The Flow of Power

Steven Low
Bootcamp: Power systems

The flow of power (S Low)
- Basic concepts and models
- Power flow and optimization

The flow of information (S Meyn)
- Distributed control architectures

The flow of money (K Poolla)
- Market structures and services

from steady state to dynamics
from engineering to economics
R. Karp’s instruction

“... the level should be sufficiently elementary that an expert on the topic will be bored.”
The flow of power I

Basic concepts and models

Why smart grid? (15 mins)

Three-phase AC transmission: 3 key ideas (30 mins)
  - Phasor representation
  - Balanced operation
  - Per-phase analysis

Device models (30 mins)
  - Transmission line
  - Transformer
  - Generator
The flow of power II

Power flow and optimization

Network models (10mins)
- Admittance matrix
- Power flow models

Optimal power flow problems (35mins)
- Formulation and example
- Convex relaxations
- Real-time OPF
Why smart grid?
Watershed moment

Energy network will undergo similar architectural transformation that phone network went through in the last two decades to become the world’s largest and most complex IoT.

1876: Bell: telephone
1888: Tesla: multi-phase AC

Both started as natural monopolies
Both provided a single commodity
Both grew rapidly through two WWs

1980-90s: deregulation started

1969: DARPA.net

Convergence to Internet
Watershed moment

Industries will be restructured
AT&T, MCI, McCaw Cellular, Qualcomm
Google, Facebook, Twitter, Amazon, eBay, Netflix

Infrastructure will be reshaped
Centralized intelligence, vertically optimized
Distributed intelligence, layered architecture
Watershed moment

The five largest companies in 2006 ...

<table>
<thead>
<tr>
<th>Rank</th>
<th>Company</th>
<th>Market Cap</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Exxon Mobil</td>
<td>$540 BILLION</td>
</tr>
<tr>
<td>2</td>
<td>General Electric</td>
<td>463</td>
</tr>
<tr>
<td>3</td>
<td>Microsoft</td>
<td>355</td>
</tr>
<tr>
<td>4</td>
<td>Citigroup</td>
<td>331</td>
</tr>
<tr>
<td>5</td>
<td>Bank of America</td>
<td>290</td>
</tr>
</tbody>
</table>
**Watershed moment**

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</tbody>
</table>

**… and now**

<table>
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<th>Rank</th>
<th>Company</th>
<th>Market Cap (Billion)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Apple</td>
<td>$794</td>
</tr>
<tr>
<td>2</td>
<td>Alphabet (Google)</td>
<td>593</td>
</tr>
<tr>
<td>3</td>
<td>Microsoft</td>
<td>506</td>
</tr>
<tr>
<td>4</td>
<td>Amazon</td>
<td>429</td>
</tr>
<tr>
<td>5</td>
<td>Facebook</td>
<td>414</td>
</tr>
</tbody>
</table>

What will drive power network transformation?
Electricity gen & transportation

They consume the most energy
- Consume 2/3 of all energy in US (2014)

They emit the most greenhouse gases
- Emit >1/2 of all greenhouse gases in US (2014)

To drastically reduce greenhouse gases
- Generate electricity from renewable sources
- Electrify transportation
World energy stats (2011)

<table>
<thead>
<tr>
<th>Consumption</th>
<th>519 quad BTU</th>
<th>per capita (mil BTU)</th>
</tr>
</thead>
<tbody>
<tr>
<td>petroleum</td>
<td>34%</td>
<td></td>
</tr>
<tr>
<td>coal</td>
<td>29%</td>
<td></td>
</tr>
<tr>
<td>gas</td>
<td>23%</td>
<td></td>
</tr>
<tr>
<td>renewable (elec)</td>
<td>8%</td>
<td></td>
</tr>
<tr>
<td>nuclear</td>
<td>5%</td>
<td></td>
</tr>
</tbody>
</table>

**Consumption**

<table>
<thead>
<tr>
<th>Consumption</th>
<th>519 (quad BTU)</th>
<th>per capita (mil BTU)</th>
</tr>
</thead>
<tbody>
<tr>
<td>China</td>
<td>20%</td>
<td>78</td>
</tr>
<tr>
<td>US</td>
<td>19%</td>
<td>313</td>
</tr>
<tr>
<td>Russia</td>
<td>6%</td>
<td>209</td>
</tr>
<tr>
<td>India</td>
<td>5%</td>
<td>20</td>
</tr>
<tr>
<td>Japan</td>
<td>4%</td>
<td>164</td>
</tr>
<tr>
<td><strong>total</strong></td>
<td><strong>54%</strong></td>
<td></td>
</tr>
</tbody>
</table>

Source: EIA
## World energy stats (2011)

### Top 5 countries

<table>
<thead>
<tr>
<th>Consumption</th>
<th>519 (quad BTU)</th>
<th>CO2 emission</th>
</tr>
</thead>
<tbody>
<tr>
<td>China</td>
<td>20%</td>
<td>27%</td>
</tr>
<tr>
<td>US</td>
<td>19%</td>
<td>17%</td>
</tr>
<tr>
<td>Russia</td>
<td>6%</td>
<td>5%</td>
</tr>
<tr>
<td>India</td>
<td>5%</td>
<td>5%</td>
</tr>
<tr>
<td>Japan</td>
<td>4%</td>
<td>4%</td>
</tr>
<tr>
<td><strong>total</strong></td>
<td><strong>54%</strong></td>
<td><strong>58%</strong></td>
</tr>
</tbody>
</table>

Source: EIA
US greenhouse gas emission 2014

Electricity generation and transportation are top-two GHG emitters (56% total)

… and they consume the most energy (66% total)

Total (2014) = 6,870 Million Metric Tons of CO2 equivalent

US energy flow 2014

Total = 98.3 quadrillion BTU

Source: EIA Monthly Energy Review March 2015
US electricity flow 2014

- Fossil: 65%
- Nuclear: 21%
- Renewable: 13%

Conversion loss: 63%
Plant use: 2%
T&D losses: 2.4%

Gross gen: 37%
End use: 33%

US total energy use: 98.3 quads
For electricity gen: 39%

Source: EIA March 2015 Monthly Energy Review
US renewable generations

750 billion kWh

As Industry Scales, Prices Fall

© 2015 GTM Research
US wind generation capacity exceeded hydro capacity in 2016.

Hydro capacity (2015): 78,956 MW
Wind capacity (2016): 82,183 MW

US wind generation capacity exceeded hydro capacity in 2016.

Source: American Wind Energy Association
US solar industry snapshot

- **US installed solar capacity by mid 2015:** ~23 GW
  - 784K homes and businesses
- **Q2 2015 solar installation:** 1.4 GW
  - Utility: 729 MW
  - Residential: 473 MW (70% growth yr-on-yr)
- **H1 2015:** a new solar installation / 2 mins

Source: SEIA 2015 (Solar Energy Industries Association)
Power the world by solar

Areas are calculated based on an assumption of 20\% operating efficiency of collection devices and a 2000 hour per year natural solar input of 1000 watts per square meter striking the surface.

These 19 areas distributed on the map show roughly what would be a reasonable responsibility for various parts of the world based on 2009 usage. They would be further divided many times, the more the better to reach a diversified infrastructure that localizes use as much as possible.

The large square in the Saharan Desert (1/4 of the overall 2030 required area) would power all of Europe and North Africa. Though very large, it is 18 times less than the total area of that desert.

The definition of “power” covers the fuel required to run all electrical consumption, all machinery, and all forms of transportation. It is based on the US Department of Energy statistics of worldwide Btu consumption and estimates the 2030 usage (678 quadrillion Btu) to be 44\% greater than that of 2008.

Area calculations do not include magenta border lines.
High Levels of Wind and Solar PV Will Present an Operating Challenge!

Source: Rosa Yang, EPRI
- 68 meters (residential)
- Sept 2012 (23 days)
- 240 volts
- +/-5% min-228/max-252
- Hourly by meter #
- A few “high” meters
- Larger # of low meters

Voltage violations are quite frequent
Power struggle: Green energy versus a grid that's not ready "Energiewende" Germany's Green Energy Destabilizing Electric Grids

"Energiewende" is a movement in Germany that promotes the transition to renewable energy sources. It aims to reduce the country's dependence on fossil fuels and nuclear power. However, this transition has led to challenges for the power grid, as the fluctuation in energy production from renewable sources like solar and wind can make it harder to keep the lights on.

The image contains a slide from a presentation, which includes text and a diagram. The slide is titled "Power struggle: Green energy versus a grid that's not ready." It explains that renewable energy sources, such as solar and wind, have increased rapidly in Germany, but the grid is not yet sufficiently robust to handle the variability in power supply.

The slide mentions "Energiewende" as the name of this movement, which translates to "Energy Transition." It highlights that while systems aren't yet running smoothly, it's crucial to continue investing in renewable energy sources and improving the grid infrastructure.
Today’s grid

- Few large generators
  - ~10K bulk generators (>90% capacity), actively controlled
- Many dump loads
  - 131M customers, 3,100 utilities, ~billion passive loads
- Control paradigm: schedule supply to match demand
  - Centralized, human-in-the-loop, worst case, deterministic
Wind and solar farms are not dispatchable
- Many small distributed generations

Network of distributed energy resources (DERs)
- EVs, smart buildings/appliances/inverters, wind turbines, storage

Control paradigm: match demand to volatile supply
- Distributed, real-time feedback, risk limiting, robust
Risk: active DERs introduce rapid random fluctuations in supply, demand, power quality increasing risk of blackouts

Opportunity: active DERs enables realtime dynamic network-wide feedback control, improving robustness, security, efficiency

Caltech research: distributed control of networked DERs

- Foundational theory, practical algorithms, concrete applications
- Integrate engineering and economics
- Active collaboration with industry
Recap

Global energy demand will continue to grow

There is more renewable energy than the world ever needs

- Someone will figure out how to capture and store it

There will be connected intelligence everywhere

- Cost of computing, storage, communication and manufacturing will continue to drop

➔ Power system will transform into the largest and most complex Internet of Things

- Generation, transmission, distribution, consumption, storage
Recap

To develop technologies that will enable and guide the historic transformation of our power system

- Materials, devices, systems, theory, algorithms
- Control, optimization, stochastics, data, economics
Motivation: Optimal power flow

\[
\begin{align*}
\text{min} & \quad \text{tr} \left( CVV^H \right) \\
\text{over} & \quad (V, s, l) \\
\text{subject to} & \quad s_j = \text{tr} \left( Y_j^H VV^H \right) \\
& \quad l_{jk} = \text{tr} \left( B_{jk}^H VV^H \right) \\
& \quad s_j \leq s_j \leq s_j \\
& \quad l_{jk} \leq l_{jk} \leq l_{jk} \\
& \quad V_j \leq |V_j| \leq \overline{V}_j
\end{align*}
\]

gen cost, power loss
power flow equation
line flow
injection limits
line limits
voltage limits

- \( Y_j^H \) describes network topology and impedances
- \( s_j \) is net power injection (generation) at node \( j \)
The flow of power I

Basic concepts and models

Why smart grid?  (15 mins)

Three-phase AC transmission: 3 key ideas  (30 mins)
- Phasor representation
- Balanced operation
- Per-phase analysis

Device models  (30 mins)
- Transmission line
- Transformer
- Generator
Visualizing the grid

adapted from

Electric(Power(Delivery(Systems)
Tutorial(at(U.C.(Berkeley)
September(11,(2009)
(  
Dr.(Alexandra(“Sascha”(von(Meier(  

Transmission lines: 190K miles
Distribution lines: 73K miles
(2002)

[Sascha von Meier]
Today’s grid

- Few large generators
  - ~10K bulk generators (>90% capacity), actively controlled
- Many dump loads
  - 131M customers, 3,100 utilities, ~billion passive loads
- Control paradigm: schedule supply to match demand
  - Centralized, human-in-the-loop, worst case, deterministic
Power System Structure with typical voltage levels

Generators

Transmission

Subtransmission

Primary Distribution

Secondary Distribution

120 V

21 kV 230 kV

60 kV 12 kV

[Sascha von Meier]
Power System Structure with typical voltage levels

- Generators
- Transmission
- Subtransmission
- Primary Distribution
- Secondary Distribution
- 120 V service drop

- 21 kV
- 230 kV
- 60 kV
- 12 kV
Power System Structure with typical voltage levels

Generators → Transmission → Subtransmission → Primary Distribution → Secondary Distribution → 120 V

- 21 kV
- 230 kV
- 60 kV
- 12 kV
Power System Structure with typical voltage levels

- Generators
- Transmission
- Subtransmission
- Primary Distribution
- Secondary Distribution

Voltage Levels:
- 21 kV
- 230 kV
- 60 kV
- 12 kV
- 120 V

distribution substation

[Sascha von Meier]
Power System Structure with typical voltage levels

- Generators
- Transmission
- Subtransmission
- Primary Distribution
- Secondary Distribution
- Transformer & distribution line

Voltage Levels:
- 21 kV
- 230 kV
- 60 kV
- 12 kV
- 120 V
Power System Structure with typical voltage levels

Generators

Transmission

Subtransmission

Primary Distribution

Secondary Distribution

120 V

21 kV

230 kV

60 kV

12 kV
Quantities of interest

- Voltage, current, power, energy
- All are sinusoidal functions of time

Mathematical model

\[ v(t) = V_{\text{max}} \cos(\omega t + \phi) \]

- Nominal frequency
  - US: 60 Hz
  - EU: 50 Hz
Quantities of interest

- Voltage, current, power, energy
- All are sinusoidal functions of time

Voltage

\[ v(t) = V_{\max} \cos(\omega t + \phi) \]

nominal frequency
North/Central Americas: 60 Hz
Most other major countries: 50 Hz

- Steady state: frequencies at all points are nominal
- Reasonable model at timescales of minute and up
- Dynamic models at sec-min timescale: S Meyn’s tutorial

this part of tutorial is all about steady state
Phasor representation

Quantities of interest
- Voltage, current, power, energy
- All are sinusoidal functions of time

Voltage

\[ v(t) = V_{\text{max}} \cos(\omega t + \phi) \]

Voltage phasor

\[ V = \frac{V_{\text{max}}}{\sqrt{2}} e^{j\phi} \]
Phasor representation

Quantities of interest

- Voltage, current, power, energy
- All are sinusoidal functions of time

Voltage

\[ v(t) = V_{\text{max}} \cos(\omega t + \phi) \]

Voltage phasor

\[ V = \frac{V_{\text{max}}}{\sqrt{2}} e^{j \phi} \]

\[ v(t) = \text{Re} \left\{ \sqrt{2} V e^{j \omega t} \right\} = \text{Re} \left\{ V_{\text{max}} e^{j(\omega t + \phi)} \right\} \]
Phasor representation

Quantities of interest
- Voltage, current, power, energy
- All are sinusoidal functions of time

Voltage

\[ v(t) = V_{\text{max}} \cos(\omega t + \phi) \]

Voltage phasor

\[ V = \frac{V_{\text{max}}}{\sqrt{2}} e^{j \phi} \]

RMS

\[ |V| = \sqrt{\frac{1}{T} \int_{0}^{T} v^2(t) \, dt} \]
Phasor representation

Voltage

\[ v(t) = V_{\text{max}} \cos(\omega t + \phi_v) \]

\[ V = \frac{V_{\text{max}}}{\sqrt{2}} e^{j \phi_v} \]

Current

\[ i(t) = I_{\text{max}} \cos(\omega t + \phi_i) \]

\[ I = \frac{I_{\text{max}}}{\sqrt{2}} e^{j \phi_i} \]
Linear circuit elements

Resistor $R$  \[ v(t) = R \times i(t) \]

Inductor $L$  \[ v(t) = L \times \frac{di}{dt}(t) \]

Capacitor $C$  \[ i(t) = C \times \frac{dv}{dt}(t) \]

these are main circuit elements to model the grid
Linear circuit elements

Resistor $R$
\[ v(t) = R \times i(t) \]
\[ V = R \times I \]

Inductor $L$
\[ v(t) = L \times \frac{di}{dt}(t) \]
\[ V = j \ L \times I \]

Capacitor $C$
\[ i(t) = C \times \frac{dv}{dt}(t) \]
\[ V = (j \omega C)^{-1} \times I \]
Linear circuit elements

Time domain

\[ v(t) \]

\[ + \quad i(t) \quad - \]

Phasor domain

\[ V \]

\[ + \quad I \quad - \]

\[ R, jL, (jC)^1 \]
Complex power

Quantities of interest
- Voltage, current, power, energy
- All are sinusoidal functions of time

Instantaneous power

\[
p(t) = v(t)i(t) = \frac{V_{\text{max}} I_{\text{max}}}{2} \left( \cos(q_V - q_I) + \cos(2\omega t + q_V + q_I) \right)
\]

average power
Complex power

Quantities of interest

- Voltage, current, power, energy
- All are sinusoidal functions of time

Instantaneous power

\[ p(t) = v(t)i(t) \]

\[
= \frac{V_{\text{max}} I_{\text{max}}}{2} \left( \cos(q_V - q_I) + \cos(2\omega t + q_V + q_I) \right)
\]

average power

Complex power

\[ S := VI^* = P + jQ \]

real (active) power

reactive power
Phasor analysis

Steady state behavior described by algebraic equations
  - Instead of dynamic equations

Circuit analysis
  - Voltages and currents are linear

Power flow analysis
  - Power flow equations are nonlinear

\[ p(t) = v(t)i(t) \]

\[ S := VI^* \]

We will describe device and network models, and analyze them, in phasor domain
3-phase AC : 3 key ideas

- Phasor representation
- Balanced operation
- Per-phase analysis
3-phase AC system

3 single-phase system:

+- E
L
+- E
L
+- E
L

single 3-phase system:

+- E
L
+- E
L
+- E
L
+- E
L
+- E
L
+- E
L

\[ I_a + I_b + I_c = 0 \]

It does not imply that \((I_a, I_b, I_c)\) are balanced since one also needs to prove that they have the same magnitude.
3-phase AC system

Y-configuration:

Voltage source

Impedance load

Delta-configuration:
3-phase AC system

Y-configuration:

Balanced 3p source
- Equal in magnitude, 120 deg difference in phase
- $E_{an} = 1 \angle q$, $E_{bn} = 1 \angle q - 120^\circ$, $E_{cn} = 1 \angle q + 120^\circ$

Balanced 3p impedance load
- Identical impedances
3-phase AC system

Balanced 3p source
- Equal in magnitude, 120 deg difference in phase
  \[ E_{ab} = 1\angle \theta, \quad E_{bc} = 1\angle \theta - 120^\circ, \quad E_{ca} = 1\angle \theta + 120^\circ \]

Balanced 3p impedance load
- Identical impedances

Delta-configuration:
Balanced 3-phase system

Balanced 3p operation
- Balanced 3p sources
- Balanced 3p loads
- Balanced (identical) transmission lines
Advantages

1-phase

\[ p(t) = v(t)i(t), \quad S := VI^* \]

3-phase

\[ S_3 := V_aI_a^* + V_bI_b^* + V_cI_c^* \]
Advantages

1-phase

\[ p(t) = v(t)i(t), \quad S := VI^* \]

3-phase

\[ S_3 := V_a I_a^* + V_b I_b^* + V_c I_c^* = 3S \]

\[ p_3(t) := v_a(t)i_a(t) + v_a(t)i_a(t) + v_a(t)i_a(t) \]
Advantages

1-phase

\[ p(t) = v(t)i(t), \quad S := VI^* \]

3-phase

\[ S_3 := V_a I_a^* + V_b I_b^* + V_c I_c^* = 3S \]

\[ p_{3\phi}(t) := v_a(t)i_a(t) + v_b(t)i_b(t) + v_c(t)i_c(t) \]
\[ = 3|V_a||I_a|\cos(\phi_V - \phi_I) = 3P \]

Advantages of balanced 3p operation

- Instantaneous power is constant in \( t \)
- Uses \( \sim 1/2 \) as much materials (wires) as three 1p system
- Incurs \( \sim 1/2 \) as much active power loss as three 1p system
3-phase AC : 3 key ideas

- Phasor representation
- Balanced operation
- Per-phase analysis
Per-phase analysis: Wye

Important properties of balanced 3p system

- All $V_{\text{neutral-neutral}} = 0$
Important properties of balanced 3p system

- All $V_{\text{neutral-neutral}} = 0$
- All voltages and currents are 3-phase balanced
- Phases are decoupled, i.e., variables in each phase depend only on quantities in that phase
Per-phase analysis: Wye

Properties:
- All \( V_{\text{neutral-neutral}} = 0 \)
- All voltages and currents are 3-phased balanced
- Phases are decoupled

per-phase equivalent circuit
Delta-Wye transformation

**Equivalent 3p sources:** same external behavior

**line-to-line voltages:** \( E_{ab}^Y = E_{ab}^\Delta, \ E_{bc}^Y = E_{bc}^\Delta, \ E_{ca}^Y = E_{ca}^\Delta \)

\[
E_{an}^Y = \frac{E_{ab}^\Delta}{\sqrt{3} e^{j\pi/6}}.
\]
Delta-Wye transformation

Equivalent 3p sources: same external behavior
line-to-line voltages: \( E_{ab}^Y = E_{ab}^\Delta, \ E_{bc}^Y = E_{bc}^\Delta, \ E_{ca}^Y = E_{ca}^\Delta \)

\[
E_{an}^Y = \frac{E_{ab}^\Delta}{\sqrt{3} e^{j\pi/6}},
\]
\[
E_{bn}^Y = \frac{E_{bc}^\Delta}{\sqrt{3} e^{j\pi/6}},
\]
\[
E_{cn}^Y = \frac{E_{ca}^\Delta}{\sqrt{3} e^{j\pi/6}}.
\]
Delta-Wye transformation

**Equivalent 3p sources:** same external behavior
same terminal currents on same line-to-line voltages

\[
Z^Y = \frac{Z^D}{3}
\]
Per-phase analysis

- Convert all Delta sources and loads into Wye
- Solve phase \( a \) circuit with all neutrals connected for desired variables
- Phase \( b / c \) variables: subtract / add 120deg to phase \( a \) variables
- If variables internal to Delta configurations are desired, solve them from original circuit
where

Applying loads Figure EE nonzero.

Since $W_n$ "we have admittances the $E_2$ resistances $a_1$ admittances the $H_2$ node determinant $c_2$.

Three-phase system 2018 balanced $V_2$.

and (a) = $a_1 n_2$ admittances $i_1(t)$.

The three-phase $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittances $v_2(t)$ admittance

$|T_{a c 1}| \cos (w t + \theta_{a c 1}) (2.30)$

\[
\begin{align*}
\text{Solve for } V_2 \\
V_2(t) &= \text{Re} \left( \sqrt{2} V_2 e^{j t} \right)
\end{align*}
\]
Per-phase analysis

Solve for $V_1$

$$V_{ca} = \sqrt{3} e^{j\frac{\pi}{6}} V_1 e^{j\frac{\pi}{3}}$$

$$I_{ca} = L_1 V_{ca}$$

$$i_1(t) = \text{Re} \left( \sqrt{2} I_{ca} e^{j t} \right)$$

Finally, steady-state admittances $y_{ab}$ and $y_{ac}$ are given by:

$$y_{ab} = \frac{1}{y_{ac}}$$

and $\omega$ is the steady-state system frequency and $V_2$ is given by (2.29).
Recap: basic concepts

3-phase AC transmission system

- Phasor representation
- Balanced operation
- Per-phase analysis

We will describe device and network models, and analyze them, in phasor domain, using per-phase analysis
The flow of power I

Basic concepts and models

Why smart grid? (15 mins)

Three-phase AC transmission: 3 key ideas (30 mins)
- Phasor representation
- Balanced operation
- Per-phase analysis

Device models (30 mins)
- Transmission line
- Transformer
- Generator
Transmission line model

-terminal behavior

What do line parameters depend on?

What about a 3-phase line?

What are some implications?

Terminal behavior \((V_2, I_2) \rightarrow (V_1, I_1)\)

What do line parameters \((Z', Y')\) depend on?

What about a 3-phase line?

What are some implications?

Note

January 24, 2018

\[
\hat{g} = (r_g - w^2l_c) + jw(r_c + g_l).
\]

Note that \(\text{Im} \hat{g} > 0\) and hence \(|\hat{g}|^2(0, p/2)\).

If we write \(g = a + jb\) then \(a > 0\).

Hence \(\cosh(g) = \frac{1}{2}e^g + e^{-g}\) and \(\sinh(g) = \frac{1}{2}e^g - e^{-g}\) and \(\tanh(g) = \frac{e^g - e^{-g}}{e^g + e^{-g}} = \frac{1}{1 + \frac{e^{-2g}}{e^g}} = 1\) as \(g \rightarrow \infty\).

Hence \(V_1/I_1 = \frac{1}{1 + Z_0Y_0/2 Z_0 Y_0(1 + Z_0Y_0/4)}\) and \(V_2/I_2 = (3.10)\)
Transmission line model

Line inductance \( l \)

Total flux linkages \( l(t) = l \times i(t) \)

Multiple conductors

\[
\lambda_k = i_k \frac{\mu_0}{2\pi} \ln \frac{1}{r_k'} + \sum_{k' \neq k} l_{k'} \frac{\mu_0}{2\pi} \ln \frac{1}{d_{kk'}}
\]

- \( \lambda_k \): self-inductance
- \( \sum_{k' \neq k} l_{k'} \frac{\mu_0}{2\pi} \ln \frac{1}{d_{kk'}} \): mutual inductance

radius \( r_k \)

separation \( d_{kk'} \)
Transmission line model

Conditions

- Symmetric 3-phase line
- \( i_a(t) + i_b(t) + i_c(t) = 0 \)

Multiple conductors

\[
a(t) = \frac{0}{2} \ln \frac{D}{r'} x_i a(t)
\]

"self-inductance" \( l \) H/m

The phases are decoupled!
Transmission line model

Line capacitance $c$

\[ q(t) = c \times v(t) \]

Multiple conductors

\[ v_k = q_k \underbrace{\frac{1}{2\pi\varepsilon} \ln \frac{1}{r_k}}_{\text{self inductance}} + \sum_{k' \neq k} q_{k'} \underbrace{\frac{1}{2\pi\varepsilon} \ln \frac{1}{d_{kk'}}}_{\text{mutual inductance}} \]

radius $r_k$

separation $d_{kk'}$
Transmission line model

Conditions
- Symmetric 3-phase line
- \( q_a(t) + q_b(t) + q_c(t) = 0 \)

Multiple conductors

\[
\nu_k(t) = \frac{1}{2} \ln \frac{D}{r} \times q_k(t)
\]

“self-capacitance” \( 1/c \) F/m

The phases are decoupled!
Transmission line model

Line parameters (balanced 3p line)

- Phases are decoupled

- Series impedance
  \[ z = r + jwL \quad / \text{m} \]

- Shunt admittance (to neutral)
  \[ y = g + jwC \quad 1 / \text{m} \]

- Line inductance and capacitance

  \[ l = \frac{\mu_0}{2\pi} \ln \frac{D}{r'} \quad \text{H/m} \]

  \[ c = \frac{2\pi\varepsilon}{\ln(D/r)} \quad \text{F/m} \]

- Line resistance \( r \) / conductance \( g \) depend on wire material & size
Transmission line model

per-phase model of phase voltage:

\[ V_1 + - V_2 + - x I_2 I_1 \]
Transmission line model

per-phase model of phase voltage:

\[
\begin{align*}
I_1 & \quad \cdots \quad I(x) \quad \cdots \quad I_2 \\
V_1 & \quad \vdots \quad V(x) \quad \vdots \quad V_2 \\
\vdots & \quad \vdots \quad \vdots \quad \vdots
\end{align*}
\]

\[
\begin{align*}
\frac{dV}{dx} & = 0 \quad Z \quad V(x) \\
\frac{dI}{dx} & = y \quad 0 \quad I(x)
\end{align*}
\]

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix}
= \begin{bmatrix}
\cosh(\gamma \ell) & Z_c \sinh(\gamma \ell) \\
Z_c^{-1} \sinh(\gamma \ell) & \cosh(\gamma \ell)
\end{bmatrix}
\begin{bmatrix}
V_2 \\
I_2
\end{bmatrix}
\]

\[
Z_c := \sqrt{\frac{z}{y}} \quad \text{and} \quad \gamma := \sqrt{zy}
\]
Transmission line model

model of transmission line

\[ Z' = Z \times \frac{\sinh(l)}{l} \]

\[ Y' = Y \times \frac{\tanh(l/2)}{l/2} \]

\[ Z := z \ell \]

\[ Y := y \ell \]
Transmission line model

Long line (I > 150mi):

\[ Z' = Z \times \frac{\sinh(l)}{l} \]
\[ Y' = Y \times \frac{\tanh(l/2)}{l/2} \]

Long line (50 < I < 150mi):

\[ Z' = Z \]
\[ Y' = Y \]

Long line (I < 50mi):

\[ Z' = Z \]
\[ Y' = 0 \]

\[ Z := z\ell \]
\[ Y := y\ell \]
Transmission line model

High voltage min transmission line loss

Specified: required load power $|S_2|$ and voltage $|V_2|$

$$|I| = \frac{|S_2|}{|V_2|}$$

line loss = $R|I|^2$
Transmission line model

Recap

- Line characteristics depend on materials, size, and geometry of 3-phase line
- Linear per-phase circuit model \((V_2, I_2) \leftrightarrow (V_1, I_1)\)
- Circuit model: series impedance + shunt admittance

\[
\begin{align*}
V_1 & = \frac{Y_0^2}{2} V_2 + I_2 \\
V_2 & = \frac{Z_0^2}{2} \left( 1 + \frac{Y_0^2}{4} \right) V_1 - \frac{Y_0^2}{2} I_2
\end{align*}
\]
The flow of power I

Basic concepts and models

Why smart grid? (15 mins)

Three-phase AC transmission: 3 key ideas (30 mins)
- Phasor representation
- Balanced operation
- Per-phase analysis

Device models (30 mins)
- Transmission line
- Transformer
- Generator
Transformer model

Single-phase ideal transformer $n$

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} = \begin{bmatrix}
a & 0 \\
0 & n
\end{bmatrix} \begin{bmatrix}
V_2 \\
I_2
\end{bmatrix}
\]

\[
\frac{-S_{21}}{S_{12}} := \frac{V_2 I_2^{*}}{V_1 I_1^{*}} = n \ a = 1
\]
Transformer model

Single-phase (non-ideal) transformer

\[ T_{\text{ideal}} \]

\[
\begin{align*}
V_1 I_1 &= a_0 V_2 I_2 \\
\end{align*}
\]

The ratio of the complex receiving-end to sending-end power is:

\[
\frac{-S_{21}}{S_{12}} = \frac{V_2 I_2^\ast}{V_1 I_1^\ast} = n \cdot a = 1
\]

i.e., an ideal transformer has no power loss.

4.2 Equivalent circuit of single-phase transformer

A real transformer has power losses due to resistance in windings \((r | I |^2)\), eddy currents, and hysteresis losses. It also has nonzero leakage fluxes and finite permeability of the magnetic core. A more realistic transformer model includes a series resistance to model the power losses, a series inductance to model the leakage fluxes, and a shunt admittance to model the finite permeability of the magnetic core, in both the primary circuit \((Z_p, Y_m)\) and the secondary circuit \((Z_s)\), as shown in Figure 4.2(a).

This circuit is simplified by referring the impedance \(Z_s\) on the secondary side to the primary to obtain the equivalent primary side series impedance \(Z_l\):

\[
Z_l = Z_p + a_2 Z_s
\]

This is shown in Figure 4.2(b).

See Chapter 4.5 for equivalence of impedances in primary and secondary circuits.

\[ V_1 + V_2 = I_1 + I_2 \]

\[ Y_m \]

\[ N_1 \quad N_2 \]

Parameters \((n, Z_l, Y_m)\)

\((Z_l, Y_m)\) can be easily measured.
We will hence use phasor quantities in the following by default. We define the transmission matrix $T$ for an ideal transformer as

$$V_1 I_1 = a \cdot 0 \cdot 0 = a \cdot 0 \cdot 0 = \frac{n}{Z_{l}} \cdot a \cdot Y_{m} \cdot n \cdot Z_{l} \cdot V_2$$

$$I_1 = a \cdot Y_{m} \cdot n \cdot I_2$$

The ratio of the complex receiving-end to sending-end power is

$$\frac{V_2 I_2}{V_1 I_1} = \frac{V_2 I_2}{n \cdot a} \cdot 1 \cdot 1 = n \cdot a$$

i.e., an ideal transformer has no power loss.
Transformer model

3-phase ideal transformer

![Transformer Diagrams]
## Transformer model

### 3-phase ideal transformer

<table>
<thead>
<tr>
<th>Property</th>
<th>Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Voltage gain</td>
<td>$K(n)$</td>
</tr>
<tr>
<td>Current gain</td>
<td>$\frac{1}{K^*(n)}$</td>
</tr>
<tr>
<td>Power gain</td>
<td>1</td>
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</tbody>
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<td>$K_{YY}(n) := n$</td>
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Transformer model

3-phase ideal transformer

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per-phase properties
Transformer model

3-phase ideal transformer

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</tr>
<tr>
<td>$\Delta \Delta$</td>
<td>$K_{\Delta \Delta}(n) := n$</td>
</tr>
<tr>
<td>$\Delta Y$</td>
<td>$K_{\Delta Y}(n) := \sqrt{3n} e^{j\pi/6}$</td>
</tr>
</tbody>
</table>

per-phase properties
### Transformer model

#### 3-phase ideal transformer

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<td>$K_{\Delta\Delta}(n) := n$</td>
</tr>
<tr>
<td>$\Delta Y$</td>
<td>$K_{\Delta Y}(n) := \sqrt{3}n \ e^{j\pi/6}$</td>
</tr>
<tr>
<td>$Y \Delta$</td>
<td>$K_{Y \Delta}(n) := \frac{n}{\sqrt{3}} \ e^{j\pi/6}$</td>
</tr>
</tbody>
</table>

**per-phase properties**
Transformer model

Per-phase equivalent circuit

\[ \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 + Z_p Y_p & Z_p \\ Y_p & 1 \end{bmatrix} \begin{bmatrix} K^{-1}(n) & 0 \\ 0 & K^*(n) \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \]
Transformed model

Recap

- Four configurations: YY, DD, DY, YD
- Linear per-phase circuit model \((V_2, I_2) \leftrightarrow (V_1, I_1)\)
Generator model

\[ V_a = E_a - r I_a - jX_s I_a \] (5.1)

Where:

- \( V_a \): terminal voltage
- \( E_a \): open-circuit (internal) voltage
- \( r \): winding resistance
- \( X_s \): synchronous reactance

This can be represented by an open-circuit Thévenin equivalent voltage source in series with an impedance \( r + jX_s \); see Figure 5.1.

\[ V_a \] : terminal voltage
\[ E_a \] : open-circuit (internal) voltage
Putting everything together

\[ V_{\text{line}} \]

3p generator (terminal voltage) 3p transformer (stepup) 3p transmission line 3p transformer (stepdown)
Putting everything together

\[ V_{\text{line}} \]

\[ Z_{\text{line}} \]

\[ Z_{\text{load}} \]

single-phase equivalent circuit
The flow of power II

Power flow and optimization

Network models (10mins)
- Admittance matrix
- Power flow models

Optimal power flow problems (35mins)
- Formulation and example
- Convex relaxations
- Real-time OPF
Example circuit model

\[ V_1 \quad y^s \quad y^n \quad n(n-1)y \quad (1-n)y \quad y^s \quad V_2 \]

\[ I_1 \quad y \quad y \quad I_2 \]

- Generator
- Transformer
- Ideal Transformer
- Line

\[ y_1^s \quad y_2^s \quad I_1 \quad I_2 \]

\[ V_1 \quad y \quad V_2 \]

- Generator
- Load
Network model

Each line modeled as a P-model with:
- Series impedance
- Shunt admittance at each end
- They may not be equal

Relevant expressions for admittance matrix:
- $y_{jk}$ and $y_{mk}$
- $y_{jk}$ and $y_{km}$

Applying Kirchhoff's current law at buses:

$I_{12} = y_{12}^{s} (V_1 - V_2) + y_{12}^{m} V_1$
$I_{13} = y_{13}^{s} (V_1 - V_3) + y_{13}^{m} V_1$
$I_{21} = y_{21}^{s} (V_2 - V_1) + y_{21}^{m} V_2$
$I_{11} = I_{12} + I_{13} = (y_{12}^{s} + y_{13}^{s}) V_1 - y_{12}^{m} V_2 - y_{13}^{m} V_3$

$I_{22} = I_{21} = y_{22}^{s} V_2 - y_{21}^{m} V_1$

$I_{31} = I_{32} = (y_{31}^{s} + y_{32}^{s}) V_3 - y_{31}^{m} V_2 - y_{32}^{m} V_1$

Note that the admittances are not equal due to complex line loss in the P-model.
Network admittance matrix

\[ \mathbf{Y} : \text{network graph + admittances} \]

\[ \mathbf{I} = \mathbf{YV} \]

Each line is modeled by a \( \mathbf{P} \)-model with a series admittance \( y_{\text{s}jk} \) and shunt admittances \( y_{\text{m}jk} \) and \( y_{\text{m}kj} \) (not necessarily equal) at two ends of the line. Note that the \( y \) are not negatives of each other because of the complex line loss due to the series and shunt admittances in the \( \mathbf{P} \)-model.

Applying Kirchhoff's current law at buses:

\[ I_{\text{12}} = y_{\text{s}12} (V_1 - V_2) + y_{\text{m}12} V_1 \]

\[ I_{\text{13}} = y_{\text{s}13} (V_1 - V_3) + y_{\text{m}13} V_1 \]

\[ I_{\text{21}} = y_{\text{s}21} (V_1 - V_2) + y_{\text{m}21} V_1 \]

\[ I_{\text{31}} = y_{\text{s}31} (V_3 - V_1) + y_{\text{m}31} V_1 \]

Hence

\[ I_{\text{1}} = I_{\text{12}} + I_{\text{13}} \]

\[ I_{\text{2}} = I_{\text{21}} \]

\[ I_{\text{3}} = I_{\text{31}} \]

\[ \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \begin{bmatrix} y_{\text{s}12} & y_{\text{m}12} \\ y_{\text{s}13} & y_{\text{m}13} \\ y_{\text{s}21} & y_{\text{m}21} \end{bmatrix} \begin{bmatrix} I_{\text{1}} \\ I_{\text{2}} \end{bmatrix} = \begin{bmatrix} y_{\text{s}12} & y_{\text{m}12} \\ y_{\text{s}13} & y_{\text{m}13} \\ y_{\text{s}21} & y_{\text{m}21} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \]

Figure 6.7: Network admittance matrix.
Network admittance matrix

\[ Y_{jk} = \begin{cases} 
-y_{jk}^s, & j \sim k \ (j \neq k) \\
\sum_{k:j \sim k} y_{jk}^s + y_{jj}^m, & j = k \\
0, & \text{otherwise}
\end{cases} \]

where

\[ y_{jj}^m := \sum_{k:j \sim k} y_{jk}^m \]
The flow of power II

Power flow and optimization

Network models (10mins)
- Admittance matrix
- Power flow models

Optimal power flow problems (35mins)
- Formulation and example
- Convex relaxations
- Real-time OPF
Bus injection model

admittance matrix:

\[ Y_{ij} := \begin{cases} y_{ik} & \text{if } i = j \\ y_{ij} & \text{if } i \sim j \\ 0 & \text{else} \end{cases} \]

graph \( G \): undirected

\( Y \) specifies topology of \( G \) and impedances \( z \) on lines
Bus injection model

\[ I = YV \]

\[ s_j = V_j I_j^* \quad \text{for all } j \]

Kirchhoff law

power balance

admittance matrix:

\[ Y_{ij} := \begin{cases} 
  y_{ik} & \text{if } i = j \\
  y_{ij} & \text{if } i \sim j \\
  0 & \text{else} 
\end{cases} \]

\( I_j \): nodal current

\( V_j \): voltage
Bus injection model

\[ I = YV \]  
Kirchhoff law

\[ s_j = V_j I^*_j \]  
for all \( j \)  
power balance

Eliminate \( I \) :

\[ s_j = y_{jk}^* \left( |V_j|^2 - V_j V_k^* \right) \]  
for all \( j \)
Bus injection model

Complex form:

\[ S_j = \sum_{k \neq j}^* y_{jk} \left( |V_j|^2 V_j V_k^* \right) \quad \text{for all } j \]

Polar form:

\[ p_j = \left( \sum_{k=0}^n g_{jk} \right) |V_j|^2 - \sum_{k \neq j} |V_j||V_k| (g_{jk} \cos \theta_{jk} - b_{jk} \sin \theta_{jk}) \]
\[ q_j = \left( \sum_{k=0}^n b_{jk} \right) |V_j|^2 - \sum_{k \neq j} |V_j||V_k| (b_{jk} \cos \theta_{jk} + g_{jk} \sin \theta_{jk}) \]

Cartesian form:

\[ p_j = \sum_{k=0}^n \left( g_{jk} (e_j^2 + f_j^2) - g_{jk}(e_e e_k + f_e f_k) + b_{jk}(f_e e_k - e_e f_k) \right) \]
\[ q_j = \sum_{k=0}^n \left( b_{jk} (e_j^2 + f_j^2) - b_{jk}(e_e e_k + f_e f_k) - g_{jk}(f_e e_k - e_e f_k) \right) \]
Bus injection model

DC power flow

\[ p_j = \sum_{k=0}^{n} b_{jk} |V_j||V_k|(\theta_j - \theta_k) \]

Assumptions:
• Lossless short line
• Small angle difference
• Fixed voltage magnitude
• Ignore reactive power
The flow of power II

Power flow and optimization

Network models (10 mins)
- Admittance matrix
- Power flow models

Optimal power flow problems (35 mins)
- Formulation and example
- Convex relaxations
- Real-time OPF
Optimal power flow (OPF)

OPF is solved routinely for
- network control & optimization decisions
- market operations & pricing
- at timescales of mins, hours, days, ...

Non-convex and hard to solve
- Huge literature since 1962
- Common practice: DC power flow (LP)
- Also: Newton-Raphson, interior point, ...

$$\min c(x) \quad \text{s. t.} \quad F(x) = 0, \quad x \in \bar{x}$$
Optimal power flow

\[
\begin{align*}
\text{min} & \quad \text{tr} \left( CVV^H \right) \\
\text{over} & \quad (V, s, l) \\
\text{subject to} & \quad s_j = \text{tr} \left( Y_j^H VV^H \right) \\
& \quad l_{jk} = \text{tr} \left( B_{jk}^H VV^H \right) \\
\end{align*}
\]

- \( Y_j^H \) describes network topology and impedances
- \( s_j \) is net power injection (generation) at node \( j \)

- gen cost, power loss
- power flow equation
- line flow
- injection limits
- line limits
- voltage limits
Optimal power flow

\[
\begin{align*}
\text{min} & \quad \text{tr} \left( CVV^H \right) \\
\text{over} & \quad (V, s, l) \\
\text{subject to} & \quad s_j = \text{tr} \left( Y_j^H V V^H \right) \\
& \quad l_{jk} = \text{tr} \left( B_{jk}^H V V^H \right) \\
& \quad s_j \leq \underline{s}_j, \quad s_j \geq \overline{s}_j \\
& \quad l_{jk} \leq \underline{l}_{jk}, \quad l_{jk} \geq \overline{l}_{jk} \\
& \quad V_j \leq |V_j| \leq \overline{V}_j
\end{align*}
\]

- gen cost, power loss
- power flow equation
- line flow
- injection limits
- line limits
- voltage limits

**nonconvex** feasible set (nonconvex QCQP)
- \( Y_j^H \) not Hermitian (nor positive semidefinite)
- \( C \) is positive semidefinite (and Hermitian)
Optimal power flow

OPF problem underlies numerous applications
• nonlinearity of power flow equations $\Rightarrow$ nonconvexity

Ian Hiskens, Michigan
Dealing with nonconvexity

Linearization
- DC approximation

Convex relaxations
- Semidefinite relaxation (Lasserre hierarchy)
- QC relaxation (van Hentenryck)
- Strong SOCP (Sun)
Dealing with nonconvexity

Linearization
- DC approximation

Convex relaxations
- Semidefinite relaxation (Lasserre hierarchy)
- QC relaxation (van Hentenryck)
- Strong SOCP (Sun)

Realtime OPF
- Online algorithm, as opposed to offline
- Also tracks time-varying OPF
Relaxations of AC OPF
dealing with nonconvexity

Bose (UIUC)  Chandy  Farivar (Google)  Gan (FB)  Lavaei (UCB)  Li (Harvard)

many others at & outside Caltech …

Low, Convex relaxation of OPF, 2014
http://netlab.caltech.edu
Equivalent feasible sets

\[
\begin{align*}
\min & \quad \text{tr } CVV^H \\
\text{subject to} & \quad s_j \quad \text{tr } \left( Y_j^H VV^H \right) \quad \tilde{s}_j \quad v_j \quad |V_j|^2 \quad \tilde{v}_j
\end{align*}
\]

Equivalent problem:

\[
\begin{align*}
\min & \quad \text{tr } CW \\
\text{subject to} & \quad s_j \quad \text{tr } \left( Y_j^H W \right) \quad \tilde{s}_j \quad v_j \quad W_{ij} \quad \tilde{v}_j
\end{align*}
\]

\[
W \quad 0, \quad \text{rank } W = 1
\]

convex in W 

except this constraint

quadratic in V

linear in W
Solution strategy

OPF: \[ \min_{x \in X} f(x) \]

relaxation: \[ \min_{\hat{x} \in x^+} f(\hat{x}) \]

If optimal solution \( \hat{x}^* \) satisfies easily checkable conditions, then optimal solution \( x^* \) of OPF can be recovered
Equivalent relaxations

\[ V \leftrightarrow W \leftrightarrow W_G \]

**Theorem**

- Radial \( G \): SOCP is equivalent to SDP \( (v \subseteq w^+ \quad w_G^+ \subseteq w^+ \) \)
- Mesh \( G \): SOCP is strictly coarser than SDP

For radial networks: always solve SOCP!
Exact relaxation

For *radial* networks, *sufficient* conditions on
- power injections bounds, or
- voltage upper bounds, or
- phase angle bounds
Exact relaxation

For *radial* networks, *sufficient* conditions on
- power injections bounds, or
- voltage upper bounds, or
- phase angle bounds
Exact relaxation

QCQP \((C, C_k)\)

\[
\begin{align*}
\min & \quad \text{tr} \left( Cxx^H \right) \\
\text{over} & \quad x \quad C^n \\
\text{s.t.} & \quad \text{tr} \left( C_kxx^H \right) \quad b_k \quad k \quad K
\end{align*}
\]

Graph of QCQP

\(G(C, C_k)\) has edge \((i, j)\) \iff \(C_{ij} \neq 0\) or \([C_k]_{ij} \neq 0\) for some \(k\)

QCQP over tree

\(G(C, C_k)\) is a tree
Exact relaxation

**QCQP** $(C, C_k)$

$$\min \quad \text{tr}(Cxx^H)$$

over $x \in \mathbb{C}^n$

s.t. $\text{tr}(C_kxx^H) \leq b_k \quad k \in K$

**Key condition**

$i \sim j : \left( C_{ij}, [C_k]_{ij}, k \right)$ lie on half-plane through 0

**Theorem**

SOCP relaxation is exact for QCQP over tree

Bose et al 2012, 2014
Sojoudi, Lavaei 2013
Implication on OPF

Not both lower & upper bounds on real & reactive powers at both ends of a line can be finite
Example

\begin{itemize}
\item Relaxation is exact if $X$ and $Y$ have same Pareto front
\item SOCP is faster but coarser than SDP
\end{itemize}
## Potential benefits

<table>
<thead>
<tr>
<th>IEEE test systems</th>
<th>rank $\left(\overline{X}_0\right)$</th>
<th>SDP cost</th>
<th>MATPOWER cost</th>
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<td>9093.8</td>
</tr>
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</table>

12.4% lower cost than solution from nonlinear solver MATPOWER

[Louca, Seiler, Bitar 2013]
Potential benefits

Case study on an SCE feeder

- Southern California
- 1,400 residential houses, ~200 commercial buildings
- Controllable loads: EV, pool pumps, HVAC, PV inverters
- Formulated as an OPF problem, multiphase unbalanced radial network

baseline

peak load reduction: 8%
energy cost reduction: 4%

optimized
Realtime AC OPF for tracking

Gan (FB)  Tang (Caltech)  Dvijotham (DeepMind)

See also: Dall’Anese et al, Bernstein et al, Hug & Dorfler et al, Callaway et al

Gan & L, JSAC 2016
Tang et al, TSG 2017
Motivations

Simplify OPF simulation/solution
- Solving static OPF with simulator in the loop
- Avoid modifying GridLab-D during ARPA-E GENI (2012-15)

Deal with nonconvexity
- Network computes power flow solutions in real time at scale for free
- Exploit it for our optimization/control

Track optimal solution of time-varying OPF
- Uncertainty will continue to increase
- Real-time measurements increasingly become available on seconds timescale
- Must, and can, close the loop in the future
Dealing with nonconvexity

Linearization
  - DC approximation

Convex relaxations
  - Semidefinite relaxation (Lasserre hierarchy)
  - QC relaxation (van Hentenryck)
  - Strong SOCP (Sun)

Realtime OPF
  - Online algorithm, as opposed to offline
  - Also tracks time-varying OPF
Literature

Static OPF:
- Gan and Low, JSAC 2016
- Dall’Anese, Dhople and Giannakis, TPS 2016
- Arnold et al, TPS 2016

Time-varying OPF:
- Dall’Anese and Simonetto, TSG 2016
- Wang et al, TPS 2016
- Tang, Dvijotham and Low, TSG 2017
- Tang and Low, CDC 2017

Earlier relevant work on voltage control
- Survey: Molzahn et al, TSG 2017
OPF

\[ \min \quad c_0(y) + c(x) \]

over \( x, y \)

s. t.

controllable devices

uncontrollable state
OPF

\[
\begin{align*}
\min & \quad c_0(y) + c(x) \\
\text{over} & \quad x, y \\
\text{s. t.} & \quad F(x, y) = 0
\end{align*}
\]

power flow equations
\textbf{OPF}

\[
\begin{align*}
\min & \quad c_0(y) + c(x) \\
\text{over} & \quad x, \ y \\
\text{s. t.} & \quad F(x, y) = 0 \\
& \quad y \leq \bar{y} \\
& \quad x \in X := \{x, \ x, \ \bar{x}\}
\end{align*}
\]

\text{power flow equations} \quad \text{operational constraints} \quad \text{capacity limits}

Assume: \[\frac{\partial F}{\partial y} \geq 0\] \quad \Rightarrow \quad y(x) \over X
OPF: eliminate $y$

$$\min_x c_0(y(x)) + c(x)$$

s.t. $y(x) \leq \bar{y}$

$x \in X := \{x \mid x \leq \bar{x} \}$

**Theorem** [Huang, Wu, Wang, & Zhao. TPS 2016] For DistFlow model, controllable (feasible) region

$$\{x \mid y(x) \leq \bar{y}, x \in X\}$$

is convex (despite nonlinearity of $y(x)$)
min \limits_x c_0(y(x)) + c(x)

s. t. \quad y(x) \leq \bar{y}

x \quad X := \{x \quad x \quad \bar{x}\}

add barrier or penalty function
to remove operational constraints

\min \limits_{x \quad X} f(x, y(x); \quad )

f: nonconvex
Online (feedback) perspective

Network: power flow solver

\[ y(t) : F(x(t), y(t)) = 0 \]

DER: gradient update

\[ x(t+1) = G(x(t), y(t)) \]

- Explicitly exploits network as power flow solver
- Naturally tracks changing network conditions

cyber network

physical network

control \( x(t) \)

measurement, communication \( y(t) \)
Outline: realtime OPF

Motivation

Problem formulation

Static OPF
- 1st order algorithm
- Optimality properties

Time-varying OPF
- 2nd order algorithm
- Tracking performance
- Distributed implementation

[Gan & Low, JSAC 2016]
[Tang, Dj, & Low, TSG 2017]
[Tang & Low, CDC 2017]
Static OPF

\[
\begin{align*}
\min & \quad f(x, y(x); \quad) \\
onumber
\text{over} & \quad x \quad X
\end{align*}
\]

gradient projection algorithm:

\[
\begin{align*}
x(t + 1) &= x(t) \frac{f(x(t))}{x(t)} \\
y(t) &= y(x(t))
\end{align*}
\]

active control

law of physics

[Gan & Low, JSAC 2016]
Local optimality

Under appropriate assumptions

- $x(t)$ converges to set of local optima
- if #local optima is finite, $x(t)$ converges
Global optimality

Assume: $p_0(x)$ convex over $X$
$v_k(x)$ concave over $X$

$$A := \{x \in X : v(x) \leq av + (1-a)v\}$$

**Theorem**

If co\{local optima\} are in $A$ then
- $x(t)$ converges to the set of global optima
- $x(t)$ itself converges a global optimum if 
  #local optima is finite
Global optimality

Assume: \( p_0(x) \) convex over \( X \)
\( v_k(x) \) concave over \( X \)

\[ A := \{ x \in X : v(x) \leq av + (1-a)v^\star \} \]

**Theorem**

- Can choose \( a \) s.t.
  \[ A \rightarrow \text{original feasible set} \]
- If SOCP is exact over \( X \), then assumption holds

Incidentally, this turns out to be the convergence condition in Arnold, et al, “Model-Free Optimal Control of VAR Resources in Distribution Systems: An Extremum Seeking Approach,”
Informally, a local minimum is almost as good as any strictly interior feasible point.

\[ f(x^*) - f(\hat{x}) \leq r \]

any local optimum
any original feasible pt
slightly away from \( X \) boundary

\[ f(x^*) \quad f(\hat{x}) \quad 0 \]
## Simulations

<table>
<thead>
<tr>
<th># bus</th>
<th>CVX</th>
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</table>
Outline: realtime OPF

Motivation

Problem formulation

Static OPF

Dynamic OPF

- 2\textsuperscript{nd} order algorithm
- Tracking performance
- Distributed implementation

See also: Dall’Anese and Simonetto, TSG 2016
Wang et al, TPS 2016
Tracking performance

realtime OPF algorithms can track time-varying OPF well
Tracking performance

realtime OPF algorithms can track time-varying OPF well

IEEE 300 bus
Drifting OPF

\[
\min_x c_0(y(x)) + c(x)
\]
\[
\text{s. t. } y(x) \geq \bar{y}
\]
\[
x \in X
\]

\[
\min_x c_0(y(x), t) + c(x, t)
\]
\[
\text{s. t. } y(x, t) \geq \bar{y}
\]
\[
x \in X
\]

static OPF

drifting OPF
Drifting OPF

\[
\begin{align*}
\min & \quad f_t(x, y(x); t) \\
\text{over} & \quad x \in X_t
\end{align*}
\]

Quasi-Newton algorithm:

\[
\begin{align*}
x(t+1) &= x(t) - \left( H(t) \right)^{-1} \frac{f_t}{x}(x(t)) \\
y(t) &= y(x(t))
\end{align*}
\]

[_active control]

[law of physics]

[Tang, Dj & Low, 2017]
Drifting OPF

\[
\min_{x} f_t(x, y(x); t)
\]
over \( x \in X_t \)

Computing \( x(t+1) \) by solving convex QP:

\[
\begin{align*}
\min_{x} \quad & \left( \nabla f_t(x(t)) \right)^T (x \quad x(t)) \\
+ \quad & \frac{1}{2} (x \quad x(t))^T B_t(x(t)) (x \quad x(t)) \\
\text{s. t.} \quad & x \in X_t 
\end{align*}
\]

e.g. approx Hessian

[Tang, Dj & Low, 2017]
Tracking performance

\[
\text{error} := \frac{1}{T} \sum_{t=1}^{T} \left\| x^{\text{online}}(t) - x^*(t) \right\|
\]

control error

(assuming \( x^{\text{online}}(0) = x^*(0) \))
Tracking performance

\[
\text{error} := \frac{1}{T} \sum_{t=1}^{T} \| x^{\text{online}}(t) - x^*(t) \|
\]

**Theorem**

\[
\text{error} \leq \sqrt{\frac{m}{M}} \cdot \frac{1}{T} \sum_{t=1}^{T} \| x^*(t) - x^*(t-1) \| + \left( \frac{1}{T} \sum_{t=1}^{T} \| x^*(t) - x^*(t-1) \| \right)
\]

[avg rate of drifting of optimal solution of feasible set]

[Tang, Dj, & Low, TSG 2017]
Tracking performance

error := \frac{1}{T} \sum_{t=1}^{T} \left\| x^{\text{online}}(t) - x^*(t) \right\|

**Theorem**

error \leq \sqrt{\frac{m}{M}} \cdot \frac{1}{T} \sum_{t=1}^{T} \left( \left\| x^*(t) - x^*(t-1) \right\| + \right)

error in Hessian approx

[Tang, Dj, & Low, TSG 2017]
Tracking performance

\[
\text{error} := \frac{1}{T} \sum_{t=1}^{T} \left\| x^{\text{online}}(t) - x^*(t) \right\|
\]

**Theorem**

\[
\text{error} \leq \sqrt{\frac{m}{M}} \cdot \frac{1}{T} \sum_{t=1}^{T} \left( \left\| x^*(t) - x^*(t-1) \right\| + \sum_t \right)
\]

"condition number" of Hessian

[Tang, Dj, & Low, TSG 2017]
Implementation

Implement L-BFGS-B

- More scalable
- Handles (box) constraints $X$

Simulations

- IEEE 300 bus
Tracking performance

IEEE 300 bus
Tracking performance

IEEE 300 bus
Key message

Large network of DERs
- Real-time optimization at scale
- Computational challenge: power flow solution

Online optimization [feedback control]
- Network computes power flow solutions in real time at scale for free
- Exploit it for our optimization/control
- Naturally adapts to evolving network conditions

Examples
- Slow timescale: OPF
- Fast timescale: frequency control