* power source: terminal power s_i = internal power
- s_i is variable if the source is controllable, or given otherwise

Impedance $z_i \in \mathbb{C}$

• Impedance z_i : constrains its terminal voltage & current $V_i = -z_i I_i$

• Nodal vars at each bus $j: \; s_j = V_j ar{I}_j$

- Nodal vars at different buses :
 - Current balance: I = YV

• Power balance:
$$s_j = f_j(V)$$

Single-phase OPF Assumptions

Network: $G := (\overline{N}, E)$ with N + 1 buses in $\overline{N} := \{0, 1, ..., N\}$ and M lines in E

• Line $(j,k) \in E$: characterized by $\left(y_{jk}^{s}, y_{jk}^{m}\right) \in \mathbb{C}^{2}$ and $\left(y_{kj}^{s}, y_{kj}^{m}\right) \in \mathbb{C}^{2}$

• Special case:
$$y_{jk}^s = y_{kj}^s$$
; $y_{jk}^m = y_{kj}^m = 0$

Assume WLOG

- Single-phase devices: voltage sources and power sources only
- Each bus has a single device with $\left(s_{j}, V_{j}\right)$

Formulate the simplest OPF to study general computational properties

Optimization variable: $(s, C) := \left(s_j, V_j, j \in \overline{N}\right)$

• Represents voltage sources V_i and power sources s_i only

Cost function C(s, V)

• Fuel cost :
$$C(s, V) := \sum_{i:\text{gens}} c_j \operatorname{Re}(s_j)$$

• Total real power loss: $C(s, V) := \sum_{i} \operatorname{Re}(s_i)$

Power flow equations in BIM

• Equality constraints on (*V*, *s*)

$$s_j = \sum_{k:j\sim k} S_{jk}(V) := \sum_{k:j\sim k} \bar{y}_{jk}^s \left(|V_j|^2 - V_j \bar{V}_k \right) + \bar{y}_{jj}^m |V_j|^2, \qquad j \in \overline{N}$$

where $y_{jj}^m := \sum_{k:j\sim k} y_{jk}^m$

• Derivation:

$$I_{jk}(V) := y_{jk}^{s}(V_{j} - V_{k}) + y_{jk}^{m}V_{j}$$

$$S_{jk}(V) := V_{j}\bar{I}_{jk}(V) := \bar{y}_{jk}^{s}\left(|V_{j}|^{2} - V_{j}\bar{V}_{k}\right) + \bar{y}_{jk}^{m}|V_{j}|^{2}$$

- Can also use polar form and Cartesian form
- Nonlinear and global equality constraints, resulting in nonconvexity of OPF

Operational constraints

- Injection limits (e.g. gen. or load capacity limits): $s_j^{\min} \leq s_j \leq s_j^{\max}$
- Voltage limits: $v_j^{\min} \leq |V_j|^2 \leq v_j^{\max}$
- Line limits: $|I_{jk}(V)|^2 \le \ell_{jk}^{\max}$, $|I_{kj}(V)|^2 \le \ell_{kj}^{\max}$

$$\begin{vmatrix} y_{jk}^{s}(V_{j} - V_{k}) + y_{jk}^{m}V_{j} \end{vmatrix}^{2} \leq \ell_{jk}^{\max}, \quad (j,k) \in E \\ \begin{vmatrix} y_{kj}^{s}(V_{k} - V_{j}) + y_{kj}^{m}V_{k} \end{vmatrix}^{2} \leq \ell_{kj}^{\max}, \quad (j,k) \in E \end{aligned}$$

Line limits can also be on line powers $(S_{jk}(V), S_{kj}(V))$ or apparent powers $(|S_{jk}(V)|, |S_{kj}(V)|)$

OPF in BIM

 $\begin{array}{ll} \min_{(s,V)} & C(s,V) \\ \text{subject to} & f(s,V) = 0 & \text{power flow equations} \\ & g(s,V) \leq 0 & \text{operational constraints} \end{array}$

- Does not need assumption $y_{jk}^s = y_{kj}^s$
- Can accommodate single-phase transformers with complex turns ratios
- Can allow voltages or power injections be fixed and given; e.g., $s_i^{\min} = s_i^{\max}$
- ... or unconstrained, e.g., $s_0^{\min} := -\infty i\infty$, $s_0^{\max} := \infty + i\infty$

Single-phase OPF

- 1. Other devices
 - Can include other devices such as current sources, impedances, capacity taps
 - Allow multiple devices connected to same bus
- 2. Can formulate OPF in terms of V only
 - Use power flow equations to express injections $s_j(V)$ as functions of V
 - Eliminate s_i and power flow equations (equality constraints)

Next: explain each in turn

Single-phase OPF Including other devices

Examples

- Current source (controllable): variable I_j with local constraints $|I_j|^2 \le I_j^{\text{max}}$, $s_j = V_j \bar{I}_j$
- Impedance z_j : imposes additional constraint $s_j = |V_j|^2 / \bar{z}_j$
- Capacitor tap (controllable): variable y_j with local constraints $y_j^{\min} \le y_j \le y_j^{\max}$, $s_j = \bar{y}_j |V_j|^2$ • Multiple devices: injection variables s_{jk} with local constraints $s_{jk}^{\min} \le s_{jk} \le s_{jk}^{\max}$, $s_j = \sum s_{jk}$

Including other devices at bus j imposes additional local constraints

- Additional optimization var u_i may be introduced
- Equality constraints relating (s_j, V_j) and u_j (if present) : $f_j(u_j, s_j, V_j) = 0$
- Inequality (operational) constraints (e.g., capacity limits): $g_i(u_i) \le 0$

Single-phase OPF In terms of V only

Equality constraints (BIM in complex form)

• Expresses s_i in terms of voltages V

$$s_{j}(V) = \sum_{k:j \sim k} S_{jk}(V) := \sum_{k:j \sim k} \bar{y}_{jk}^{s} \left(|V_{j}|^{2} - V_{j}\bar{V}_{k} \right) + \bar{y}_{jj}^{m} |V_{j}|^{2}, \quad j \in \overline{N}$$

Cost C(V) := C(s(V), V) expressed as function of V

• Fuel cost:

$$C(V) := \sum_{j:\text{gens}} c_j \operatorname{Re}(s_j(V)) = \sum_{j:\text{gens}} c_j \operatorname{Re}\left(\sum_{k:j\sim k} \bar{y}_{jk}^s \left(|V_j|^2 - V_j \bar{V}_k\right) + \bar{y}_{jj}^m |V_j|^2\right)$$

Total real power loss:

$$C(V) := \sum_{j} \operatorname{Re}(s_j(V))$$

Single-phase OPF Operational constraints

Injection limits (e.g. generation or load capacity limits) $s_j^{\min} \leq s_j(V) \leq s_j^{\max}$:

$$s_j^{\min} \leq \sum_{k:j\sim k} \bar{y}_{jk}^s \left(|V_j|^2 - V_j \bar{V}_k \right) + \bar{y}_{jj}^m |V_j|^2 \leq s_j^{\max}, \qquad j \in \overline{N}$$

• Or in polar form:

$$p_{j}^{\min} \leq \sum_{k:k\sim j} \left(g_{jk}^{s} + g_{jk}^{m} \right) |V_{j}|^{2} - \sum_{k:k\sim j} |V_{j}| |V_{k}| \left(g_{jk}^{s} \cos \theta_{jk} + b_{jk}^{s} \sin \theta_{jk} \right) \leq p_{j}^{\max}$$

$$q_{j}^{\min} \leq -\sum_{k:k\sim j} \left(b_{jk}^{s} + b_{jk}^{m} \right) |V_{j}|^{2} - \sum_{k:k\sim j} |V_{j}| |V_{k}| \left(g_{jk}^{s} \sin \theta_{jk} - b_{jk}^{s} \cos \theta_{jk} \right) \leq q_{j}^{\max}$$

Single-phase OPF Operational constraints

Voltage limits (same as before):

$$v_j^{\min} \leq |V_j|^2 \leq v_j^{\max}, \quad j \in \overline{N}$$

Line limits (same as before):

$$\left| y_{jk}^{s}(V_{j} - V_{k}) + y_{jk}^{m} V_{j} \right|^{2} \leq \ell_{jk}^{\max}, \qquad (j,k) \in E$$
$$\left| y_{kj}^{s}(V_{k} - V_{j}) + y_{kj}^{m} V_{k} \right|^{2} \leq \ell_{kj}^{\max}, \qquad (j,k) \in E$$

• Line limits can also be on line powers $\left(S_{jk}(V), S_{kj}(V)\right)$ or apparent powers $\left(\left|S_{jk}(V)\right|, \left|S_{kj}(V)\right|\right)$

Single-phase OPF In terms of V only

Feasible set

 $\mathbb{V} := \left\{ V \in \mathbb{C}^{N+1} \mid V \text{ satisfies operational constraints} \right\}$

OPF in BIM

 $\min_{V \in \mathbb{V}} \quad C(V)$

- Does not need assumption $y_{jk}^s = y_{kj}^s$
- Can accommodate single-phase transformers with complex turns ratios

Single-phase OPF In terms of V only

Feasible set

 $\mathbb{V} := \left\{ V \in \mathbb{C}^{N+1} \mid V \text{ satisfies operational constraints} \right\}$

OPF in BIM

 $\min_{V \in \mathbb{V}} \quad C(V)$

We will mostly study this simple OPF Can express it as a QCQP

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1. Bus injection model

- Single-phase devices
- Single-phase OPF
- Single-phase OPF as QCQP
- 2. Branch flow model
- 3. NP-hardness
- 4. Global optimality
- 5. Techniques for scalability: case study

Quadratically constrained quadratic program:

 $\min_{x \in \mathbb{C}^n} \quad x^{\mathsf{H}} C_0 x$

- s.t. $x^{\mathsf{H}}C_{l}x \leq b_{l}, \qquad l = 1, ..., L$
- $C_l: n \times n$ Hermitian matrix $\Rightarrow x^H C_l x \in \mathbb{R}$
- $b_l \in \mathbb{R}$
- Homogeneous QCQP : all monomials are of degree 2

Inhomogeneous QCQP

$$\min_{x \in \mathbb{C}^n} \quad x^{\mathsf{H}} C_0 x + \left(c_0^{\mathsf{H}} x + x^{\mathsf{H}} c_0 \right)$$

s.t.
$$x^{\mathsf{H}} C_l x + \left(c_l^{\mathsf{H}} x + x^{\mathsf{H}} c_l \right) \leq b_l, \qquad l = 1, \dots, L$$

Homogenization: introduce scalar var $t \in \mathbb{C}$

• Set
$$x := \hat{x}\overline{t}$$
 and require $|t|^2 = 1$ (i.e., $t = e^{i\theta}$ for some θ). Then

$$x^{\mathsf{H}}C_{l}x + c_{l}^{\mathsf{H}}x + x^{\mathsf{H}}c_{l} = \hat{x}^{\mathsf{H}}C_{l}\hat{x} + c_{l}^{\mathsf{H}}(\hat{x}\bar{t}) + (\hat{x}\bar{t})^{\mathsf{H}}c_{l} = \begin{bmatrix} \hat{x}^{\mathsf{H}} & t^{\mathsf{H}} \end{bmatrix} \begin{bmatrix} C_{l} & c_{l} \\ c_{l}^{\mathsf{H}} & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ t \end{bmatrix}$$

Equivalent homogeneous QCQP

$$\begin{split} \min_{\hat{x} \in \mathbb{C}^{n}, t \in \mathbb{C}} & \left[\hat{x}^{\mathsf{H}} \ t^{\mathsf{H}} \right] \begin{bmatrix} C_{0} \ c_{0} \\ c_{0}^{\mathsf{H}} \ 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ t \end{bmatrix} \\ \text{s.t.} & \left[\hat{x}^{\mathsf{H}} \ t^{\mathsf{H}} \right] \begin{bmatrix} C_{l} \ c_{l} \\ c_{l}^{\mathsf{H}} \ 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ t \end{bmatrix} \leq b_{l}, \qquad l = 1, \dots, L \\ & \left[\hat{x}^{\mathsf{H}} \ t^{\mathsf{H}} \right] \begin{bmatrix} 0 \ 0 \\ 0 \ 1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ t \end{bmatrix} = 1 \end{split}$$

• If $(\hat{x}^{\text{opt}}, t^{\text{opt}})$ is optimal for homogeneous QCQP, then product $x^{\text{opt}} := \hat{x}^{\text{opt}} t^{\text{opt}}$ is optimal for original inhomogeneous QCQP

Steven Low OPF Bus injection model

OPF as QCQP Equivalent real QCQP

Even though OPF is often formulated in \mathbb{C} , it is converted to \mathbb{R} before being solved iteratively

 $\min_{x \in \mathbb{C}^n} \quad x^{\mathsf{H}} C_0 x$ s.t. $x^{\mathsf{H}} C_l x \leq b_l, \qquad l = 1, \dots, L$

- $C_l: n \times n$ complex Hermitian matrix
- $b_l \in \mathbb{R}$

Equivalent to:

$$\min_{\substack{(x_r, x_i) \in \mathbb{R}^{2n} \\ \text{s.t.}}} \begin{bmatrix} x_r \\ x_i \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} C_{0r} & -C_{0i} \\ C_{0i} & C_{0r} \end{bmatrix} \begin{bmatrix} x_r \\ x_i \end{bmatrix}$$
$$s.t. \begin{bmatrix} x_r \\ x_i \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} C_{lr} & -C_{li} \\ C_{li} & C_{lr} \end{bmatrix} \begin{bmatrix} x_r \\ x_i \end{bmatrix} \le b_l, \quad l = 1, \dots, L$$

• $2n \times 2n$ real symmetric matrices

To write OPF as QCQP:

- Assume cost function $C(V) = V^{H}C_{0}V$ can be written as a quadratic form
- Need to rewrite operational constraints in terms of quadratic forms

Injection limits $s_j^{\min} \le s_j(V) \le s_j^{\max}$

$$s_{j}(V) = V_{j}I_{j}^{\mathsf{H}} = \left(e_{j}^{\mathsf{H}}V\right)\left(e_{j}^{\mathsf{H}}I\right)^{\mathsf{H}} = e_{j}^{\mathsf{H}}VV^{\mathsf{H}}Y^{\mathsf{H}}e_{j}$$
$$s_{j}(V) = \operatorname{tr}\left(e_{j}^{\mathsf{H}}VV^{\mathsf{H}}Y^{\mathsf{H}}e_{j}\right) = \operatorname{tr}\left(\left(Y^{\mathsf{H}}e_{j}e_{j}^{\mathsf{H}}\right)VV^{\mathsf{H}}\right) =: V^{\mathsf{H}}Y_{j}^{\mathsf{H}}V$$

Injection limits $s_j^{\min} \le s_j(V) \le s_j^{\max}$

$$s_{j}(V) = V_{j}I_{j}^{\mathsf{H}} = \left(e_{j}^{\mathsf{H}}V\right)\left(e_{j}^{\mathsf{H}}I\right)^{\mathsf{H}} = e_{j}^{\mathsf{H}}VV^{\mathsf{H}}Y^{\mathsf{H}}e_{j}$$
$$s_{j}(V) = \operatorname{tr}\left(e_{j}^{\mathsf{H}}VV^{\mathsf{H}}Y^{\mathsf{H}}e_{j}\right) = \operatorname{tr}\left(\left(Y^{\mathsf{H}}e_{j}e_{j}^{\mathsf{H}}\right)VV^{\mathsf{H}}\right) =: V^{\mathsf{H}}Y_{j}^{\mathsf{H}}V$$

- Y_j is not Hermitian so $V^{\mathsf{H}}Y_j^{\mathsf{H}}V$ is generally complex
- Define $\Phi_j := \frac{1}{2} \left(Y_j^{\mathsf{H}} + Y_j \right), \qquad \Psi_j := \frac{1}{2i} \left(Y_j^{\mathsf{H}} Y_j \right)$

• Then
$$\operatorname{Re}(s_j) = V^{\mathsf{H}} \Phi_j V$$
, $\operatorname{Im}(s_j) = V^{\mathsf{H}} \Psi_j V$

Hence
$$s_j^{\min} \le s_j(V) \le s_j^{\max}$$
 is equivalent to:
 $p_j^{\min} \le V^{\mathsf{H}} \Phi_j V \le p_j^{\max}, \quad q_j^{\min} \le V^{\mathsf{H}} \Psi_j V \le q_j^{\max}$

Steven Low OPF Bus injection model

OPF as QCQP Voltage limits

Voltage magnitude is: $|V_j|^2 = V^H E_j V$ where $E_j := e_j e_j^T$

Hence voltage limits are: $v_j^{\min} \leq V^{\mathsf{H}} E_j V \leq v_j^{\max}$

OPF as QCQP Line limits

Write I_{jk} in terms of voltage vector V:

$$I_{jk} = y_{jk}^{s}(V_{j} - V_{k}) + y_{jk}^{m}V_{j} = \left(y_{jk}^{s}(e_{j} - e_{k})^{\mathsf{T}} + y_{jk}^{m}e_{j}^{\mathsf{T}}\right)V$$

Hence current limit is: $|I_{jk}|^2 = V^{\mathsf{H}} \hat{Y}_{jk} V \leq \ell_{jk}^{\max}$ where

$$\hat{Y}_{jk} := \left(\bar{y}_{jk}^{s}(e_{j} - e_{k}) + \bar{y}_{jk}^{m}e_{j} \right) \left(y_{jk}^{s}(e_{j} - e_{k})^{\mathsf{T}} + y_{jk}^{m}e_{j}^{\mathsf{T}} \right)$$

OPF as QCQP Simplest formulation

$$\begin{split} \min_{V \in \mathbb{C}^{N+1}} & V^{\mathsf{H}} C_0 V \\ \text{s.t.} & p_j^{\min} \leq V^{\mathsf{H}} \Phi_j V \leq p_j^{\max}, \qquad j \in \overline{N} \\ & q_j^{\min} \leq V^{\mathsf{H}} \Psi_j V \leq q_j^{\max}, \qquad j \in \overline{N} \\ & v_j^{\min} \leq V^{\mathsf{H}} E_j V \leq v_j^{\max}, \qquad j \in \overline{N} \\ & V^{\mathsf{H}} \hat{Y}_{jk} V \leq \ell_{jk}^{\max}, \qquad (j,k) \in E \\ & V^{\mathsf{H}} \hat{Y}_{kj} V \leq \ell_{kj}^{\max}, \qquad (j,k) \in E \end{split}$$

Outline

1. Bus injection model

- 2. Branch flow model
 - Radial network
- 3. NP-hardness
- 4. Global optimality

Radial network Assumptions: DistFlow model

Radial network

• BFM most useful for modeling distribution systems which are mostly radial (and unbalanced)

$$z_{jk}^s = z_{kj}^s$$
 or equivalently $y_{jk}^s = y_{kj}^s$

- Does not apply to 3-phase transformers in ΔY or $Y\Delta$ configuration or their per-phase equivalent with complex gains

 $y_{jk}^m = y_{kj}^m = 0$

• Reasonable assumption for distribution line where $|y_{jk}^m|, |y_{kj}^m| \ll |y_{jk}^s|$

Includes only voltage sources and power sources

- Optimization variables are voltages (squared magnitudes) v_i and power injections s_i respectively
- Can include current sources or an impedances with additional vars and constraints.

DistFlow model

Power flow equations

• All lines point away from bus 0 (root)

$$\sum_{k:j \to k} S_{jk} = S_{ij} - z_{ij}^{s} \ell_{ij} + s_{j}, \qquad j \in \overline{N}$$
$$v_{j} - v_{k} = 2 \operatorname{Re} \left(\overline{z}_{jk}^{s} S_{jk} \right) - |z_{jk}^{s}|^{2} \ell_{jk}, \qquad j \to k \in E$$
$$v_{j} \ell_{jk} = |S_{jk}|^{2}, \qquad j \to k \in E$$

Operational constraints

$$s_{j}^{\min} \leq s_{j} \leq s_{j}^{\max}$$
$$v_{j}^{\min} \leq v_{j} \leq v_{j}^{\max}$$
$$\ell_{jk} \leq \ell_{jk}^{\max}$$

Single-phase OPF DistFlow model

Feasible set

 $\mathbb{X}_{df} := \left\{ x := (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies PF equations & operational constraints} \right\}$

OPF in BFM

 $\min_{x} C(x) \qquad \text{s.t.} \qquad x \in \mathbb{X}_{df}$

Single-phase OPF Equivalence

Recall for BIM:

- Feasible set: $\mathbb{V} := \{ V \in \mathbb{C}^{N+1} \mid V \text{ satisfies operational constraints} \}$
- OPF: $\min_{V \in \mathbb{V}} C(V)$

OPF in BFM is equivalent to OPF in BIM:

- Feasible sets X_{df} and V are equivalent (Ch 5)
- ... provided cost functions C(x) and C(V) are the same

General radial network

Does not assume $z_{jk}^s = z_{kj}^s$ nor $y_{jk}^m = y_{kj}^m = 0$

Need branch quantities in both directions

•
$$\ell := (\ell_{jk}, \ell_{kj}, (j,k) \in E), \ S := (S_{jk}, S_{kj}, (j,k) \in E)$$

• $\alpha_{jk} := 1 + z_{jk}^{s} y_{jk}^{m}, \ \alpha_{kj} := 1 + z_{kj}^{s} y_{kj}^{m}$

BFM for general radial network

$$s_j = \sum_{k:j \sim k} S_{jk}, \qquad j \in \overline{N}$$

$$|\alpha_{jk}|^2 v_j - v_k = 2 \operatorname{Re} \left(\alpha_{jk} \bar{z}_{jk}^s S_{jk} \right) - |z_{jk}^s|^2 \ell_{jk}, \qquad (j,k) \in E$$

$$|\alpha_{kj}|^{2}v_{k} - v_{j} = 2 \operatorname{Re}\left(\alpha_{kj}\bar{z}_{kj}^{s}S_{kj}\right) - |z_{kj}^{s}|^{2}\ell_{kj}, \qquad (j,k) \in E$$

$$\left|S_{jk}\right|^{2} = v_{j}\ell_{jk}, \qquad \left|S_{kj}\right|^{2} = v_{k}\ell_{kj}, \qquad (j,k) \in E$$

$$\bar{\alpha}_{jk}v_j - \bar{z}_{jk}^s S_{jk} = \left(\bar{\alpha}_{kj}v_k - \bar{z}_{kj}^s S_{kj}\right)^{\mathsf{H}}, \qquad (j,k) \in E$$

Single-phase OPF General radial network

Operational constraints (same as before but line limits in both directions)

 $s_j^{\min} \le s_j \le s_j^{\max}, \quad v_j^{\min} \le v_j \le v_j^{\max}, \quad \ell_{jk} \le \ell_{jk}^{\max}, \quad \ell_{kj} \le \ell_{kj}^{\max}$

Feasible set

 $X_{\text{tree}} := \left\{ x := (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies PF equations & operational constraints} \right\}$

OPF in BFM

 $\min_{x} C(x) \qquad \text{s.t.} \qquad x \in \mathbb{X}_{\text{tree}}$

OPF in BFM is equivalent to OPF in BIM:

- Feasible sets \mathbb{X}_{tree} and \mathbb{V} are equivalent (Ch 5)
- ... provided cost functions C(x) and C(V) are the same

Outline

- 1. Bus injection model
- 2. Branch flow model

3. NP-hardness

- OPF feasibility
- OPF is NP-hard
- 4. Global optimality

OPF feasibility

Tree network

Star network (\overline{N}, E) with N + 1 buses and M = N lines

- $y_{jk}^s = y_{kj}^s$ and $y_{jk}^m = y_{kj}^m = 0$
- Fixed voltage magnitudes $|V_j| := 1$ pu
- Fixed and given injections $(p_j, q_j), j \in N_L \subset \overline{N}$
- Dispatchable generation (p_j, q_j) with $p_j \ge 0, \ j \in N_G \subset \overline{N}$
- Line limits: $|\theta_j \theta_k| \le \overline{\theta} \in (0, \pi/2], \ (j, k) \in E$

Each instance of OPF feasibility problem is specified by

- Tree network $(N_G \cup N_L, E)$
- Line admittances $(g_{jk}, b_{jk}, (j, k) \in E)$
- Line limits $\overline{\theta} \in (0, \pi/2]$
- Fixed injections $(p_j, q_j, j \in N_L)$

OPF feasibility Tree network

Find

- Real power generations $\left(p_j, j \in N_G\right) \ge 0$
- Voltage angles $\left(\theta_{j}, j \in \overline{N}\right)$
- . Line flows $\left(P_{jk},Q_{jk},(j,k)\in E\right)$

that satisfy the polar form power flow equation and line limits:

NP-hardness P and NP

Let

- Σ : finite set of symbols
- Σ^* : set of all finite strings of symbols in Σ
- $L \subseteq \Sigma^*$: language over Σ

Deterministic Turing machine (DTM): computation model that takes an input $\sigma \in \Sigma^*$, performs computation (read, write, state transition), and either halts in "yes" or "no" state, or does not halt

Given DTM *M*, time complexity function $c_M : \mathbb{N}_+ \to \mathbb{N}_+$:

 $c_M(n) := \max\{m : \exists \sigma \in \Sigma^* \text{ with } |\sigma| = n \text{ s.t. } M \text{ takes } m \text{ steps to halt on } \sigma\}$

M is called a polynomial time DTM if \exists a polynomial *p* s.t. $c_M(n) \le p(n)$ for all *n*

Language recognized by (DTM or NDTM) M is

 $L_{M} := \{ \sigma \in \Sigma^{*} : M \text{ halts on } \sigma \text{ in "yes" state} \}$

NP-hardness P and NP

The class P of languages is

 $\mathsf{P} := \{L \subseteq \Sigma^* : \exists \text{ polynomial time DTM } M \text{ for which } L = L_M \}$

Informally: P consists of all language over Σ that are recognized by a DTM in polynomial time

While P captures "solvability" of a problem, NP captures "verifiability"

• It is difficult (NP-complete) to find a cycle in an arbitrary graph that visits every node exactly once, but easy to verify if a candidate is a solution

Given NDTM *M*, time complexity function $c_M : \mathbb{N}_+ \to \mathbb{N}_+$:

 $c_M(n) := \max\{m : \exists \sigma \in \Sigma^* \text{ with } |\sigma| = n \text{ s.t. } M \text{ takes } m \text{ steps to halt on } \sigma \text{ in "yes" state} \}$

M is called a polynomial time NDTM if \exists a polynomial *p* s.t. $c_M(n) \le p(n)$ for all *n*

The class NP of languages is

NP := { $L \subseteq \Sigma^*$: \exists polynomial time NDTM *M* for which $L = L_M$ }

 $\mathsf{P} \subseteq \mathsf{NP}$

Informally: NP consists of all language recognized by a NDTM (or verifiable by a DTM) in polynomial time

NP-hardness NP-hard and NP-complete

A function $f: \Sigma_1^* \to \Sigma_2^*$ is a language $L_j := \{(\sigma, f(\sigma)) : \sigma \in \Sigma_1^*\} \subseteq \Sigma_1^* \times \Sigma_2^*$ DTM *M* computes *f* if $L_M = L_f$

A polynomial reduction from $L_1 \subseteq \Sigma_1^*$ to $L_2 \subseteq \Sigma_2^*$ is a function $f: \Sigma_1^* \to \Sigma_2^*$ which can be computed by a polynomial time DTM s.t.

$$\sigma \in L_1 \iff f(\sigma) \in L_2, \qquad \sigma \in \Sigma_1$$

A language L is NP-hard if for every $L' \in$ NP there exists a polynomial reduction from L' to L

It is NP-complete if *L* is NP-hard and $L \in NP$

• NP-complete languages are in a sense the "hardest" languages in NP

NP-hardness Decision problems

A decision problem is a problem whose solution is either "yes" or "no"

• It is defined by a set of finite instances, e.g. specified in terms of sets, graphs, functions, real numbers

Let Π be a decision problem (or its instances) that can be "encoded" into a language problem over some alphabet Σ

• Informally, an encoding is $\sigma: \Pi \to \Sigma^*$ that maps each instance $y \in \Pi$ to a string $\sigma(y) \in \Sigma^*$

Let $Y \subseteq \Pi$ be the subset of instances whose solutions are "yes"

• We will refer to Y either as a set of problem instances or simply a problem by itself

Let $L_Y := \{\sigma(y) : y \in Y\}$ be the language defined by instances in Y

• Solution of instance $y \in \Pi$ is "yes" if and only if $y \in Y$ if and only if $\sigma(y) \in L_Y$

Hardness properties of Y are then defined in terms of hardness properties of its encoding L_Y

- e.g. *Y* is in P if $L_Y \in P$, *Y* is NP-complete if L_Y is NP-complete
- OPF feasibility problem is such a decision problem

NP-hardness

Theorem

OPF feasibility problem on a tree network is NP-hard

Remarks:

- OPF feasibility is not proved to be in NP, because solution can be irrational
- Proved by polynomial reduction of NP-complete subset sum problem to OPF feasibility
- OPF feasibility can be proved to be strongly NP-hard by polynomial reduction of strongly NP-complete one-in-three 3SAT problem to OPF feasibility

NP-hardness is worst-case result

- Subclasses of OPF cane polynomial time solvable
- e.g., those satisfied sufficient conditions for exact relaxations or global optimality

Outline

- 1. Bus injection model
- 2. Branch flow model
- 3. NP-hardness
- 4. Global optimality
 - Convex relaxation
 - Lyapunov-lik condition for global optimality
 - Application to OPF on radial network

ing, if for any relaxed point, there exists a path connecting it to the non-convex reasible set and the path satisfies

- along the path the cost is non-increasing,
- along the path the 'distance' to the non-convex feasible? set is non-increasing.

set is non-increasing; then the problem must have exact relaxation and no spurious local optima simultaneously. Here the distance can be any properly constructed function, as we will define later as a properly constructed function, as we will define later as a properly constructed function. The second part is on the necessary condition, which says that if a problem does have exact relaxation and no spurious local optima simultaneously, then there must exist such Lyapunov-like function and paths satisfying the requirements above.¹ Though Lyapunov-like functions and paths are guaranteed to exist; for specific problems it could still be difficult to find to exist; for specific problems it could still be difficult to find to exist; for specific problems it could still be difficult to find to exist; for specific problems it could still be difficult to find

Consider

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- X: notherweith compact approach the types same for the type state of the time to the type of the time to the type of type of the type of type o
- X: compact and spinyax a upper aptimate the periori explain Phe
- $f: \mathbb{R}^n$ to it begins conversional contraction of the second widespreachenepiwoak experience that local algorithms for OPF

Optimal solutions exist for both problems P1 and P2

C. Background for Power Systems

C. Recker oundefappficeationSystemain motivation of this work, Optimale Bowter applica (GRBS) and an appropriate institution worthe Optimal power spacements in a property of the second secon at appropriation prohibing the providence of the property of the providence of the p op anninear ophysical descent and in period on already strain sublects the normalite are physically as such a Beaution aits Astrangulation knowlage Bel hotherefore, there ip nake owner facient also sitter [9], [20], [21]. Therefore, there is no known efficient algorithm The necessary condition is based upon some stronger assumptions so the second part is not the exact converse of the first part. The necessary condition is based upon some stronger assumptions so the

III. PRELIMINARIES

In this paper, we will use \mathbb{K} to denote the set \mathbb{R} d anumbers or the set C of complex unmpers. For any inerer Contrais a Banach space.

Consider a (potentially non-convex) optimization pro

minimize minimize	f(x)
subject to subject to	$x \in \mathcal{X}$
and its convex relaxation and its convex relaxation	
	- ()

minimize	f(x) f(x)
subject to subject to	$\begin{array}{c} x \in \hat{\mathcal{X}} \\ x \in \hat{\mathcal{X}} \end{array}$

Here \mathcal{X} is a nonempty compact subset of \mathbb{K}^n , not nece Here Knice Repart and a to the set of the se

convex while X. The superiset of Yver The a centrement rest Rente efficiently represented **Definition 1.** A point Definitionerel.exispoin (1) if there herists give $x \in \mathcal{X}$ with $||x - x^{\mathrm{lo}}|$



ocal optim $\leq f(x) f$

Definition 2 (Strong Exactness). We say the relaxation Definition 2ehStrong Exartness bethe alay othe of (2) atiofe exact weither espect to bin in any for timal point of (2) is fee and hence globally optimal, for (1). Unless otherwise specified, we will always use the

extendens reflectivisauspectified, exactively a Definition the inaparticrefer that such (2) triang x arac these. (Definition 2, if imparticular that, if (2) is exact, then $\forall \hat{x} \in \mathcal{X} \setminus \mathcal{X}, f$ **Definition** f(x). A path in $S \subseteq \mathbb{K}^n$ connecting point a to Definition Brugusaturition & KO, donne Singe go that ho basis dh (b) tin hous function $h: [0,1] \to S$ such that h(0)and we have before to a path by the corresponding function the we main defee ft the papeby the corresponding function

the nemainder not the paper are equivalent:

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Our conditions have two parts. The first part, which also appeared in [18], is on the sufficient condition. Roughly speaking, if for any relaxed point, there exists a path connecting it to the non-convex deasible set and the path satisfies Exact a feasible set and the path satisfies

• along the path the 'distance' to the non-convex feasibile set is non-increasing, set is non-increasing.

Set is non-intereasing, then the problem must have exact relaxation and no spurious Definition for an optimal simultaneously. Here the distance can be any property constructed function, as we will define later as a property constructed function for as we will define later as a property constructed function. The second patrix is on the necessary condition, which says that if a problem does have the necessary condition, which says that if a problem does have the necessary condition, which says that if a problem does have the necessary condition which says that if a problem does have the necessary condition which says that if a problem does have the necessary condition which says that if a problem does have the necessary condition which says that if a problem does have the necessary condition which says that if a problem does have the necessary condition and no spurious local eptimal simultaneously then there must exist such Lyapunoy-like function and paths then there must exist such Lyapunoy-like function and paths then there must exist such Lyapunoy-like function and paths then there must exist such Lyapunoy-like function and paths then there must exist such Lyapunoy-like function and paths then there must exist such Lyapunoy-like function and paths then there must exist such Lyapunoy-like function and paths then there must exist such Lyapunoy-like function and paths then there must exist such Lyapunoy-like function and paths then there must exist such Lyapunoy-like function and paths then there must exist such Lyapunoy-like function and paths then there must exist such Lyapunoy-like function and paths the there functions and paths are guaranteed to exist, for specific problems it could still be difficult to find, to exist, for specific problems it could still be difficult to find, to exist, for specific problems it could still be difficult to find, to exist, for specific problems it could still be difficult to find, to exist, for specific problems it could still be difficult to to exist for specific problems it could still be difficult to find The Pystpurtov-fike Whethen and pather of a fulles problems from the Lyapunoy-like function and pathy of a new problem from the primitives problems with known by apunovalike function and Pathats This herpeople allowanges tand eurowand in an end whappy resultsrap the people active sources of the people active states of the people active states and the people active states active sta the water prosed approach the type pocific of the proses approach the type pocific of the proses approach the prosest of the p Rowern Flowdattsh lowarankn SDP. chukenork proves the OTPF knowavecondisionridusatoean drainalaeckad in perioriexforaioPRFe twittespread empirical logatrienthaanocat anglystlanglain OPF widespread compinional experiments that local algorithms for OPF

problems often work extremely well.

C. Background for Power Systems

C. Recker of the fap fice a for state that motivation of this work, Optimale Bowter applyica (GRRS) and an admentative to the Optimal power systems OPIFstiproposed problem Select in thes

or room apprinting. In this paper, we show that a the sufficient condition for redexation exactness is also suffic for local optima to be globally optimal. To the best of authors' knowledge, this is the first analytical result of itsk and we hope that the approaches proposed in this paper thelp derive more sufficient conditions along this direction

II. PRELIMINARIES

In this paper, we will use \mathbb{K} to denote the set \mathbb{R} of numbers or the set C of complex numbers. For any f nergert, c. Ran is a Banach space.

Consider a (potentially non-convex) optimization problem



Here \mathcal{X} is a nonempty compact subset of \mathbb{K}^n , not necess Here \mathcal{X} is a nonempty compact subset of \mathbb{K}^n not necess convex, while \mathcal{X} be the necessary compact and compac

convexet while X. The superset of Jver The a continuinity represente efficiently represented **Definition 1.** A point Definition-1.exispoint (1) if there herists give

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) cal optimy $\leq f(x)$ for

 $x \in \mathcal{X}$ with $||x - x^{\mathrm{lo}}|$ Definition 2 (Strong Exactness). We say the relaxation (Definition 2e Strong Exaptans by Washwith of an information exact heither especitive (b) if any for (in al point of (2) is feasi and hence globally optimal, for (1). Unless otherwise specified, we will always use the

extortess reflectivisauspectified, exactively a Defysition them anaparticrefer that such String xacac these. Definition 2, if in imparticular that, if (2) is exact, then $\forall \hat{x} \in \mathcal{X} \setminus \mathcal{X}, f(\hat{x})$ **Definition** f(x). A path in $S \subset \mathbb{K}^n$ connecting point a to t



Path

Definition

- 1. A path in $Y \subseteq \mathbb{R}^n$ connecting *a* to be *b* in *Y* is a continuous function $h : [0,1] \to Y$ s.t. h(0) = a and h(1) = b
- 2. An arbitrary set $\{h_i : i \in I\}$ of paths in *Y* is called
 - uniformly bounded if \exists finite H s.t. $\|h_i(t)\|_{\infty} \leq H$ for all $t \in [0,1]$ and $i \in I$
 - uniformly equicontinuous if for any $\epsilon > 0$, $\exists \ \delta > 0$ s.t. $\|h_i(t_2) - h_i(t_1)\|_{\infty} \leq \epsilon$ for all $i \in I$ whenever $\|t_2 - t_1\| < \delta$

Example: If all paths in $\{h_i : i \in I\}$ are linear, then $\{h_i : i \in I\}$ is both uniformly bounded and uniformly equicontinuous





Lyapunov-like function

Definition

A Lyapunov-like function associated with problems P1 and P2 is a continuous function $V: \hat{X} \to \mathbb{R}_+$ s.t. V(x) = 0 if $x \in X$ and V(x) > 0 if $x \in \hat{X} \setminus X$



Global optimality Optimality conditions

1. There is a Lyapunov-like function V and, for every infeasible point $x \in \hat{X} \setminus X$, \exists path h_x s.t.

(a) $h_x(0) = x, h_x(1) \in X, f(h_x(1)) < f(x)$

(b) Both $f(h_x(t))$ and $V(h_x(t))$ are nonincreasing for $t \in [0,1]$

Every infeasible pt x can be brought back to X with a lower cost

Nonincreasing cost or certificate along path to feasibility

- 2. The set $\{h_x : x \in \hat{X} \setminus x\}$ of paths in 1 is uniformly bounded and uniformly equicontinuous
- 3. At least one of the following holds:
 - (d) All local optima of P1 are isolated (i.e., every local optimum has a neighborhood with no other local optimum)

(e) For
$$\{h_x : x \in \hat{X} \setminus x\}$$
 in 1, $\exists \alpha > 0$ s.t. for all $x \in \hat{X} \setminus X$ and all $0 \le s < t \le 1$,
 $f(h_x(s)) - f(h_x(t)) \ge \alpha ||h_x(s) - h_x(t)||$ Cost must defined by the cost must defined by

Cost must decrease sufficiently along path to feasibility

for some norm $\|\cdot\|$

Global optimality

Theorem [Sufficiency]

Suppose conditions 1, 2, 3 hold.

- 1. The convex relaxation P2 is exact wrt P1
- 2. Every local optimum of P1 is a global optimum

Moreover if condition 3(a) holds, then the optimal point is unique

Remarks

- Exactness \iff existence of $\{h_x : x \in \hat{X} \setminus x\}$ that satisfies condition 1
- Other conditions are to prove that there is no spurious local optimum

Global optimality

A set $y \subseteq \mathbb{R}^n$ is semianalytic if every $x \in \mathbb{R}^n$ has a neighborhood U s.t. Kand U can be represented as a finite Boolean combination of sets $\{x : g(x) = 0\}$ and $\{x \in \mathcal{X}, y \in \mathcal{X}\}$ of this ford there exists $k > x \in \mathcal{X}, y \in \mathcal{X}, y \in \mathcal{X}\}$ is the set of the some analytic functions g, h (usually satisfied by engineering problems)

Theorem [Necessity]

Suppose X is semianalytic and f is analytic. If

- 1. The convex relaxation P2 is exact wrt P1, and
- 2. Every local optimum of P1 is a global optimum

then \exists Lyapunov-like function V and a family of paths $\{h_x : x \in \hat{X} \setminus x\}$ that satisfy cond 1 and 2

Now we are in a good position to discuss sor that rule out pseudo local optima and therefore g any local optimum must be a global optimum.

Corollary 3. If all local optima of (1) are i Condition (C) implies that any local optimum of (optimum.

Here, local optima being isolated means any lo of (1) has an open neighborhood which contains optimum. The proof is straightforward as by defin local optimum could not be pseudo local optimu this case the optimum can be proved to be also

Another way to eliminate pseudo local o strengthening the monotonicity of $f(h_x(t))$ in C Consider the following condition which is slig

In Condition (C'), $\|\cdot\|$ could be any norm

 $f(h_x(t)) - f(h_x(s)) \ge k ||h_x(t) - h|$





Lyapunov-like optimality condition Comparison with Lyapunov stability

Consider the dynamical system

 $\dot{x} = f(x(t)), \qquad t \ge 0, \ x(0) = x_0$

Let x^* be an equilibrium point where $f(x^*) = 0$

Lyapunov stability theory

- 1. Lyapunov function V(x) is a continuously differentiable function s.t. $V(x) > V(x^*)$ and $\dot{V}(x) < 0$ for all $x \neq x^*$ in \mathbb{R}^n
- 2. *V* certifies stability of x^* : x^* is globally asymptotically stable if a Lyapunov function V(x) exists

Lapunov-like optimality condition

- 1. *V* certifies global optimality of a local optimum $x^* \in X$
- 2. No dynamics to specify path : no requirement on differentiability of V, but
- 3. Need to construct both V and paths $\{h_x : x \in \hat{X} \setminus x\}$ (no general method known)

Application to OPF Recall: OPF in DistFlow model

DistFlow equations (radial network):

$$\sum_{k:j \to k} S_{jk} = S_{ij} - z_{ij}^{s} \ell_{ij} + s_{j}, \qquad j \in \overline{N}$$

$$v_{j} - v_{k} = 2 \operatorname{Re}\left(\overline{z}_{jk}^{s} S_{jk}\right) - |z_{jk}^{s}|^{2} \ell_{jk}, \qquad j \to k \in E$$

$$v_{j} \ell_{jk} = |S_{jk}|^{2}, \qquad j \to k \in E \qquad \text{Nonconvex constraint}$$

Operational constraints:

$$s_j^{\min} \le s_j \le s_j^{\max}, \quad v_j^{\min} \le v_j \le v_j^{\max}, \quad \ell_{jk} \le \ell_{jk}^{\max}$$

Feasible set

 $X := \{x := (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies DistFlow equations & operational constraints} \}$

Application to OPF Convex relaxation

Replace

$$v_j \mathcal{C}_{jk} = |S_{jk}|^2, \qquad j \to k \in E$$

by

 $v_j \mathcal{C}_{jk} \ge |S_{jk}|^2, \qquad j \to k \in E$

Convex superset

 $\hat{X} := \{x : x \text{ satisfies constraints with SOC replacement}\}$

Consider

Nonconvex optimization P1:	$\min_{x} f(x)$	s.t.	$x \in X \subseteq \mathbb{R}^n$
Convex relaxation P2:	$\min_{x} f(x)$	s.t.	$x \in \hat{X} \subseteq \mathbb{R}^n$

Optimality conditions OPF in DistFlow model

- 4. *X* is nonempty compact, \hat{X} is compact, cost function *f* is convex and continuous
- 5. Cost function $f(x) = f(p, q, v, \ell)$ is independent of S = (P, Q), continuously differentiable strongly inc. in ℓ with $\nabla f(x) \ge 0$. Moreover $\exists c > 0$ s.t. $\frac{\partial f}{\partial \ell_l}(x) \ge 0$ for all $l \in E$ and all $x \in \hat{X}$ demand large enough not to pose constraints 6. No lower bounds on injections: $s_j^{\min} = -\infty i\infty$ usually satisfied
- 7. $z_{jk} =: (r_{jk}, x_{jk}) > 0$ and line limit satisfies $|z_{jk}|^2 \ell^{\max} \le v_j^{\min}$

Remarks

• Differentiability is not necessary and can be replaced by subgradient (which always exist since f is convex)

Global optimality OPF in DistFlow model

Theorem

Suppose conditions 4-7 hold on radial network.

- 1. Convex relaxation P2 is exact wrt P1
- 2. Every local optimum of P1 is a global optimum

Remarks

• Exactness is proved in Ch 11 on Semidefinite relaxations of OPF in BFM

Global optimality Construction: V

Proof requires construction of Lyapunov-like function V and family of paths $\{h_x : x \in \hat{X} \setminus x\}$

Lyapunov-like function:

$$V(x) := \sum_{j \to k \in E} \left(v_j \ell_{jk} - |S_{jk}|^2 \right)$$

•
$$V(x) \ge 0$$
 for all $x \in \hat{X}$, with "=" iff $x \in X$

Global optimality Construction: h_x

Define quadratic function

$$\phi_{jk}(a) := \frac{|z_{jk}|^2}{4} a^2 + \left(v_j - \operatorname{Re}\left(\bar{z}_{jk}S_{jk}\right)\right) a + \left(|S_{jk}|^2 - v_j \mathscr{C}_{jk}\right)$$

Define Δ_{jk} := positive root of $\phi_{jk}(a) = 0$ if $v_j \ell_{jk} > |S_{jk}|^2$, or $\Delta_{jk} := 0$ otherwise

For infeasible pt $x \in \hat{X} \setminus X$, define path $h_x(t) := \left(\tilde{s}(t), \tilde{v}(t), \tilde{\ell}(t), \tilde{S}(t)\right) = x - tA\Delta(x)$ for $t \in [0,1]$:

$$\tilde{s}_{j}(t) := s_{j} - \frac{t}{2} \sum_{i:i \to j} z_{ij} \Delta_{ij} - \frac{t}{2} \sum_{k:j \to k} z_{jk} \Delta_{jk}, \qquad j \in \overline{N}$$

$$\tilde{v}_j(t) := v_j, \qquad j \in \overline{N}$$

$$\tilde{\ell}_{jk}(t) := \ell_{jk} - t\Delta_{jk}, \qquad j \to k \in E$$

$$\tilde{S}_{jk}(t) := S_{jk} - \frac{t}{2} z_{jk} \Delta_{jk}, \qquad j \to k \in E$$

Global optimality Proof idea

Prove the Lyapunov-like function V and family of paths $\{h_x : x \in \hat{X} \setminus x\}$ defined above satisfy conditions 1, 2, 3