Power System Analysis

Chapter 9 Optimal power flow

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Outline

1. Bus injection model
2. Branch flow model
3. NP-hardness
4. Global optimality

Outline

1. Bus injection model
   - Single-phase devices
   - Single-phase OPF
   - Single-phase OPF as QCQP
2. Branch flow model
3. NP-hardness
4. Global optimality

Single-phase OPF

Optimal power flow (OPF) is fundamental because it underlies numerous power system applications

- Unit commitment, optimal dispatch, state estimation, contingency analysis, voltage control, …

OPF is a constrained optimization problem

$$\min_{u,x} \quad c(u, x) \qquad \text{subject to} \qquad f(u, x) = 0, \quad g(u, x) \leq 0$$

- Control u : generation commitment, generation set points, transformer taps, EV charging levels, inverter reactive power, …
- Network state x : voltages, line currents, power flows, …
- Cost function $c(u, x)$: generation cost, voltage deviation, power loss, user disutility, …
- Equality constraint $f(u, x) = 0$: power flow equations, …
- Inequality constraint $g(u, x) \leq 0$: operation constraints, e.g., generation/consumption limits, voltage limits, line limits, security constraints, …

Single-phase devices

Voltage source $V_j \in \mathbb{C}$

- *Ideal* voltage source: terminal voltage V_j = internal voltage
- V_j is variable if the source is controllable, or fixed and given otherwise

Current source $I_j \in \mathbb{C}$

- *Ideal* current source: terminal current I_j = internal current
- I_j is variable if the source is controllable, or given otherwise

Power source $s_j \in \mathbb{C}$

- *Ideal* power source: terminal power s_j = internal power
- s_j is variable if the source is controllable, or given otherwise

Impedance $z_j \in \mathbb{C}$

- Impedance z_j: constrains its terminal voltage & current $V_j = -z_j I_j$

- Nodal vars at each bus j : $s_j = V_j \bar{I}_j$
- Nodal vars at different buses :
  - Current balance: $I = YV$
  - Power balance: $s_j = f_j(V)$

Single-phase OPF

Assumptions

Network: $G := (\overline{N}, E)$ with $N+1$ buses in $\overline{N} := \{0,1,\ldots,N\}$ and M lines in E

- Line $(j,k) \in E$: characterized by $\left(y_{jk}^s, y_{jk}^m\right) \in \mathbb{C}^2$ and $\left(y_{kj}^s, y_{kj}^m\right) \in \mathbb{C}^2$
- Special case: $y_{jk}^s = y_{kj}^s$; $y_{jk}^m = y_{kj}^m = 0$

Assume WLOG

- Single-phase devices: voltage sources and power sources only
- Each bus has a single device with $\left(s_j, V_j\right)$

Formulate the simplest OPF to study general computational properties

Single-phase OPF

Simplest formulation

Optimization variable: $(s, C) := \left(s_j, V_j, j \in \overline{N}\right)$

- Represents voltage sources V_j and power sources s_j only

Cost function $C(s, V)$

- Fuel cost : $C(s, V) := \sum_{j:\text{gens}} c_j \operatorname{Re}(s_j)$
- Total real power loss: $C(s, V) := \sum_{j} \operatorname{Re}(s_j)$

Single-phase OPF

Simplest formulation

Power flow equations in BIM

- Equality constraints on (V, s)

$$s_j \;=\; \sum_{k:j\sim k} S_{jk}(V) \quad := \quad \sum_{k:j\sim k} \bar{y}^s_{jk}\left(|V_j|^2 - V_j\bar{V}_k \right) \;+\; \bar{y}^m_{jj}\,|V_j|^2, \qquad j \in \overline{N}$$

where $y^m_{jj} := \sum_{k:j\sim k} y^m_{jk}$

- Derivation:

$$I_{jk}(V) \;:=\; y^s_{jk}(V_j - V_k) \;+\; y^m_{jk}\,V_j$$

$$S_{jk}(V) \;:=\; V_j\bar{I}_{jk}(V) \quad := \quad \bar{y}^s_{jk}\left(|V_j|^2 - V_j\bar{V}_k \right) + \bar{y}^m_{jk}\,|V_j|^2$$

- Can also use polar form and Cartesian form
- Nonlinear and global equality constraints, resulting in nonconvexity of OPF

Single-phase OPF

Simplest formulation

Operational constraints

- Injection limits (e.g. gen. or load capacity limits): $s_j^{\min} \le s_j \le s_j^{\max}$
- Voltage limits: $v_j^{\min} \le |V_j|^2 \le v_j^{\max}$
- Line limits: $|I_{jk}(V)|^2 \le \ell_{jk}^{\max}, \ |I_{kj}(V)|^2 \le \ell_{kj}^{\max}$

$$\left| y_{jk}^s (V_j - V_k) + y_{jk}^m V_j \right|^2 \le \ell_{jk}^{\max}, \qquad (j,k) \in E$$

$$\left| y_{kj}^s (V_k - V_j) + y_{kj}^m V_k \right|^2 \le \ell_{kj}^{\max}, \qquad (j,k) \in E$$

Line limits can also be on line powers $\left(S_{jk}(V), S_{kj}(V)\right)$ or apparent powers $\left(\left|S_{jk}(V)\right|, \left|S_{kj}(V)\right| \right)$

Single-phase OPF

Simplest formulation

OPF in BIM

$$\min_{(s,V)} \quad C(s, V)$$

$$\text{subject to} \quad f(s, V) = 0 \qquad \text{power flow equations}$$

$$g(s, V) \leq 0 \qquad \text{operational constraints}$$

- Does not need assumption $y_{jk}^s = y_{kj}^s$
- Can accommodate single-phase transformers with *complex* turns ratios
- Can allow voltages or power injections be fixed and given; e.g., $s_j^{\min} = s_j^{\max}$
- … or unconstrained, e.g., $s_0^{\min} := -\infty - i\infty, \quad s_0^{\max} := \infty + i\infty$

Single-phase OPF

1. Other devices
   - Can include other devices such as current sources, impedances, capacity taps
   - Allow multiple devices connected to same bus
2. Can formulate OPF in terms of V only
   - Use power flow equations to express injections $s_j(V)$ as functions of V
   - Eliminate s_j and power flow equations (equality constraints)

Next: explain each in turn

Single-phase OPF

Including other devices

Examples

- Current source (controllable): variable I_j with local constraints $|I_j|^2 \leq I_j^{\max}, \quad s_j = V_j \bar{I}_j$
- Impedance z_j: imposes additional constraint $s_j = |V_j|^2 / \bar{z}_j$
- Capacitor tap (controllable): variable y_j with local constraints $y_j^{\min} \leq y_j \leq y_j^{\max}, \quad s_j = \bar{y}_j |V_j|^2$
- Multiple devices: injection variables s_{jk} with local constraints $s_{jk}^{\min} \leq s_{jk} \leq s_{jk}^{\max}, \quad s_j = \sum_k s_{jk}$

Including other devices at bus j imposes additional local constraints

- Additional optimization var u_j may be introduced
- Equality constraints relating $\left(s_j, V_j\right)$ and u_j (if present) : $f_j\left(u_j, s_j, V_j\right) = 0$
- Inequality (operational) constraints (e.g., capacity limits): $g_j(u_j) \leq 0$

Single-phase OPF

In terms of V only

Equality constraints (BIM in complex form)

- Expresses s_j in terms of voltages V

$$s_j(V) \;=\; \sum_{k:j\sim k} S_{jk}(V) \;\;:=\;\; \sum_{k:j\sim k} \bar{y}^s_{jk}\left(|V_j|^2 - V_j\bar{V}_k\right) \;+\; \bar{y}^m_{jj}|V_j|^2, \qquad j \in \overline{N}$$

Cost $C(V) := C(s(V), V)$ expressed as function of V

- Fuel cost:

$$C(V) \;:=\; \sum_{j:\text{gens}} c_j \,\text{Re}(s_j(V)) \;=\; \sum_{j:\text{gens}} c_j \,\text{Re}\left(\sum_{k:j\sim k} \bar{y}^s_{jk}\left(|V_j|^2 - V_j\bar{V}_k\right) \;+\; \bar{y}^m_{jj}|V_j|^2\right)$$

- Total real power loss:

$$C(V) := \sum_j \text{Re}(s_j(V))$$

Single-phase OPF

Operational constraints

Injection limits (e.g. generation or load capacity limits) $s_j^{\min} \le s_j(V) \le s_j^{\max}$:

$$s_j^{\min} \le \sum_{k:j\sim k} \bar{y}_{jk}^s \left(|V_j|^2 - V_j \bar{V}_k \right) + \bar{y}_{jj}^m |V_j|^2 \le s_j^{\max}, \qquad j \in \overline{N}$$

- Or in polar form:

$$p_j^{\min} \le \sum_{k:k\sim j} \left(g_{jk}^s + g_{jk}^m \right) |V_j|^2 - \sum_{k:k\sim j} |V_j||V_k| \left(g_{jk}^s \cos\theta_{jk} + b_{jk}^s \sin\theta_{jk} \right) \le p_j^{\max}$$

$$q_j^{\min} \le -\sum_{k:k\sim j} \left(b_{jk}^s + b_{jk}^m \right) |V_j|^2 - \sum_{k:k\sim j} |V_j||V_k| \left(g_{jk}^s \sin\theta_{jk} - b_{jk}^s \cos\theta_{jk} \right) \le q_j^{\max}$$

Single-phase OPF

Operational constraints

Voltage limits (same as before):

$$v_j^{\min} \ \le \ |V_j|^2 \ \le \ v_j^{\max}, \qquad j \in \overline{N}$$

Line limits (same as before):

$$\left| y_{jk}^s (V_j - V_k) \ + \ y_{jk}^m V_j \right|^2 \le \ \ell_{jk}^{\max}, \qquad (j,k) \in E$$

$$\left| y_{kj}^s (V_k - V_j) \ + \ y_{kj}^m V_k \right|^2 \le \ \ell_{kj}^{\max}, \qquad (j,k) \in E$$

- Line limits can also be on line powers $\left(S_{jk}(V), S_{kj}(V)\right)$ or apparent powers $\left(\left| S_{jk}(V) \right|, \left| S_{kj}(V) \right| \right)$

Single-phase OPF

In terms of V only

Feasible set

$$\mathbb{V} := \left\{ V \in \mathbb{C}^{N+1} \mid V \text{ satisfies operational constraints} \right\}$$

OPF in BIM

$$\min_{V \in \mathbb{V}} \quad C(V)$$

- Does not need assumption $y_{jk}^s = y_{kj}^s$
- Can accommodate single-phase transformers with *complex* turns ratios

Single-phase OPF

In terms of V only

Feasible set

$$\mathbb{V} := \left\{ V \in \mathbb{C}^{N+1} \mid V \text{ satisfies operational constraints} \right\}$$

OPF in BIM

$$\min_{V \in \mathbb{V}} \quad C(V)$$

We will mostly study this simple OPF

Can express it as a QCQP

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1. Bus injection model
   - Single-phase devices
   - Single-phase OPF
   - Single-phase OPF as QCQP
2. Branch flow model
3. NP-hardness
4. Global optimality
5. Techniques for scalability: case study

OPF as QCQP

QCQP

Quadratically constrained quadratic program:

$$\begin{aligned} \min_{x \in \mathbb{C}^n} \quad & x^{\mathsf{H}} C_0 x \\ \text{s.t.} \quad & x^{\mathsf{H}} C_l x \ \leq \ b_l, \qquad l = 1, \ldots, L \end{aligned}$$

- C_l : $n \times n$ Hermitian matrix $\Rightarrow x^{\mathsf{H}} C_l x \in \mathbb{R}$
- $b_l \in \mathbb{R}$
- Homogeneous QCQP : all monomials are of degree 2

OPF as QCQP

QCQP

Inhomogeneous QCQP

$$\begin{aligned} \min_{x \in \mathbb{C}^n} \quad & x^{\mathsf{H}} C_0 x \;+\; \left(c_0^{\mathsf{H}} x + x^{\mathsf{H}} c_0\right) \\ \text{s.t.} \quad & x^{\mathsf{H}} C_l x \;+\; \left(c_l^{\mathsf{H}} x + x^{\mathsf{H}} c_l\right) \;\le\; b_l, \qquad l = 1, \dots, L \end{aligned}$$

Homogenization: introduce scalar var $t \in \mathbb{C}$

- Set $x := \hat{x}\bar{t}$ and require $|t|^2 = 1$ (i.e., $t = e^{i\theta}$ for some θ). Then

$$x^{\mathsf{H}} C_l x \;+\; c_l^{\mathsf{H}} x + x^{\mathsf{H}} c_l \;=\; \hat{x}^{\mathsf{H}} C_l \hat{x} \;+\; c_l^{\mathsf{H}} (\hat{x}\bar{t}) \;+\; (\hat{x}\bar{t})^{\mathsf{H}} c_l \;=\; \begin{bmatrix} \hat{x}^{\mathsf{H}} & t^{\mathsf{H}} \end{bmatrix} \begin{bmatrix} C_l & c_l \\ c_l^{\mathsf{H}} & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ t \end{bmatrix}$$

OPF as QCQP

QCQP

Equivalent homogeneous QCQP

$$\min_{\hat{x}\in\mathbb{C}^n,\, t\in\mathbb{C}} \quad \begin{bmatrix}\hat{x}^{\mathsf{H}} & t^{\mathsf{H}}\end{bmatrix} \begin{bmatrix} C_0 & c_0 \\ c_0^{\mathsf{H}} & 0 \end{bmatrix} \begin{bmatrix}\hat{x} \\ t\end{bmatrix}$$

$$\text{s.t.} \quad \begin{bmatrix}\hat{x}^{\mathsf{H}} & t^{\mathsf{H}}\end{bmatrix} \begin{bmatrix} C_l & c_l \\ c_l^{\mathsf{H}} & 0 \end{bmatrix} \begin{bmatrix}\hat{x} \\ t\end{bmatrix} \le b_l, \qquad l = 1,\ldots,L$$

$$\begin{bmatrix}\hat{x}^{\mathsf{H}} & t^{\mathsf{H}}\end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix}\hat{x} \\ t\end{bmatrix} = 1$$

- If $\left(\hat{x}^{\text{opt}}, t^{\text{opt}}\right)$ is optimal for homogeneous QCQP, then product $x^{\text{opt}} := \hat{x}^{\text{opt}}\, t^{\text{opt}}$ is optimal for original inhomogeneous QCQP

OPF as QCQP
Equivalent real QCQP

Even though OPF is often formulated in $\mathbb{C}$, it is converted to $\mathbb{R}$ before being solved iteratively

QCQP

$$\min_{x \in \mathbb{C}^n} \quad x^{\mathsf{H}} C_0 x$$
$$\text{s.t.} \quad x^{\mathsf{H}} C_l x \ \le \ b_l, \qquad l = 1, \dots, L$$

- C_l : $n \times n$ complex Hermitian matrix
- $b_l \in \mathbb{R}$

Equivalent to:

$$\min_{(x_r, x_i) \in \mathbb{R}^{2n}} \quad \begin{bmatrix} x_r \\ x_i \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} C_{0r} & -C_{0i} \\ C_{0i} & C_{0r} \end{bmatrix} \begin{bmatrix} x_r \\ x_i \end{bmatrix}$$
$$\text{s.t.} \quad \begin{bmatrix} x_r \\ x_i \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} C_{lr} & -C_{li} \\ C_{li} & C_{lr} \end{bmatrix} \begin{bmatrix} x_r \\ x_i \end{bmatrix} \le b_l, \quad l = 1, \dots, L$$

- $2n \times 2n$ real symmetric matrices

OPF as QCQP

To write OPF as QCQP:

- Assume cost function $C(V) = V^{\mathsf{H}} C_0 V$ can be written as a quadratic form
- Need to rewrite operational constraints in terms of quadratic forms

OPF as QCQP

Injection limits $s_j^{\min} \le s_j(V) \le s_j^{\max}$

$$s_j(V) = V_j I_j^{\mathsf{H}} = \left(e_j^{\mathsf{H}} V\right)\left(e_j^{\mathsf{H}} I\right)^{\mathsf{H}} = e_j^{\mathsf{H}} VV^{\mathsf{H}} Y^{\mathsf{H}} e_j$$

$$s_j(V) = \mathrm{tr}\left(e_j^{\mathsf{H}} VV^{\mathsf{H}} Y^{\mathsf{H}} e_j\right) = \mathrm{tr}\left(\left(Y^{\mathsf{H}} e_j e_j^{\mathsf{H}}\right) VV^{\mathsf{H}}\right) =: V^{\mathsf{H}} Y_j^{\mathsf{H}} V$$

OPF as QCQP

Injection limits $s_j^{\min} \le s_j(V) \le s_j^{\max}$

$$s_j(V) = V_j I_j^{\mathsf{H}} = \left(e_j^{\mathsf{H}} V\right)\left(e_j^{\mathsf{H}} I\right)^{\mathsf{H}} = e_j^{\mathsf{H}} \, V V^{\mathsf{H}} Y^{\mathsf{H}} e_j$$

$$s_j(V) = \mathrm{tr}\left(e_j^{\mathsf{H}} V V^{\mathsf{H}} Y^{\mathsf{H}} e_j\right) = \mathrm{tr}\left(\left(Y^{\mathsf{H}} e_j e_j^{\mathsf{H}}\right) V V^{\mathsf{H}}\right) =: V^{\mathsf{H}} Y_j^{\mathsf{H}} V$$

- Y_j is not Hermitian so $V^{\mathsf{H}} Y_j^{\mathsf{H}} V$ is generally complex
- Define $\Phi_j := \frac{1}{2}\left(Y_j^{\mathsf{H}} + Y_j\right), \qquad \Psi_j := \frac{1}{2i}\left(Y_j^{\mathsf{H}} - Y_j\right)$
- Then $\mathrm{Re}(s_j) = V^{\mathsf{H}} \Phi_j V, \qquad \mathrm{Im}(s_j) = V^{\mathsf{H}} \Psi_j V$

Hence $s_j^{\min} \le s_j(V) \le s_j^{\max}$ is equivalent to:

$$p_j^{\min} \le V^{\mathsf{H}} \Phi_j V \le p_j^{\max}, \qquad q_j^{\min} \le V^{\mathsf{H}} \Psi_j V \le q_j^{\max}$$

OPF as QCQP

Voltage limits

Voltage magnitude is: $|V_j|^2 = V^{\mathsf{H}} E_j V$ where $E_j := e_j e_j^{\mathsf{T}}$

Hence voltage limits are: $v_j^{\min} \leq V^{\mathsf{H}} E_j V \leq v_j^{\max}$

OPF as QCQP

Line limits

Write I_{jk} in terms of voltage vector V:

$$I_{jk} \; = \; y_{jk}^s(V_j - V_k) \; + \; y_{jk}^m V_j \; = \; \left(y_{jk}^s(e_j - e_k)^{\mathsf{T}} \; + \; y_{jk}^m \, e_j^{\mathsf{T}} \right) V$$

Hence current limit is: $|I_{jk}|^2 = V^{\mathsf{H}} \hat{Y}_{jk} V \; \leq \; \ell_{jk}^{\max}$ where

$$\hat{Y}_{jk} \; := \; \left(\bar{y}_{jk}^s(e_j - e_k) \; + \; \bar{y}_{jk}^m \, e_j \right) \left(y_{jk}^s(e_j - e_k)^{\mathsf{T}} + y_{jk}^m \, e_j^{\mathsf{T}} \right)$$

OPF as QCQP

Simplest formulation

$$\min_{V \in \mathbb{C}^{N+1}} \quad V^{\mathsf{H}} C_0 V$$

$$\begin{aligned}
\text{s.t.} \quad p_j^{\min} \ \le \ & V^{\mathsf{H}} \Phi_j V \ \le \ p_j^{\max}, && j \in \overline{N} \\
q_j^{\min} \ \le \ & V^{\mathsf{H}} \Psi_j V \ \le \ q_j^{\max}, && j \in \overline{N} \\
v_j^{\min} \ \le \ & V^{\mathsf{H}} E_j V \ \le \ v_j^{\max}, && j \in \overline{N} \\
& V^{\mathsf{H}} \hat{Y}_{jk} V \ \le \ \ell_{jk}^{\max}, && (j,k) \in E \\
& V^{\mathsf{H}} \hat{Y}_{kj} V \ \le \ \ell_{kj}^{\max}, && (j,k) \in E
\end{aligned}$$

Outline

1. Bus injection model
2. Branch flow model
   - Radial network
3. NP-hardness
4. Global optimality

Radial network

Assumptions: DistFlow model

Radial network

- BFM most useful for modeling distribution systems which are mostly radial (and unbalanced)

$z^s_{jk} = z^s_{kj}$ or equivalently $y^s_{jk} = y^s_{kj}$

- Does not apply to 3-phase transformers in ΔY or $Y\Delta$ configuration or their per-phase equivalent with complex gains

$y^m_{jk} = y^m_{kj} = 0$

- Reasonable assumption for distribution line where $|y^m_{jk}|, |y^m_{kj}| \ll |y^s_{jk}|$

Includes only voltage sources and power sources

- Optimization variables are voltages (squared magnitudes) v_j and power injections s_j respectively
- Can include current sources or an impedances with additional vars and constraints.

DistFlow model

Power flow equations

- All lines point away from bus 0 (root)

$$\sum_{k:j\to k} S_{jk} = S_{ij} - z^s_{ij}\ell_{ij} + s_j, \qquad j \in \overline{N}$$

$$v_j - v_k = 2\,\mathrm{Re}\left(\bar{z}^s_{jk} S_{jk}\right) - |z^s_{jk}|^2 \ell_{jk}, \qquad j \to k \in E$$

$$v_j \ell_{jk} = |S_{jk}|^2, \qquad j \to k \in E$$

Operational constraints

$$s_j^{\min} \le s_j \le s_j^{\max}$$

$$v_j^{\min} \le v_j \le v_j^{\max}$$

$$\ell_{jk} \le \ell_{jk}^{\max}$$

Single-phase OPF

DistFlow model

Feasible set

$$\mathbb{X}_{\text{df}} := \left\{ x := (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies PF equations \& operational constraints} \right\}$$

OPF in BFM

$$\min_{x} \quad C(x) \qquad \text{s.t.} \qquad x \in \mathbb{X}_{\text{df}}$$

Single-phase OPF

Equivalence

Recall for BIM:

- Feasible set: $\mathbb{V} := \left\{ V \in \mathbb{C}^{N+1} \mid V \text{ satisfies operational constraints} \right\}$
- OPF: $\min_{V \in \mathbb{V}} \; C(V)$

OPF in BFM is equivalent to OPF in BIM:

- Feasible sets $\mathbb{X}_{\text{df}}$ and $\mathbb{V}$ are equivalent (Ch 5)
- … provided cost functions $C(x)$ and $C(V)$ are the same

General radial network

Does not assume $z^s_{jk} = z^s_{kj}$ nor $y^m_{jk} = y^m_{kj} = 0$

Need branch quantities in both directions

- $\ell := \left(\ell_{jk}, \ell_{kj}, (j,k) \in E\right), \ S := \left(S_{jk}, S_{kj}, (j,k) \in E\right)$
- $\alpha_{jk} \ := \ 1 + z^s_{jk}\, y^m_{jk}, \quad \alpha_{kj} \ := \ 1 + z^s_{kj}\, y^m_{kj}$

BFM for general radial network

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad j \in \overline{N}$$

$$|\alpha_{jk}|^2 v_j - v_k = 2\,\text{Re}\left(\alpha_{jk}\bar{z}^s_{jk}S_{jk}\right) - |z^s_{jk}|^2 \ell_{jk}, \qquad (j,k) \in E$$

$$|\alpha_{kj}|^2 v_k - v_j = 2\,\text{Re}\left(\alpha_{kj}\bar{z}^s_{kj}S_{kj}\right) - |z^s_{kj}|^2 \ell_{kj}, \qquad (j,k) \in E$$

$$\left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}, \qquad (j,k) \in E$$

$$\bar{\alpha}_{jk} v_j - \bar{z}^s_{jk} S_{jk} = \left(\bar{\alpha}_{kj} v_k - \bar{z}^s_{kj} S_{kj}\right)^{\mathsf{H}}, \qquad (j,k) \in E$$

Single-phase OPF

General radial network

Operational constraints (same as before but line limits in both directions)

$$s_j^{\min} \le s_j \le s_j^{\max}, \quad v_j^{\min} \le v_j \le v_j^{\max}, \quad \ell_{jk} \le \ell_{jk}^{\max}, \quad \ell_{kj} \le \ell_{kj}^{\max}$$

Feasible set

$$\mathbb{X}_{\text{tree}} := \left\{ x := (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies PF equations \& operational constraints} \right\}$$

OPF in BFM

$$\min_x \quad C(x) \qquad \text{s.t.} \qquad x \in \mathbb{X}_{\text{tree}}$$

OPF in BFM is equivalent to OPF in BIM:

- Feasible sets $\mathbb{X}_{\text{tree}}$ and $\mathbb{V}$ are equivalent (Ch 5)
- … provided cost functions $C(x)$ and $C(V)$ are the same

Outline

1. Bus injection model
2. Branch flow model
3. NP-hardness
    - OPF feasibility
    - OPF is NP-hard
4. Global optimality

OPF feasibility

Tree network

Star network $(\overline{N}, E)$ with $N+1$ buses and $M = N$ lines

- $y^s_{jk} = y^s_{kj}$ and $y^m_{jk} = y^m_{kj} = 0$
- Fixed voltage magnitudes $|V_j| := 1$ pu
- Fixed and given injections $(p_j, q_j),\ j \in N_L \subset \overline{N}$
- Dispatchable generation (p_j, q_j) with $p_j \geq 0,\ j \in N_G \subset \overline{N}$
- Line limits: $|\theta_j - \theta_k| \leq \overline{\theta} \ \in \ (0, \pi/2], \ \ (j,k) \in E$

Each instance of OPF feasibility problem is specified by

- Tree network $(N_G \cup N_L, E)$
- Line admittances $(g_{jk}, b_{jk}, (j,k) \in E)$
- Line limits $\overline{\theta} \in (0, \pi/2]$
- Fixed injections $(p_j, q_j, j \in N_L)$

OPF feasibility

Tree network

Find

- Real power generations $\left(p_j, j \in N_G\right) \geq 0$
- Voltage angles $\left(\theta_j, j \in \overline{N}\right)$
- Line flows $\left(P_{jk}, Q_{jk}, (j,k) \in E\right)$

that satisfy the polar form power flow equation and line limits:

OPF feasibility:

$$
\begin{aligned}
p_j &= \sum_{k:j\sim k} P_{jk}, \qquad q_j = \sum_{k:j\sim k} Q_{jk}, && j \in N_L \\
p_j &\geq 0, && j \in N_G \\
P_{jk} &= g_{jk}(1 - \cos\theta_{jk}) - b_{jk}\sin\theta_{jk}, && (j,k) \in E \\
Q_{jk} &= -b_{jk}(1 - \cos\theta_{jk}) - g_{jk}\sin\theta_{jk}, && (j,k) \in E \\
|\theta_j - \theta_k| &\leq \overline{\theta}, && (j,k) \in E
\end{aligned}
$$

NP-hardness

P and NP

Let

- Σ : finite set of symbols
- Σ^* : set of all finite strings of symbols in Σ
- $L \subseteq \Sigma^*$: language over Σ

Deterministic Turing machine (DTM): computation model that takes an input $\sigma \in \Sigma^*$, performs computation (read, write, state transition), and either halts in "yes" or "no" state, or does not halt

Given DTM M, time complexity function $c_M : \mathbb{N}_+ \to \mathbb{N}_+$:

$$c_M(n) \ := \ \max\{m : \exists \sigma \in \Sigma^* \text{ with } |\sigma| = n \text{ s.t. } M \text{ takes } m \text{ steps to halt on } \sigma\}$$

M is called a polynomial time DTM if $\exists$ a polynomial p s.t. $c_M(n) \leq p(n)$ for all n

Language recognized by (DTM or NDTM) M is

$$L_M \ := \ \{\sigma \in \Sigma^* : M \text{ halts on } \sigma \text{ in "yes" state}\}$$

NP-hardness

P and NP

The class P of languages is

$$\text{P} \;:=\; \{L \subseteq \Sigma^* : \exists \text{ polynomial time DTM } M \text{ for which } L = L_M\}$$

Informally: P consists of all language over Σ that are recognized by a DTM in polynomial time

While P captures "solvability" of a problem, NP captures "verifiability"

- It is difficult (NP-complete) to find a cycle in an arbitrary graph that visits every node exactly once, but easy to verify if a candidate is a solution

Given NDTM M, time complexity function $c_M : \mathbb{N}_+ \to \mathbb{N}_+$:

$$c_M(n) \;:=\; \max\{m : \exists \sigma \in \Sigma^* \text{ with } |\sigma| = n \text{ s.t. } M \text{ takes } m \text{ steps to halt on } \sigma \text{ in "yes" state}\}$$

M is called a polynomial time NDTM if $\exists$ a polynomial p s.t. $c_M(n) \leq p(n)$ for all n

The class NP of languages is

$$\text{NP} \;:=\; \{L \subseteq \Sigma^* : \exists \text{ polynomial time NDTM } M \text{ for which } L = L_M\}$$

P ⊆ NP

Informally: NP consists of all language recognized by a NDTM (or verifiable by a DTM) in polynomial time

NP-hardness

NP-hard and NP-complete

A function $f : \Sigma_1^* \to \Sigma_2^*$ is a language $L_j := \{(\sigma, f(\sigma)) : \sigma \in \Sigma_1^*\} \subseteq \Sigma_1^* \times \Sigma_2^*$

DTM M computes f if $L_M = L_f$

A polynomial reduction from $L_1 \subseteq \Sigma_1^*$ to $L_2 \subseteq \Sigma_2^*$ is a function $f : \Sigma_1^* \to \Sigma_2^*$ which can be computed by a polynomial time DTM s.t.

$$\sigma \in L_1 \iff f(\sigma) \in L_2, \qquad \sigma \in \Sigma_1$$

A language L is NP-hard if for every $L' \in$ NP there exists a polynomial reduction from L' to L

It is NP-complete if L is NP-hard and $L \in$ NP

- NP-complete languages are in a sense the “hardest” languages in NP

NP-hardness

Decision problems

A decision problem is a problem whose solution is either "yes" or "no"

- It is defined by a set of finite instances, e.g. specified in terms of sets, graphs, functions, real numbers

Let Π be a decision problem (or its instances) that can be "encoded" into a language problem over some alphabet Σ

- Informally, an encoding is $\sigma : \Pi \to \Sigma^*$ that maps each instance $y \in \Pi$ to a string $\sigma(y) \in \Sigma^*$

Let $Y \subseteq \Pi$ be the subset of instances whose solutions are "yes"

- We will refer to Y either as a set of problem instances or simply a problem by itself

Let $L_Y := \{\sigma(y) : y \in Y\}$ be the language defined by instances in Y

- Solution of instance $y \in \Pi$ is "yes" if and only if $y \in Y$ if and only if $\sigma(y) \in L_Y$

Hardness properties of Y are then defined in terms of hardness properties of its encoding L_Y

- e.g. Y is in P if $L_Y \in$ P, Y is NP-complete if L_Y is NP-complete
- OPF feasibility problem is such a decision problem

NP-hardness

Theorem

OPF feasibility problem on a tree network is NP-hard

Remarks:

- OPF feasibility is not proved to be in NP, because solution can be irrational
- Proved by polynomial reduction of NP-complete subset sum problem to OPF feasibility
- OPF feasibility can be proved to be strongly NP-hard by polynomial reduction of strongly NP-complete one-in-three 3SAT problem to OPF feasibility

NP-hardness is worst-case result

- Subclasses of OPF cane polynomial time solvable
- e.g., those satisfied sufficient conditions for exact relaxations or global optimaiity

Outline

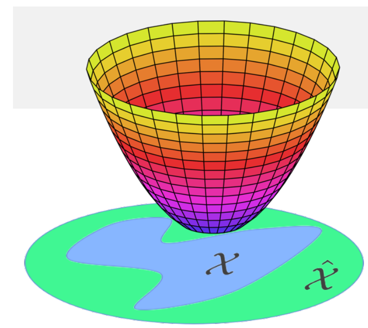
1. Bus injection model
2. Branch flow model
3. NP-hardness
4. Global optimality
   - Convex relaxation
   - Lyapunov-lik condition for global optimality
   - Application to OPF on radial network

Optimization and relaxation

Consider

Nonconvex optimization P1: $\min_x \; f(x) \quad \text{s.t.} \quad x \in X \subseteq \mathbb{R}^n$

Convex relaxation P2: $\min_x \; f(x) \quad \text{s.t.} \quad x \in \hat{X} \subseteq \mathbb{R}^n$



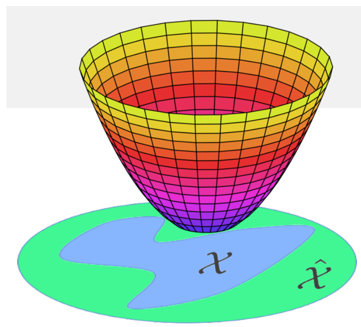
- X : nonempty, compact (not necessarily convex)
- $\hat{X}$: compact and convex superset $\hat{X} \supseteq X$
- $f : \mathbb{R}^n \to \mathbb{R}$ is convex (and hence continuous) function on $\mathbb{R}^n$

Optimal solutions exist for both problems P1 and P2

Exact relaxation

Definition

1. $x^* \in X$ is a local optimum of P1 if $\exists \delta > 0$ s.t. $f(x^*) \leq f(x)$ for all $\|x - x^*\| < \delta$
2. $x^* \in X$ is a global optimum of P1 if $f(x^*) \leq f(x)$ for all $x \in X$
3. P2 is exact wrt P1 if every optimal x^* of P2 is feasible (and hence optimal) for P1



Path

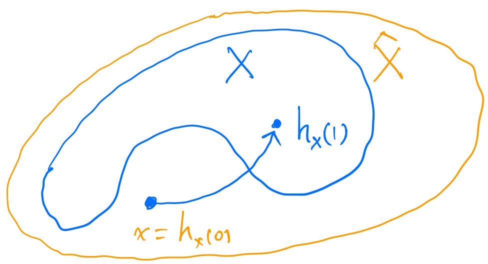
Definition

1. A path in $Y \subseteq \mathbb{R}^n$ connecting a to be b in Y is a continuous function $h : [0,1] \to Y$ s.t. $h(0) = a$ and $h(1) = b$

2. An arbitrary set $\{h_i : i \in I\}$ of paths in Y is called

   - uniformly bounded if $\exists$ finite H s.t. $\|h_i(t)\|_\infty \leq H$ for all $t \in [0,1]$ and $i \in I$

   - uniformly equicontinuous if for any $\epsilon > 0$, $\exists\, \delta > 0$ s.t. $\|h_i(t_2) - h_i(t_1)\|_\infty \leq \epsilon$ for all $i \in I$ whenever $|t_2 - t_1| < \delta$

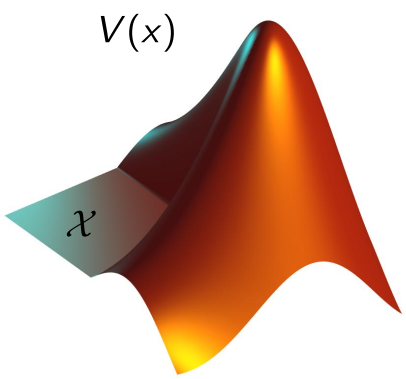


Example: If all paths in $\{h_i : i \in I\}$ are linear, then $\{h_i : i \in I\}$ is both uniformly bounded and uniformly equicontinuous

Lyapunov-like function

Definition

A Lyapunov-like function associated with problems P1 and P2 is a continuous function $V : \hat{X} \to \mathbb{R}_+$ s.t. $V(x) = 0$ if $x \in X$ and $V(x) > 0$ if $x \in \hat{X} \backslash X$



Global optimality

Optimality conditions

1. There is a Lyapunov-like function V and, for every infeasible point $x \in \hat{X} \backslash X$, $\exists$ path h_x s.t.

   Every infeasible pt x can be brought back to X with a lower cost

   (a) $h_x(0) = x$, $h_x(1) \in X$, $f(h_x(1)) < f(x)$

   (b) Both $f(h_x(t))$ and $V(h_x(t))$ are nonincreasing for $t \in [0,1]$

   Nonincreasing cost or certificate along path to feasibility

2. The set $\{h_x : x \in \hat{X} \backslash x\}$ of paths in 1 is uniformly bounded and uniformly equicontinuous

3. At least one of the following holds:

   (d) All local optima of P1 are isolated (i.e., every local optimum has a neighborhood with no other local optimum)

   (e) For $\{h_x : x \in \hat{X} \backslash x\}$ in 1, $\exists \alpha > 0$ s.t. for all $x \in \hat{X} \backslash X$ and all $0 \leq s < t \leq 1$,

$$f(h_x(s)) - f(h_x(t)) \geq \alpha \| h_x(s) - h_x(t) \|$$

   for some norm $\| \cdot \|$

   Cost must decrease sufficiently along path to feasibility

Global optimality

Theorem [Sufficiency]

Suppose conditions 1, 2, 3 hold.

1. The convex relaxation P2 is exact wrt P1
2. Every local optimum of P1 is a global optimum

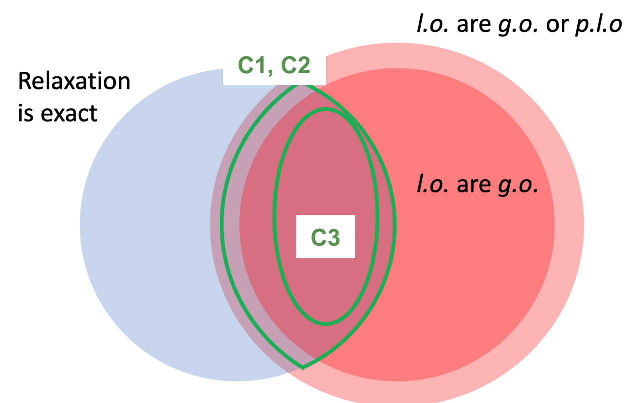
Moreover if condition 3(a) holds, then the optimal point is unique

Remarks

- Exactness $\iff$ existence of $\{h_x : x \in \hat{X} \backslash x\}$ that satisfies condition 1
- Other conditions are to prove that there is no spurious local optimum

Global optimality

A set $y \subseteq \mathbb{R}^n$ is semianalytic if every $x \in \mathbb{R}^n$ has a neighborhood U s.t. $Y \cap U$ can be represented as a finite Boolean combination of sets $\{x : g(x) = 0\}$ and $\{x : h(x) < 0\}$ for some analytic functions g, h (usually satisfied by engineering problems)



Theorem [Necessity]

Suppose X is semianalytic and f is analytic. If

1. The convex relaxation P2 is exact wrt P1, and
2. Every local optimum of P1 is a global optimum

then ∃ Lyapunov-like function V and a family of paths $\{h_x : x \in \hat{X} \backslash x\}$ that satisfy cond 1 and 2

Lyapunov-like optimality condition

Comparison with Lyapunov stability

Consider the dynamical system

$$\dot{x} = f(x(t)), \qquad t \geq 0,\ x(0) = x_0$$

Let x^* be an equilibrium point where $f(x^*) = 0$

Lyapunov stability theory

1. Lyapunov function $V(x)$ is a continuously differentiable function s.t. $V(x) > V(x^*)$ and $\dot{V}(x) < 0$ for all $x \neq x^*$ in $\mathbb{R}^n$
2. V certifies stability of x^* : x^* is globally asymptotically stable if a Lyapunov function $V(x)$ exists

Lapunov-like optimality condition

1. V certifies global optimality of a local optimum $x^* \in X$
2. No dynamics to specify path : no requirement on differentiability of V, but
3. Need to construct both V and paths $\{h_x : x \in \hat{X} \backslash x\}$ (no general method known)

Application to OPF

Recall: OPF in DistFlow model

DistFlow equations (radial network):

$$\sum_{k:j\to k} S_{jk} = S_{ij} - z^s_{ij}\ell_{ij} + s_j, \qquad j \in \overline{N}$$

$$v_j - v_k = 2\,\mathrm{Re}\left(\bar{z}^s_{jk} S_{jk}\right) - |z^s_{jk}|^2 \ell_{jk}, \qquad j \to k \in E$$

$$v_j \ell_{jk} = |S_{jk}|^2, \qquad j \to k \in E$$

Nonconvex constraint

Operational constraints:

$$s_j^{\min} \le s_j \le s_j^{\max}, \quad v_j^{\min} \le v_j \le v_j^{\max}, \quad \ell_{jk} \le \ell_{jk}^{\max}$$

Feasible set

$$X := \left\{ x := (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies DistFlow equations \& operational constraints} \right\}$$

Application to OPF

Convex relaxation

Replace

$$v_j \ell_{jk} = |S_{jk}|^2, \qquad j \to k \in E$$

by

$$v_j \ell_{jk} \geq |S_{jk}|^2, \qquad j \to k \in E$$

Convex second-order cone (SOC) constraint

Convex superset

$\hat{X} := \{x : x \text{ satisfies constraints with SOC replacement}\}$

Consider

Nonconvex optimization P1: $\min_x \; f(x)$ s.t. $x \in X \subseteq \mathbb{R}^n$

Convex relaxation P2: $\min_x \; f(x)$ s.t. $x \in \hat{X} \subseteq \mathbb{R}^n$

Optimality conditions

OPF in DistFlow model

4. X is nonempty compact, $\hat{X}$ is compact, cost function f is convex and continuous

5. Cost function $f(x) = f(p, q, v, \ell)$ is independent of $S = (P, Q)$, continuously differentiable with $\nabla f(x) \geq 0$. Moreover $\exists c > 0$ s.t. $\dfrac{\partial f}{\partial \ell_l}(x) \geq 0$ for all $l \in E$ and all $x \in \hat{X}$

   strongly inc. in ℓ

   demand large enough not to pose constraints

6. No lower bounds on injections: $s_j^{\min} = -\infty - i\infty$

   usually satisfied

7. $z_{jk} =: (r_{jk}, x_{jk}) > 0$ and line limit satisfies $|z_{jk}|^2 \ell^{\max} \leq v_j^{\min}$

Remarks

- Differentiability is not necessary and can be replaced by subgradient (which always exist since f is convex)

Global optimality
OPF in DistFlow model

Theorem

Suppose conditions 4-7 hold on radial network.

1. Convex relaxation P2 is exact wrt P1
2. Every local optimum of P1 is a global optimum

Remarks

- Exactness is proved in Ch 11 on Semidefinite relaxations of OPF in BFM

Global optimality

Construction: V

Proof requires construction of Lyapunov-like function V and family of paths $\{h_x : x \in \hat{X} \backslash x\}$

Lyapunov-like function:

$$V(x) \ := \sum_{j \to k \in E} \left(v_j \ell_{jk} - |S_{jk}|^2 \right)$$

- $V(x) \geq 0$ for all $x \in \hat{X}$, with "=" iff $x \in X$

Global optimality

Construction: h_x

Define quadratic function

$$\phi_{jk}(a) \ := \ \frac{|z_{jk}|^2}{4}\, a^2 \ + \ \left(v_j - \text{Re}\left(\bar{z}_{jk} S_{jk}\right)\right) a \ + \ \left(|S_{jk}|^2 - v_j \ell_{jk}\right)$$

Define Δ_{jk} := positive root of $\phi_{jk}(a) = 0$ if $v_j \ell_{jk} > |S_{jk}|^2$, or $\Delta_{jk} := 0$ otherwise

For infeasible pt $x \in \hat{X} \backslash X$, define path $h_x(t) \ := \ \left(\tilde{s}(t), \tilde{v}(t), \tilde{\ell}(t), \tilde{S}(t)\right) \ = \ x - tA\Delta(x)$ for $t \in [0,1]$:

$$\begin{aligned}
\tilde{s}_j(t) &:= \ s_j - \frac{t}{2} \sum_{i:i\to j} z_{ij}\Delta_{ij} - \frac{t}{2} \sum_{k:j\to k} z_{jk}\Delta_{jk}, && j \in \overline{N} \\
\tilde{v}_j(t) &:= \ v_j, && j \in \overline{N} \\
\tilde{\ell}_{jk}(t) &:= \ \ell_{jk} - t\Delta_{jk}, && j \to k \in E \\
\tilde{S}_{jk}(t) &:= \ S_{jk} - \frac{t}{2} z_{jk}\Delta_{jk}, && j \to k \in E
\end{aligned}$$

Global optimality

Proof idea

Prove the Lyapunov-like function V and family of paths $\{h_x : x \in \hat{X} \backslash x\}$ defined above satisfy conditions 1, 2, 3